SPECTRAL BOUNDS FOR SINGULAR INDEFINITE STURM-LIOUVILLE OPERATORS WITH $L^1$-POTENTIALS

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Abstract. The spectrum of the singular indefinite Sturm-Liouville operator
$$A = \text{sgn}(\cdot)\left(-\frac{d^2}{dx^2} + q\right)$$
with a real potential $q \in L^1(\mathbb{R})$ covers the whole real line and, in addition, non-real eigenvalues may appear if the potential $q$ assumes negative values. A quantitative analysis of the non-real eigenvalues is a challenging problem, and so far only partial results in this direction were obtained. In this paper the bound
$$|\lambda| \leq \|q\|_{L^1}^2$$
on the absolute values of the non-real eigenvalues $\lambda$ of $A$ is obtained. Furthermore, separate bounds on the imaginary parts and absolute values of these eigenvalues are proved in terms of the $L^1$-norm of the negative part of $q$.

1. Introduction

The aim of this paper is to prove bounds on the absolute values of the non-real eigenvalues of the singular indefinite Sturm-Liouville operator
$$Af = \text{sgn}(\cdot)(-f'' + qf),$$
$$\text{dom } A = \{f \in L^2(\mathbb{R}) : f, f' \in AC(\mathbb{R}), -f'' + qf \in L^2(\mathbb{R})\},$$
where $AC(\mathbb{R})$ stands for space of all locally absolutely continuous functions. It will always be assumed that the potential $q$ is real-valued and belongs to $L^1(\mathbb{R})$.

The operator $A$ is not symmetric nor self-adjoint in an $L^2$-Hilbert space due to the sign change of the weight function $\text{sgn}(\cdot)$. However, $A$ can be interpreted as a self-adjoint operator with respect to the Krein space inner product $(\text{sgn} \cdot, \cdot)$ in $L^2(\mathbb{R})$. We summarize the qualitative spectral properties of $A$ in the next theorem, which follows from [4, Theorem 4.2] or [16, Proposition 2.4] and the well-known spectral properties of the definite Sturm-Liouville operator $-\frac{d^2}{dx^2} + q$; cf. [23, 24, 25].

Theorem 1.1. The essential spectrum of $A$ coincides with $\mathbb{R}$ and the non-real spectrum of $A$ consists of isolated eigenvalues with finite algebraic multiplicity which are symmetric with respect to $\mathbb{R}$.

Indefinite Sturm-Liouville operators have been studied for more than a century, and have again attracted a lot of attention in the recent past. Early works in this context usually deal with the regular case, that is, the operator $A$ is studied on a finite interval with appropriate boundary conditions at the endpoints; cf. [15, 22] and, e.g., [11, 18, 26]. In this situation the spectrum of $A$ is purely discrete and various estimates on the real and imaginary parts of the non-real eigenvalues were

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obtained in the last few years; cf. [2, 9, 10, 14, 17, 21]. The singular case is much less studied, due to the technical difficulties which, very roughly speaking, are caused by the presence of continuous spectrum.

Explicit bounds on non-real eigenvalues for singular Sturm-Liouville operators with $L^\infty$-potentials were obtained with Krein space perturbation techniques in [5] and under additional assumptions for $L^1$-potentials in [6, 7], see also [3] for the absence of real eigenvalues and [19] for the accumulation of non-real eigenvalues of a very particular family of potentials. In this paper we substantially improve the earlier bounds in [6, 7] and relax the conditions on the potential. More precisely, here we prove for arbitrary real $q \in L^1(\mathbb{R})$ the following bound.

**Theorem 1.2.** Let $q \in L^1(\mathbb{R})$ be real. Every non-real eigenvalue $\lambda$ of the indefinite Sturm-Liouville operator $A$ satisfies

$$|\lambda| \leq ||q||^2_{L^1}, \quad (1.1)$$

Moreover, we prove two bounds in terms of the negative part $q_-$ of $q$.

