THE IMAGINARY AIRY OPERATOR WITH A ONE-CENTER δ-INTERACTION

JUSSI BEHRNDT, IVETA SEMORÁDOVÁ, AND PETR SIEGL

Happy Birthday, Fritz! With great pleasure we dedicate this note to our friend and colleague Fritz Gesztesy on the occasion of his 70th birthday.

Abstract. The aim of this paper is to investigate spectral properties of the imaginary Airy operator with an additional δ-interaction. We find the asymptotic behavior of large eigenvalues and the resolvent norm, and thereby also establish the completeness of the eigensystem. The results are compared, in particular, with the properties of the imaginary Airy operator with an additional δ'-interaction studied in [18].

1. Introduction

Schrödinger operators with singular interactions have been a popular topic in spectral theory and mathematical physics for many decades. The first rigorous mathematical contributions in this field is [10] by Berezin and Faddeev from 1961, but the best known reference is certainly the classical monograph *Solvable Models in Quantum Mechanics* by Fritz and his coauthors published first in 1988, with a second issue from 2005 [2]. In this note we go beyond the self-adjoint models treated in Fritz's monograph and in numerous other papers, and investigate a particular non-self-adjoint spectral problem; cf. [1, 3, 8, 22, 26, 32, 33]. More precisely, here our main objective is to study the imaginary Airy operator with an additional δ-interaction in $L^2(\mathbb{R})$. Note that the Airy differential expression $-\partial_x^2 + \mathrm{i}x$ was investigated in many works, e.g. [4, 18, 19, 21]. In particular, the work of Grebenkov, Helffer and Henry [18] is an extensive study of various operator realizations, including the additional δ'-interaction, but to the best of our knowledge the case of δ-interactions has not been treated in the mathematical literature.

Recall first that the $L^2(\mathbb{R})$ realization $A^{\mathbb{R}}$ of the Airy differential expression $-\partial_x^2 + \mathrm{i}x$ is m-accretive and with compact resolvent. Moreover, the spectrum is empty and the resolvent norm diverges super-exponentially, that is,

$$\|(A^{\mathbb{R}} - \lambda)^{-1}\| = \sqrt{\frac{\pi}{2}}\lambda^{-\frac{1}{4}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right)(1 + o(1)), \quad \lambda \to +\infty, \tag{1.1}$$

2020 *Mathematics Subject Classification.* 34L15, 34L40, 47A10.

Key words and phrases. imaginary Airy operator, δ-interaction, boundary triple for a dual pair, eigenvalues, resolvent estimate.

see, e.g., [19, Sec. 14]. The last two properties are in stark contrast to the behavior of the decoupled operators $A^{\mathbb{R}_-}_{\mathrm{D,N}} \oplus A^{\mathbb{R}_+}_{\mathrm{D,N}}$ in $L^2(\mathbb{R})$ with an additional Dirichlet or Neumann condition at zero, whose spectra comprise countably many non-real eigenvalues on two rays $e^{\pm\frac{\pi}{3}\mathrm{i}}\mathbb{R}_+$ and the resolvent norm decays as $\lambda \to +\infty$, see Section 2.2 for more details.

It was established in [18] that the spectral properties of the free case $A^{\mathbb{R}}$ are substantially affected not only by the complete decoupling, but also by a δ'-interaction at 0 (also called semi-permeable barrier)

$$A_{\beta\delta'} = -\partial_x^2 + \mathrm{i}x + \beta\delta', \quad \beta \in \mathbb{R}.$$

Notice that while for $\beta = 0$ the free operator $A^{\mathbb{R}}$ is recovered, the case $\beta = \pm\infty$ corresponds to the Neumann decoupling $A^{\mathbb{R}_-}_{\mathrm{N}} \oplus A^{\mathbb{R}_+}_{\mathrm{N}}$. Thus $A_{\beta\delta'}$, $\beta \in \mathbb{R} \setminus \{0\}$, lies between these two extremes, nonetheless the spectral properties are closer to the decoupled case. In particular, the spectrum contains countably many non-real eigenvalues (asymptotically close to those of Neumann decoupling) and the resolvent norm satisfies

$$\|(A_{\beta\delta'} - \lambda)^{-1}\| = \mathcal{O}(\lambda^{-\frac{1}{4}}), \quad \lambda \to +\infty, \tag{1.2}$$

see Section 2.2 for more details.