**Theorem 1.3.** Let $q \in L^1(\mathbb{R})$ be real. Every non-real eigenvalue $\lambda$ of the indefinite Sturm-Liouville operator $A$ satisfies

$$|\text{Im}\lambda| \leq 24 \cdot \sqrt{3} ||q_-||_{L^1}^2 \quad \text{and} \quad |\lambda| \leq (24 \cdot \sqrt{3} + 18) ||q_-||_{L^1}^2, \quad (1.2)$$

The bound (1.1) is proved in Section 2. Its proof is based on the Birman-Schwinger principle using similar arguments as in [1, 13], [12, Chapter 14.3]; see also [8]. The bounds in (1.2) are obtained in Section 3 by adapting the techniques from the regular case in [2, 9, 21] to the present singular situation.

2. **Proof of Theorem 1.2**

In this section we prove the bound (1.1) for the non-real eigenvalues of $A$. We adapt a technique similar to the Birman-Schwinger principle in [12] and apply it to the indefinite operator $A$. The main ingredient is a bound for the integral kernel of the resolvent of the operator

$$B_0 f = \text{sgn}(\cdot)(-f''), \quad \text{dom} B_0 = \{ f \in L^1(\mathbb{R}) : f, f' \in AC(\mathbb{R}), -f'' \in L^1(\mathbb{R}) \},$$

in $L^1(\mathbb{R})$.

**Lemma 2.1.** The operator $B_0$ is closed in $L^1(\mathbb{R})$ and for all $\lambda$ in the open upper half-plane $\mathbb{C}^+$ the resolvent of $B_0$ is an integral operator

$$(B_0 - \lambda)^{-1} g = \int_{\mathbb{R}} K_\lambda(x, y) g(y) dy, \quad g \in L^1(\mathbb{R}),$$

where the kernel $K_\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is bounded by $|K_\lambda(x, y)| \leq |\lambda|^{-\frac{1}{2}}$ for all $x, y \in \mathbb{R}$.

**Proof.** Here and in the following we define $\sqrt{\lambda}$ for $\lambda \in \mathbb{C}^+$ as the principal value of the square root, which ensures $\text{Im} \sqrt{\lambda} > 0$ and $\text{Re} \sqrt{\lambda} > 0$. For $\lambda \in \mathbb{C}^+$ consider the integral operator

$$(T_\lambda g)(x) = \int_{\mathbb{R}} K_\lambda(x, y) g(y) dy, \quad g \in L^1(\mathbb{R}), \quad (2.1)$$
with the kernel $K_\lambda(x,y) = C_\lambda(x,y) + D_\lambda(x,y)$ of the form

$$C_\lambda(x,y) = \frac{1}{2\alpha \sqrt{\lambda}} \begin{cases} \alpha e^{i\sqrt{\lambda}(x+y)}, & x \geq 0, y \geq 0, \\ -e^{i\sqrt{\lambda}(x+y)}, & x \geq 0, y < 0, \\ e^{i\sqrt{\lambda}(x+y)}, & x < 0, y \geq 0, \\ -\alpha e^{i\sqrt{\lambda}(x+y)}, & x < 0, y < 0, \end{cases}$$

and

$$D_\lambda(x,y) = \frac{1}{2\alpha \sqrt{\lambda}} \begin{cases} \alpha e^{i\sqrt{\lambda}|x-y|}, & x \geq 0, y \geq 0, \\ 0, & x \geq 0, y < 0, \\ 0, & x < 0, y \geq 0, \\ -\alpha e^{-i\sqrt{\lambda}|x-y|}, & x < 0, y < 0, \end{cases}$$

where $\alpha := \frac{1-i}{\pi}$. Hence,

$$|K_\lambda(x,y)| = |C_\lambda(x,y) + D_\lambda(x,y)| \leq \frac{1}{\sqrt{\lambda}}$$

and the integral in (2.1) converges for every $g \in L^1(\mathbb{R})$. We have

$$\sup_{y \geq 0} \int_\mathbb{R} |C_\lambda(x,y)| \, dx = \frac{1}{2\sqrt{\lambda}} \left( \frac{1}{\text{Im} \sqrt{\lambda}} + \frac{\sqrt{2}}{\text{Re} \sqrt{\lambda}} \right)$$

and

$$\sup_{y < 0} \int_\mathbb{R} |C_\lambda(x,y)| \, dx = \frac{1}{2\sqrt{\lambda}} \left( \frac{\sqrt{2}}{\text{Im} \sqrt{\lambda}} + \frac{1}{\text{Re} \sqrt{\lambda}} \right).$$

For $y \geq 0$ we estimate

$$\int_0^\infty |D_\lambda(x,y)| \, dx = \frac{1}{2\sqrt{\lambda}} \int_0^\infty e^{-\text{Im} \sqrt{\lambda}|x-y|} \, dx = \frac{2 - e^{-\text{Im} \sqrt{\lambda}y}}{2\sqrt{\lambda} \text{Im} \sqrt{\lambda}} \leq \frac{1}{\sqrt{\lambda} \text{Im} \sqrt{\lambda}},$$

and analogously for $y < 0$

$$\int_{-\infty}^0 |D_\lambda(x,y)| \, dx = \frac{1}{2\sqrt{\lambda}} \int_{-\infty}^0 e^{-\text{Re} \sqrt{\lambda}|x-y|} \, dx = \frac{2 - e^{\text{Re} \sqrt{\lambda}y}}{2\sqrt{\lambda} \text{Re} \sqrt{\lambda}} \leq \frac{1}{\sqrt{\lambda} \text{Re} \sqrt{\lambda}}.$$

Hence,

$$c := \sup_{y \in \mathbb{R}} \int_\mathbb{R} |K_\lambda(x,y)| \, dx < \infty$$

and Fubini’s theorem yields

$$\|T_\lambda g\|_{L^1} \leq \int_\mathbb{R} |g(y)| \int_\mathbb{R} |K_\lambda(x,y)| \, dx \, dy \leq c \|g\|_{L^1}.$$ 

Therefore $T_\lambda$ in (2.1) is an everywhere defined bounded operator in $L^1(\mathbb{R})$.

We claim that $T_\lambda$ is the inverse of $B_0 - \lambda$. In fact, consider the functions $u, v$ given by

$$u(x) = \begin{cases} e^{i\sqrt{\lambda}x}, & x \geq 0, \\ \overline{\alpha} e^{i\sqrt{\lambda}x} + \alpha e^{-i\sqrt{\lambda}x}, & x < 0, \end{cases} \quad \text{and} \quad v(x) = \begin{cases} \alpha e^{i\sqrt{\lambda}x} + \overline{\alpha} e^{-i\sqrt{\lambda}x}, & x \geq 0, \\ e^{i\sqrt{\lambda}x}, & x < 0, \end{cases}$$

which solve the differential equation $\text{sgn}(\cdot)(-f'') = \lambda f$, that is, $u$ and $v$, and their derivatives, belong to $AC(\mathbb{R})$ and satisfy the differential equation almost everywhere. Since the Wronskian equals $2\alpha \sqrt{\lambda}$, these solutions are linearly independent.
Note that \( u, v \notin L^1(\mathbb{R}) \) and one concludes that \( B_0 - \lambda \) is injective. A simple calculation shows the identity
\[
K_\lambda(x, y) = C_\lambda(x, y) + D_\lambda(x, y) = \frac{1}{2\alpha \sqrt{\lambda}} \begin{cases} u(x)v(y) \text{sgn}(y), & y < x, \\ v(x)u(y) \text{sgn}(y), & x < y, \end{cases}
\]
and hence we have
\[
(T_\lambda g)(x) = \frac{1}{2\alpha \sqrt{\lambda}} \left( u(x) \int_{-\infty}^{x} v(y) \text{sgn}(y) g(y) \, dy + v(x) \int_{x}^{\infty} u(y) \text{sgn}(y) g(y) \, dy \right).
\]
One verifies \( T_\lambda g, (T_\lambda g)' \in AC(\mathbb{R}) \) and \( T_\lambda g \) is a solution of \( \text{sgn}(\cdot)(-f''') - \lambda f = g \).
This implies \( (T_\lambda g)'' \in L^1(\mathbb{R}) \) and hence \( T_\lambda g \in \text{dom } B_0 \) satisfies
\[
(B_0 - \lambda)T_\lambda g = g \quad \text{for all } g \in L^1(\mathbb{R}).
\]
Therefore, \( B_0 - \lambda \) is surjective and we have \( T_\lambda = (B_0 - \lambda)^{-1} \). It follows that \( B_0 \) is a closed operator in \( L^1(\mathbb{R}) \) and that \( \lambda \) belongs to the resolvent set of \( B_0 \). \( \square \)