In this work we complement the study of Airy operators in $L^2(\mathbb{R})$ by analyzing the δ-interaction at 0, i.e.

$$A_{\alpha\delta} := -\partial_x^2 + \mathrm{i}x + \alpha\delta, \quad \alpha \in \mathbb{R},$$

which lies between the free $A^{\mathbb{R}}$ for $\alpha = 0$ and the Dirichlet decoupling $A^{\mathbb{R}_-}_{\mathrm{D}} \oplus A^{\mathbb{R}_+}_{\mathrm{D}}$ for $\alpha = \pm\infty$. It turns out that similarly to the δ'-interaction, the spectral properties are closer to the decoupled case. More precisely, we will show that for $\alpha \neq 0$ there are countably many non-real eigenvalues of $A_{\alpha\delta}$ which lie asymptotically close to those of Dirichlet decoupling, see Theorem 4.5. Moreover, for $\alpha < 0$ there is exactly one simple real eigenvalue $\lambda_0(\alpha)$ and this eigenvalue satisfies

$$\lambda_0(\alpha) = \left(\frac{3}{4}\log\frac{1}{|\alpha|}\right)^{\frac{2}{3}}(1 + o(1)), \quad \alpha \to 0-,$$
$$\lambda_0(\alpha) = -\frac{\alpha^2}{4}(1 + o(1)), \qquad \alpha \to -\infty,$$

see Theorem 4.4. In other words, the real eigenvalue disappears as we approach the free case $A_{0\delta} = A^{\mathbb{R}}$ with empty spectrum and, for a large negative coupling, we recover the behavior of the eigenvalue of $-\partial_x^2 + \alpha\delta$ in $L^2(\mathbb{R})$. It also turns out in Theorem 5.1 that for $\alpha \neq 0$ the resolvent norm satisfies

$$\|(A_{\alpha\delta} - \lambda)^{-1}\| = \frac{(2\pi)^{\frac{1}{2}}}{|\alpha|}\lambda^{\frac{1}{4}} + \mathcal{O}(\lambda^{-\frac{1}{4}}), \quad \lambda \to +\infty.$$

Although the resolvent norm is not decaying as for the δ'-interaction, see (1.2), it diverges much slower than the free case (1.1). Moreover, the obtained power-growth still allows to conclude the completeness of the eigensystem, see Corollary 5.2. In fact, as the regularity of coefficients is known to affect the behavior of the resolvent norm both in the semiclassical and non-semiclassical case, see [14, 20, 27], the observed difference in the rates reflects the intuition that the δ-interaction is less singular than δ'.

The paper is organized as follows. In Section 2, we summarize some known facts on various Airy operators, mostly relying on the results in [18]. In Section 3, we introduce the operator $A_{\alpha\delta}$ first by recalling the form based strategy and then via the construction of the boundary triple for a dual pair; cf. [9, 28, 29, 30, 31]. The latter provides, in particular, an associated Weyl function and a Krein-type resolvent formula. In Section 4 we investigate the spectrum of $A_{\alpha\delta}$ based on the analysis of the Weyl function and Section 5 contains the resolvent estimate for $\lambda \to +\infty$. Finally, Appendix A summarizes relevant properties of Airy functions and contains some technical lemmas.

Acknowledgment. The authors are most grateful to the anonymous referee for various useful comments which improved the manuscript. J.B. gratefully acknowledges financial support by the Austrian Science Fund (FWF): P33568-N. I.S. is most grateful for the stimulating research stay at Graz University of Technology and acknowledges support from NAWI Graz. This publication is based upon work from COST Action CA 18232 MAT-DYN-NET, supported by COST (European Cooperation in Science and Technology), www.cost.eu.

2. Some known results on imaginary Airy operators

We briefly recall the main facts on the domains of definition, spectra and behavior of the resolvent norm. In the following, for $a, b \geq 0$, we write $a \lesssim b$ if there exists a constant $C > 0$, independent of any relevant variable or parameter, such that $a \leq Cb$. The relation $a \gtrsim b$ is defined analogously whereas $a \approx b$ means that $a \lesssim b$ *and* $a \gtrsim b$.

2.1. **Various Airy operators.** In the following, for a multiplication operator in $L^2(\Omega)$ by a measurable function m, the maximal domain is denoted by

$$\mathrm{Dom}(m) = \{f \in L^2(\Omega) \,:\, mf \in L^2(\Omega)\}.$$

The following operators in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}_\pm)$

$$
\begin{aligned}
A^{\mathbb{R}} &= -\partial_x^2 + \mathrm{i}x, \quad \mathrm{Dom}(A^{\mathbb{R}}) = W^{2,2}(\mathbb{R}) \cap \mathrm{Dom}(x), \\
A_{\mathrm{D}}^{\mathbb{R}_\pm} &= -\partial_x^2 + \mathrm{i}x, \quad \mathrm{Dom}(A_{\mathrm{D}}^{\mathbb{R}_\pm}) = \{f \in W^{2,2}(\mathbb{R}_\pm) \cap \mathrm{Dom}(x) \; : \\
&\qquad\qquad\qquad\qquad\qquad\qquad f(0) = 0\}, \\
A_{\mathrm{N}}^{\mathbb{R}_\pm} &= -\partial_x^2 + \mathrm{i}x, \quad \mathrm{Dom}(A_{\mathrm{N}}^{\mathbb{R}_\pm}) = \{f \in W^{2,2}(\mathbb{R}_\pm) \cap \mathrm{Dom}(x) \; : \\
&\qquad\qquad\qquad\qquad\qquad\qquad f'(0) = 0\},
\end{aligned}
\tag{2.1}
$$

are m-accretive and have compact resolvents in the Schatten-class $\mathcal{S}_p$ with any $p > 3/2$, see [18]. Following [19, Sec. 14] or [17, Thm. VII.2.6], the domain of $A^{\mathbb{R}}$ can also be characterized as

$$
\mathrm{Dom}(A^{\mathbb{R}}) = \{f \in L^2(\mathbb{R}) \; : \; -f'' + \mathrm{i}xf \in L^2(\mathbb{R})\}. \tag{2.2}
$$

The separation of the domain (2.1) is related to the graph norm estimates, in particular for $A^{\mathbb{R}}$,

$$
\|A^{\mathbb{R}} f\|^2 + \|f\|^2 \approx \|f''\|^2 + \|xf\|^2 + \|f\|^2, \quad f \in \mathrm{Dom}(A^{\mathbb{R}}), \tag{2.3}
$$

and analogously for $A_{\mathrm{D,N}}^{\mathbb{R}_\pm}$, see [11, 25]. The adjoint operators read

$$
\begin{aligned}
B^{\mathbb{R}} &:= (A^{\mathbb{R}})^* = -\partial_x^2 - \mathrm{i}x, \qquad \mathrm{Dom}((A^{\mathbb{R}})^*) = \mathrm{Dom}(A^{\mathbb{R}}), \\
B_{\mathrm{D,N}}^{\mathbb{R}_\pm} &:= (A_{\mathrm{D,N}}^{\mathbb{R}_\pm})^* = -\partial_x^2 - \mathrm{i}x, \quad \mathrm{Dom}((A_{\mathrm{D,N}}^{\mathbb{R}_\pm})^*) = \mathrm{Dom}(A_{\mathrm{D,N}}^{\mathbb{R}_\pm}).
\end{aligned}
\tag{2.4}
$$

The work [18] comprises also the Robin case and, in particular, the δ'-interaction with coupling $\beta \in \mathbb{R}$, i.e.

$$
\begin{aligned}
A_{\beta\delta'}^{\mathbb{R}} &= -\partial_x^2 + \mathrm{i}x, \\
\mathrm{Dom}(A_{\beta\delta'}^{\mathbb{R}}) &= \{f \in W^{2,2}(\mathbb{R} \setminus \{0\}) \cap \mathrm{Dom}(x) \; : \; f'(0-) = f'(0+), \\
&\qquad\qquad\qquad\qquad f(0+) - f(0-) = \beta f'(0)\};
\end{aligned}
$$

note that in [18] this interaction is called a transmission boundary condition or semi-permeable barrier and the notation $\beta = 1/\kappa$ is used. It is known that $A_{\beta\delta'}^{\mathbb{R}}$ is quasi-m-accretive (i.e. there is $c_\beta > 0$ such that $A_{\beta\delta'}^{\mathbb{R}} + c_\beta$ is m-accretive) and with compact resolvent in $\mathcal{S}_p$ with any $p > 3/2$, see [18, Sec. 4].