**Proof of Theorem 1.2.** Since the non-real point spectrum of \( A \) is symmetric with respect to the real line (see Theorem 1.1) it suffices to consider eigenvalues in the upper half plane. Let \( \lambda \in \mathbb{C}^+ \) be an eigenvalue of \( A \) with a corresponding eigenfunction \( f \in \text{dom } A \). Since \( q \in L^1(\mathbb{R}) \) and \( -\frac{d^2}{dx^2} + q \) is in the limit point case at \( \pm \infty \) (see, e.g. [23, Lemma 9.37]) the function \( f \) is unique up to a constant multiple. As \( -f''' + qf = \lambda f \) on \( \mathbb{R}^+ \) and \( f''' - qf = \lambda f \) on \( \mathbb{R}^- \) with \( q \) integrable one has the well-known asymptotical behaviour
\[
\begin{align*}
f(x) &= \alpha_+ (1 + o(1)) e^{i \sqrt{\lambda} x}, & x \to +\infty, \\
f'(x) &= \alpha_+ i \sqrt{\lambda} (1 + o(1)) e^{i \sqrt{\lambda} x}, & x \to +\infty,
\end{align*}
\]
and
\[
\begin{align*}
f(x) &= \alpha_- (1 + o(1)) e^{\sqrt{\lambda} x}, & x \to -\infty, \\
f'(x) &= \alpha_- \sqrt{\lambda} (1 + o(1)) e^{\sqrt{\lambda} x}, & x \to -\infty,
\end{align*}
\]
for some \( \alpha_+, \alpha_- \in \mathbb{C} \); see, e.g. [20, § 24.2, Example a] or [23, Lemma 9.37]. These asymptotics yield \( f, qf \in L^1(\mathbb{R}) \) and \( -f''' = \lambda \text{sgn}(\cdot) f - qf \in L^1(\mathbb{R}) \), and therefore \( f \in \text{dom } B_0 \). Thus, \( f \) satisfies
\[
0 = (A - \lambda)f = \text{sgn}(\cdot)(-f''') - \lambda f + \text{sgn}(\cdot)qf = (B_0 - \lambda)f + \text{sgn}(\cdot)qf
\]
and since \( \lambda \) is in the resolvent set of \( B_0 \) we obtain
\[
-qf = q(B_0 - \lambda)^{-1} \text{sgn}(\cdot)qf.
\]
Note that \( \|qf\|_{L^1} \neq 0 \) as otherwise \( \lambda \) would be an eigenvalue of \( B_0 \). With the help of Lemma 2.1 we then conclude
\[
0 < \|qf\|_{L^1} \leq \int_{\mathbb{R}} |q(x)| \int_{\mathbb{R}} |K_\lambda(x, y)||q(y)f(y)| \, dy \, dx \leq \frac{1}{\sqrt{|\lambda|}} \|qf\|_{L^1} \|q\|_{L^1}
\]
and this yields the desired bound (1.1). \( \square \)
3. Proof of Theorem 1.3

In this section we prove the bounds in (1.2) for the non-real eigenvalues of \( A \) in Theorem 1.3, which depend only on the negative part \( q_-(x) = \max\{0, -q(x)\} \), \( x \in \mathbb{R} \), of the potential. The following lemma will be useful.