2.2. **Spectra and resolvent norms.** The spectrum of $A^{\mathbb{R}}$ is empty and the spectra of $A_{\mathrm{D,N}}^{\mathbb{R}_+}$ are explicit in terms of the Airy zeros a_n and a_n', $n \in \mathbb{N}$, of the function Ai and its derivative Ai$'$, respectively, see Appendix A,

$$
\sigma(A^{\mathbb{R}}) = \emptyset, \quad \sigma(A_{\mathrm{D}}^{\mathbb{R}_+}) = \{e^{\frac{\pi}{3}\mathrm{i}}|a_n|\}_{n\in\mathbb{N}}, \quad \sigma(A_{\mathrm{N}}^{\mathbb{R}_+}) = \{e^{\frac{\pi}{3}\mathrm{i}}|a_n'|\}_{n\in\mathbb{N}}. \tag{2.5}
$$

All eigenvalues of $A_{\mathrm{D,N}}^{\mathbb{R}_+}$ are simple (with the algebraic multiplicity one) and the corresponding eigenfunctions are complete in $L^2(\mathbb{R}_+)$.

The spectra of the other operators on the half-line can be obtained by straightforward transformations. For instance, employing

$$
\mathcal{P} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_-) \; : \; (\mathcal{P}u)(x) := u(-x),
$$

we arrive at $\mathcal{P}^{-1} A_D^{\mathbb{R}_-} \mathcal{P} = B_D^{\mathbb{R}_+}$ and $\mathcal{P}^{-1} A_N^{\mathbb{R}_-} \mathcal{P} = B_N^{\mathbb{R}_+}$, thus

$$\sigma(A_{\mathrm{D}}^{\mathbb{R}_-}) = \sigma(B_{\mathrm{D}}^{\mathbb{R}_+}) = \{\overline{\lambda} \,:\, \lambda \in \sigma(A_{\mathrm{D}}^{\mathbb{R}_+})\} = \{e^{-\frac{\pi}{3}\mathrm{i}}|a_n|\}_{n\in\mathbb{N}},$$
$$\sigma(A_{\mathrm{N}}^{\mathbb{R}_-}) = \sigma(B_{\mathrm{N}}^{\mathbb{R}_+}) = \{\overline{\lambda} \,:\, \lambda \in \sigma(A_{\mathrm{N}}^{\mathbb{R}_+})\} = \{e^{-\frac{\pi}{3}\mathrm{i}}|a_n'|\}_{n\in\mathbb{N}}.$$

Moreover, the following spectral properties of $A_{\beta\delta'}^{\mathbb{R}}$ are established in [18, Thm. 1.1].

Theorem 2.1 ([18, Thm. 1.1])**.** *For any $\beta \in \mathbb{R}\setminus\{0\}$, the spectrum of $A_{\beta\delta'}$ is discrete and non-empty. The eigenvalues $\lambda_n(\beta)$ are determined as solutions of the equation*

$$2\beta\pi \operatorname{Ai}'(e^{\frac{2}{3}\pi\mathrm{i}}\lambda) \operatorname{Ai}'(e^{-\frac{2}{3}\pi\mathrm{i}}\lambda) + 1 = 0.$$

For every $\beta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, there exists a unique eigenvalue of $A_{\beta\delta'}$ in the ball $B(e^{\pm\frac{\pi}{3}\mathrm{i}}|a_n'|, 2(\beta|a_n'|)^{-1})$. Moreover, for every $\beta > 0$, the eigensystem of $A_{\beta\delta'}$ is complete in $L^2(\mathbb{R})$.

In the theorem above, the eigensystem means the collection of all eigenfunctions and possibly finitely many root functions (generalized eigenfunctions) as finitely many Jordan blocks can be present (although the numerics suggests that all eigenvalues are simple, see [18]).

We note that the resolvents of the operators in Section 2.1 are integral operators with kernels explicit in Airy functions, see [18] for details. We recall here the behavior of the resolvent norm. For the operator $A^{\mathbb{R}}$, it is well-known that the resolvent norm is independent of $\operatorname{Im}\lambda$ and that

$$\|(A^{\mathbb{R}} - \lambda)^{-1}\| = \sqrt{\frac{\pi}{2}}\lambda^{-\frac{1}{4}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right)(1 + o(1)), \quad \lambda \to +\infty,$$

see [6, 12, 19] for details. For the Dirichlet and Neumann (and also Robin) operators on the half-line, the Hilbert-Schmidt norm has the asymptotic behavior

$$\|(A_{\mathrm{D,N}}^{\mathbb{R}_+} - \lambda)^{-1}\|_{\mathcal{S}_2} \approx \lambda^{-\frac{1}{4}}(\log\lambda)^{\frac{1}{2}}, \quad \lambda \to +\infty,$$

see [18, Prop. 3.6, 3.10]. Moreover, [7, Thm. 4.2, Rem. 5.4, Ex. 7.1] yields that the operator norm satisfies

$$\|(A_{\mathrm{D,N}}^{\mathbb{R}_+} - \lambda)^{-1}\| \lesssim \lambda^{-\frac{1}{4}}, \quad \lambda \to +\infty. \tag{2.6}$$

Finally, for the δ'-interaction with $\beta \neq 0$, it is established in [18, Prop. 8.5] that

$$\|(A_{\beta\delta'} - \lambda)^{-1} - (A_{\mathrm{N}}^{\mathbb{R}_-} \oplus A_{\mathrm{N}}^{\mathbb{R}_+} - \lambda)^{-1})\|_{\mathcal{S}_2} \lesssim \lambda^{-\frac{3}{4}}, \quad \lambda \to +\infty,$$

hence, using the estimates on the resolvent of $A_{\mathrm{N}}^{\mathbb{R}_\pm}$ above,

$$\|(A_{\beta\delta'} - \lambda)^{-1}\|_{\mathcal{S}_2} \approx \lambda^{-\frac{1}{4}}(\log\lambda)^{\frac{1}{2}}, \quad \|(A_{\beta\delta'} - \lambda)^{-1}\| \lesssim \lambda^{-\frac{1}{4}} \quad \lambda \to +\infty.$$

3. The imaginary Airy operator with δ-interaction

3.1. **Form approach.** The imaginary Airy operator with δ-interaction at 0 with coupling $\alpha \in \mathbb{R}$, i.e.

$$-\partial_x^2 + \mathrm{i}x + \alpha\delta(x), \quad \alpha \in \mathbb{R},$$

can be introduced as a densely defined closed operator $A_{\alpha\delta}$ in $L^2(\mathbb{R})$ through the corresponding sesquilinear form

$$\begin{aligned} a_{\alpha\delta}[f] &:= \|f'\|^2 + \mathrm{i}\int_{\mathbb{R}} x|f(x)|^2\,\mathrm{d}x + \alpha|f(0)|^2, \\ \mathrm{Dom}(a_{\alpha\delta}) &:= W^{1,2}(\mathbb{R}) \cap \mathrm{Dom}(|x|^{\frac{1}{2}}). \end{aligned}$$

Although this form is not coercive (or closed sectorial), it is coercive in the generalized sense of Almog and Helffer, see [5] where the free case $\alpha = 0$ is investigated in detail. For $\alpha \neq 0$, the standard inequality

$$|f(0)|^2 \leq \varepsilon\|f'\|^2 + C_\varepsilon\|f\|^2, \quad f \in W^{1,2}(\mathbb{R}),$$

valid with an arbitrary $\varepsilon > 0$ and some $C_\varepsilon > 0$, can be used to verify that the additional term is a relatively small perturbation that does not affect the generalized coercivity. Thus $a_{\alpha\delta}$ defines a closed operator $A_{\alpha\delta}$ with non-empty resolvent set and compact resolvent. Following [24, Ex. VI.2.16] and [25], one can verify that

$$\begin{aligned} A_{\alpha\delta} &= -\partial_x^2 + \mathrm{i}x, \\ \mathrm{Dom}(A_{\alpha\delta}) &= \{f \in W^{2,2}(\mathbb{R}\setminus\{0\}) \cap \mathrm{Dom}(x) \ : \\ &\qquad f(0-) = f(0+), f'(0+) - f'(0-) = \alpha f(0)\}, \end{aligned} \tag{3.1}$$

as expected. It is also straightforward to check that $A_{\alpha\delta}$ is quasi-m-accretive, thus the numerical range argument [24, Thm. V.3.2] yields a basic resolvent estimate, namely for every $\tau \in (\pi/2, 3\pi/2)$,

$$\|(A_{\alpha\delta} - e^{\mathrm{i}\tau}t)^{-1}\| \lesssim \frac{1}{t}, \quad t \to +\infty. \tag{3.2}$$

Remark 3.1. It follows from the pseudomode construction in [27, Sec. 5] that for every $\tau \in (-\pi/2, 0) \cup (0, \pi/2)$ and every $N \in N$

$$\|(A_{\alpha\delta} - e^{\mathrm{i}\tau}t)^{-1}\| \gtrsim t^N, \quad e^{\mathrm{i}\tau}t \in \rho(A_{\alpha\delta}),\ t \to +\infty. \tag{3.3}$$