**Lemma 3.1.** Let \( \lambda \in \mathbb{C}^+ \) be an eigenvalue of \( A \) and let \( f \) be a corresponding eigenfunction. Define

\[
U(x) := \int_x^\infty \text{sgn}(t)|f(t)|^2 \, dt \quad \text{and} \quad V(x) := \int_x^\infty |f'(t)|^2 + q(t)|f(t)|^2 \, dt.
\]

for \( x \in \mathbb{R} \). Then the following assertions hold:

(a) \( \lambda U(x) = f'(x)f(x) + V(x) \);
(b) \( \lim_{x \to -\infty} U(x) = 0 \) and \( \lim_{x \to -\infty} V(x) = 0 \);
(c) \( \|f\|_{L^2} \leq 2\|q_-\|_{L^1}\|f\|_{L^2} \);
(d) \( \|f\|_\infty \leq 2\sqrt{\|q_-\|_{\mathbb{R}}^2\|f\|_{L^2}} \);
(e) \( qf^2 \|_{L^1} \leq 8\|q_-\|_{L^1}\|f\|_{L^2}^2 \).

**Proof.** Note that \( f \) satisfies the asymptotics (2.2)–(2.3) and hence \( f \) and \( f' \) vanish at \( \pm \infty \) and \( f' \in L^2(\mathbb{R}) \). In particular, \( V(x) \) is well defined. We multiply the identity \( \lambda f(t) = \text{sgn}(t)(-f''(t) + q(t)f(t)) \) by \( \text{sgn}(t)f(t) \) and integration by parts yields

\[
\lambda U(x) = \int_x^\infty -f''(t)f(t) + q(t)|f(t)|^2 \, dt = f'(x)f(x) + V(x)
\]

for all \( x \in \mathbb{R} \). This shows (a). Moreover, we have

\[
\lambda \int_\mathbb{R} \text{sgn}(t)|f(t)|^2 \, dt = \lim_{x \to -\infty} \lambda U(x) = \lim_{x \to -\infty} V(x) = \int_\mathbb{R} |f'(t)|^2 + q(t)|f(t)|^2 \, dt.
\]

Taking the imaginary part shows \( \lim_{x \to -\infty} U(x) = 0 \) and, hence, \( \lim_{x \to -\infty} V(x) = 0 \). This proves (b).

As \( f \) is continuous and vanishes at \( \pm \infty \) we have \( \|f\|_\infty < \infty \). Let \( q_+(x) := \max\{0, q(x)\} \), \( x \in \mathbb{R} \). Making use of \( \lim_{x \to -\infty} V(x) = 0 \) and \( q = q_--q_- \) we find

\[
0 \leq \|f\|_{L^2}^2 = -\int_\mathbb{R} q(t)|f(t)|^2 \, dt = -\int_\mathbb{R} (q_+(t) - q_-(t))|f(t)|^2 \, dt
\]

\[
\leq \int_\mathbb{R} q_-|f(t)|^2 \, dt \leq \|q_-\|_{L^1}\|f\|_{L^\infty}^2.
\]

This implies \( \|q_+\|_{L^1} \leq \|q_-\|_{L^1} \), \( \|q_-\|_{L^1} \leq \|q_-\|_{L^1} \), \( \|f\|_{L^\infty} \) and, thus,

\[
\|qf^2\|_{L^1} = \int_\mathbb{R} |q(t)||f(t)|^2 \, dt = \int_\mathbb{R} (q_+(t) + q_-(t))|f(t)|^2 \, dt \leq 2\|q_-\|_{L^1}\|f\|_{L^\infty}^2.
\]

In order to verify (d) let \( x, y \in \mathbb{R} \) with \( x > y \). Then

\[
|f(x)|^2 - |f(y)|^2 = \int_y^x (f'(t))^2 \, dt \leq 2\int_y^x |f(t)f'(t)| \, dt \leq 2\|f\|_{L^2}\|f'\|_{L^2}.
\]

Together with \( f(y) \to 0 \), \( y \to -\infty \), leads to \( \|f\|_{L^\infty}^2 \leq 2\|f\|_{L^2}\|f'\|_{L^2} \). Since \( f \) is an eigenfunction \( \|f\|_{L^\infty} \) does not vanish and we have with (3.1)

\[
\|f\|_{L^\infty} \leq \frac{2\|f\|_{L^2}\|f'\|_{L^2}}{\|f\|_{L^\infty}} \leq 2\sqrt{\|q_-\|_{L^1}\|f\|_{L^2}},
\]
which shows (d). Moreover, the estimate in (d) applied to (3.1) and (3.2) yield (c) and (e).