On the other hand, following [7, Sec. 3,5.3], we obtain

$$\|(A_{\alpha\delta} - \mathrm{i}t)^{-1}\| \lesssim 1, \quad t \to +\infty. \tag{3.4}$$

In fact, in both cases (3.3), (3.4) a slight extension of the arguments in the given references is needed to show that local perturbations or singular interactions do not affect the claims. In the former, it is used that the pseudomodes are supported around $\sin(\tau)t$, see [27, Sec. 5], in the latter, the partition in [7, Sec. 5.3] can be employed and the estimate around 0 remains valid, see the proof of [7, Prop. 5.3].

3.2. **Boundary triple.** In the following, we introduce the operators $A_{\alpha\delta}$, $\alpha \in \mathbb{R}$, in an alternative way via a boundary triple for dual pairs, see, e.g. [29, 30] or [9, Sec. 6]. This construction yields also a convenient representation of the resolvent (which can also be obtained directly, e.g. relying on [24, Lem. III.6.36]).

First, observe that the minimal operator in $L^2(\mathbb{R})$

$$(3.5)\qquad \begin{aligned} A_{\min} &= -\partial_x^2 + \mathrm{i}x, \\ \mathrm{Dom}(A_{\min}) &= \{f \in W^{2,2}(\mathbb{R} \setminus \{0\}) \cap \mathrm{Dom}(x) \; : \\ &\qquad\qquad f(\pm 0) = f'(0\pm) = 0\}, \end{aligned}$$

is closed and its adjoint is given by

$$(3.6)\qquad A_{\min}^* = -\partial_x^2 - \mathrm{i}x, \quad \mathrm{Dom}(A_{\min}^*) = W^{2,2}(\mathbb{R} \setminus \{0\}) \cap \mathrm{Dom}(x).$$

Indeed, the closedness of $A_{\min}$ can be checked by standard arguments employing that $A_{\min} \subset A^{\mathbb{R}}$ and that the graph norm separation (2.3) allows to control the Dirichlet and Neumann trace. The claim on the adjoint is a consequence of the next lemma.

Lemma 3.2. *The adjoint of the operator*

$$\begin{aligned} A_{\min}^{\mathbb{R}_+} &= -\partial_x^2 + \mathrm{i}x, \\ \mathrm{Dom}(A_{\min}^{\mathbb{R}_+}) &= \{f \in W^{2,2}(\mathbb{R}_+) \cap \mathrm{Dom}(x) \; : \; f(0) = f'(0) = 0\}, \end{aligned}$$

in $L^2(\mathbb{R}_+)$ *reads*

$$(A_{\min}^{\mathbb{R}_+})^* = -\partial_x^2 - \mathrm{i}x, \quad \mathrm{Dom}((A_{\min}^{\mathbb{R}_+})^*) = W^{2,2}(\mathbb{R}_+) \cap \mathrm{Dom}(x).$$

Proof. The inclusion $W^{2,2}(\mathbb{R}_+) \cap \mathrm{Dom}(x) \subset \mathrm{Dom}(A_{\min}^{\mathbb{R}_+})^*$ is straightforward. For the other inclusion, let $g \in \mathrm{Dom}(A_{\min}^{\mathbb{R}_+})^*$. Then, there is $\eta \in L^2(\mathbb{R}_+)$ such that, for all $f \in \mathrm{Dom}(A_{\min}^{\mathbb{R}_+})$, we have

$$(A_{\min}^{\mathbb{R}_+} f, g)_{L^2(\mathbb{R}_+)} = (f, \eta)_{L^2(\mathbb{R}_+)}.$$

Since $C_0^\infty(\mathbb{R}_+) \subset \mathrm{Dom}(A_{\min}^{\mathbb{R}_+})$, we obtain

$$-g'' = \eta + \mathrm{i}xg \quad \text{in } \mathcal{D}'(\mathbb{R}_+).$$

As $g, \eta \in L^2(\mathbb{R}_+)$ and $xg \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}_+})$, we conclude $g'' \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}_+})$. Moreover, since $g'(x) = \int_0^x g''(t)\,\mathrm{d}t + c$ with some $c \in \mathbb{C}$, we also have $g' \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}_+})$. In summary,

$$\mathrm{Dom}(A_{\min}^{\mathbb{R}_+})^* \subset \{g \in L^2(\mathbb{R}_+) \; : \; g', g'' \in L^2_{\mathrm{loc}}(\overline{\mathbb{R}_+}), -g'' - \mathrm{i}xg \in L^2(\mathbb{R}_+)\}.$$

Next, consider $\varphi \in C^\infty(\overline{\mathbb{R}_+})$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $x \in [0,1]$ and $\varphi(x) = 0$ for $x > 2$. Define $g_1 := \varphi g$, $g_2 := (1 - \varphi)g$ and notice that $g_1 \in W^{2,2}(\mathbb{R}_+) \cap \mathrm{Dom}(x)$. Moreover, the product rule and properties of φ and g imply that the zero extension $\widetilde{g}_2$ of g_2 to $\mathbb{R}$ satisfies $\widetilde{g}_2 \in L^2(\mathbb{R})$ and $-\widetilde{g}_2'' - \mathrm{i}x\widetilde{g}_2 \in L^2(\mathbb{R})$. Hence $\widetilde{g}_2 \in \mathrm{Dom}((A^{\mathbb{R}})^*)$ by (2.2) and (2.4), and (2.1) implies $\widetilde{g}_2 \in W^{2,2}(\mathbb{R}) \cap \mathrm{Dom}(x)$, so that $g_2 \in W^{2,2}(\mathbb{R}_+) \cap \mathrm{Dom}(x)$. Finally, since $g = g_1 + g_2$, we obtain $g \in W^{2,2}(\mathbb{R}_+) \cap \mathrm{Dom}(x)$. □

Next, the densely defined closed operators

$$\begin{aligned} A &= -\partial_x^2 + \mathrm{i}x, \\ \mathrm{Dom}(A) &= \{f \in W^{2,2}(\mathbb{R} \setminus \{0\}) \cap \mathrm{Dom}(x) : \\ &\qquad f(0\pm) = 0, f'(0+) = f'(0-)\}, \\ B &= -\partial_x^2 - \mathrm{i}x, \\ \mathrm{Dom}(B) &= \mathrm{Dom}(A), \end{aligned} \tag{3.7}$$

form a dual pair $\{A, B\}$ as they satisfy

$$(Bf, g)_{L^2(\mathbb{R})} = (f, Ag)_{L^2(\mathbb{R})}, \quad f \in \mathrm{Dom}(B),\ g \in \mathrm{Dom}(A).$$

Moreover, using (3.5) and (3.6), one can check that

$$\begin{aligned} A^* &= -\partial_x^2 - \mathrm{i}x, \quad \mathrm{Dom}(A^*) = W^{2,2}(\mathbb{R} \setminus \{0\}) \cap W^{1,2}(\mathbb{R}) \cap \mathrm{Dom}(x), \\ B^* &= -\partial_x^2 + \mathrm{i}x, \quad \mathrm{Dom}(B^*) = \mathrm{Dom}(A^*). \end{aligned}$$

Lemma 3.3. *Let the dual pair of operators $\{A, B\}$ in $L^2(\mathbb{R})$ be as in* (3.7)*. Let $\mathcal{G} := \mathbb{C} \oplus \mathbb{C}$ and define the mappings*

$$\begin{aligned} \Gamma^B &= \begin{pmatrix} \Gamma_0^B \\ \Gamma_1^B \end{pmatrix} : \mathrm{Dom}(B^*) \to \mathcal{G} : f \mapsto \begin{pmatrix} f'(0-) - f'(0+) \\ f(0) \end{pmatrix}, \\ \Gamma^A &= \begin{pmatrix} \Gamma_0^A \\ \Gamma_1^A \end{pmatrix} = \begin{pmatrix} \Gamma_0^B \\ \Gamma_1^B \end{pmatrix} = \Gamma^B. \end{aligned}$$

Then $\{\mathcal{G}, \Gamma^B, \Gamma^A\}$ is a boundary triple for the dual pair $\{A, B\}$, i.e. the abstract Green's identity

$$(B^* f, g)_{L^2(\mathbb{R})} - (f, A^* g)_{L^2(\mathbb{R})} = (\Gamma_1^B f, \Gamma_0^A g)_{\mathbb{C}} - (\Gamma_0^B f, \Gamma_1^A g)_{\mathbb{C}}, \tag{3.8}$$

holds for all $f \in \mathrm{Dom}(B^*)$ and all $g \in \mathrm{Dom}(A^*)$ and the mappings Γ^B and Γ^A are surjective.