**Proof of Theorem 1.3.** Let $\lambda \in \mathbb{C}^+$ be a eigenvalue of $A$ and let $f \in \text{dom } A$ be a corresponding eigenfunction. We can assume $\|q_-\|_{L^1} > 0$ as otherwise $f = 0$ by Lemma 3.1 (d). Let $U$ and $V$ be as in Lemma 3.1, let $\delta := (24\|q_-\|_{L^1})^{-1}$ and define the function $g$ on $\mathbb{R}$ by

$$g(x) = \begin{cases} \text{sgn}(x), & |x| > \delta, \\ \frac{x}{3}, & |x| \leq \delta. \end{cases}$$

From Lemma 3.1 (a) we have

$$\lambda \int_{\mathbb{R}} g'(x)U(x) \, dx = \int_{\mathbb{R}} g'(x)(f'(x)f(x) + V(x)) \, dx. \quad (3.3)$$

Since $g$ is bounded and $U(x)$ vanishes for $x \to \pm \infty$, integration by parts leads to the estimate

$$\int_{\mathbb{R}} g'(x)U(x) \, dx = \int_{\mathbb{R}} g(x)\text{sgn}(x)|f(x)|^2 \, dx \geq \int_{|x| \leq \delta} |f(x)|^2 \, dx \geq \|f\|_{L^2}^2 - \int_{|x| > \delta} |f(x)|^2 \, dx \geq \|f\|_{L^2}^2 - 2\delta\|f\|_{L^2}^2 - 2\delta\|f\|_{L^2}^2 = \frac{2}{3}\|f\|_{L^2}^2; \quad (3.4)$$

here we have used Lemma 3.1 (d) in the last line of (3.4). Further we see with Lemma 3.1 (c)–(d)

$$\left| \int_{\mathbb{R}} g'(x)f'(x)\overline{f(x)} \, dx \right| \leq \|f\|_{L^\infty} \|f'\|_{L^2} \|g'\|_{L^2} \leq 4\|q_-\|^\frac{3}{2}_{L^2} \|f\|_{L^2}^2 \sqrt{2} \delta < 16\sqrt{3}\|q_-\|^\frac{3}{2}_{L^2} \|f\|_{L^2}^2. \quad (3.5)$$

Since $\|g\|_{L^\infty} = 1$ and $V(x)$ vanishes for $x \to \pm \infty$, integration by parts together with Lemma 3.1 (c) and (e) yields

$$\left| \int_{\mathbb{R}} g'(x)V(x) \, dx \right| = \left| \int_{\mathbb{R}} g(x)\left(|f'(x)|^2 + q(x)|f(x)|^2\right) \, dx \right| \leq \|g\|_{L^\infty} (\|f'\|_{L^2}^2 + \|qf^2\|_{L^1}) \leq 12\|q_-\|^\frac{3}{2}_{L^2} \|f\|_{L^2}^2. \quad (3.6)$$

Comparing the imaginary parts in (3.3) we have with (3.4) and (3.5)

$$\frac{2}{3}\text{Im } \lambda \|f\|_{L^2}^2 \leq \text{Im } \lambda \left| \int_{\mathbb{R}} g'(x)U(x) \, dx \right| \leq \left| \int_{\mathbb{R}} g'(x)f'(x)\overline{f(x)} \, dx \right| \leq 16\sqrt{3}\|q_-\|^\frac{3}{2}_{L^2} \|f\|_{L^2}^2. \quad \square$$

In the same way we obtain from (3.4), (3.3) and (3.5)–(3.6) that

$$\frac{2}{3}|\lambda|\|f\|_{L^2}^2 \leq \left| \lambda \int_{\mathbb{R}} g'(x)U(x) \, dx \right| \leq \left| \int_{\mathbb{R}} g'(x)(f'(x)\overline{f(x)} + V(x)) \, dx \right| \leq \left( 16\sqrt{3} + 12 \right) \|q_-\|^\frac{3}{2}_{L^2} \|f\|_{L^2}^2.$$

This shows the bounds in (1.2). \square
References