Moreover, we have (see (2.1), (2.4)*)*

$$\begin{aligned} A_0 &:= B^* \restriction \ker(\Gamma_0^B) = A^{\mathbb{R}}, \\ B_0 &:= A^* \restriction \ker(\Gamma_0^A) = B^{\mathbb{R}}, \\ A_1 &:= B^* \restriction \ker(\Gamma_1^B) = A_{\mathrm{D}}^{\mathbb{R}_-} \oplus A_{\mathrm{D}}^{\mathbb{R}_+}, \\ B_1 &:= A^* \restriction \ker(\Gamma_1^A) = B_{\mathrm{D}}^{\mathbb{R}_-} \oplus B_{\mathrm{D}}^{\mathbb{R}_+}, \end{aligned} \tag{3.9}$$

and, in particular, $\rho(A_0) = \rho(B_0) = \mathbb{C}$.

Proof. The surjectivity of Γ^B and Γ^A can be justified by a standard argument. The verification of the Green's identity reduces to integration by parts. Indeed, for all $f \in \mathrm{Dom}(B^*)$ and all $g \in \mathrm{Dom}(A^*)$ we have that

$$\begin{aligned}
&(B^* f, g)_{L^2(\mathbb{R})} - (f, A^* g)_{L^2(\mathbb{R})} \\
&\quad = -f'(0-)\overline{g(0-)} + f'(0+)\overline{g(0+)} + f(0-)\overline{g'(0-)} - f(0+)\overline{g'(0+)} \\
&\quad = f(0)\overline{(g'(0-) - g'(0+))} - (f'(0-) - f'(0+))\overline{g(0)} \\
&\quad = (\Gamma_1^B f, \Gamma_0^A g)_{\mathbb{C}} - (\Gamma_0^B f, \Gamma_1^A g)_{\mathbb{C}}.
\end{aligned}$$

Finally, (3.9) follows from (2.1), (2.4) and (with $\iota = A, B$)

$$\begin{aligned}
\ker(\Gamma_0^\iota) &= \{f \in \mathrm{Dom}(B^*) \,:\, f'(0+) = f'(0-)\}, \\
\ker(\Gamma_1^\iota) &= \{f \in \mathrm{Dom}(B^*) \,:\, f(0+) = f(0-) = 0\},
\end{aligned}$$

and $\rho(A_0) = \rho(B_0) = \mathbb{C}$ follows from (2.5) and (2.4). □

In the next step, we find the associated γ-fields and Weyl-Titchmarsh function, see [9, Def. 6.2]. To this end, notice that

$$\ker(B^* - \lambda) = \mathrm{span}\{f_\lambda\}, \quad \ker(A^* - \lambda) = \mathrm{span}\{g_\lambda\}, \quad \lambda \in \mathbb{C}, \tag{3.10}$$

with

$$\begin{aligned}
f_\lambda(x) &= 2\pi\Big(u_+(0;\lambda)u_-(x;\lambda)\vartheta(-x) + u_-(0;\lambda)u_+(x;\lambda)\vartheta(x)\Big), \\
g_\lambda(x) &= f_\lambda(-x), \quad x \in \mathbb{R},
\end{aligned} \tag{3.11}$$

where ϑ denotes the Heaviside function and

$$u_\pm(x;\lambda) := \mathrm{Ai}\left(e^{\pm 2\pi \mathrm{i}/3}(-\mathrm{i}x + \lambda)\right);$$

see Appendix A for more details. Notice also that (A.7) implies

$$\Gamma_0^B f_\lambda = \Gamma_0^A g_\lambda = 2\pi W[u_+(0;\lambda), u_-(0;\lambda)] = 1, \quad \lambda \in \mathbb{C}. \tag{3.12}$$

Lemma 3.4. *Let $\{\mathcal{G}, \Gamma^B, \Gamma^A\}$ be the boundary triple for the dual pair $\{A, B\}$ from Lemma 3.3 and let the functions f_λ, g_λ, $\lambda \in \mathbb{C}$, be as in* (3.11)*. Then, for all $\lambda \in \mathbb{C}$, the associated γ-fields are given by*

$$\begin{aligned}
\gamma(\lambda) &:= \left(\Gamma_0^B \restriction \ker(B^* - \lambda)\right)^{-1} : \mathbb{C} \to \ker(B^* - \lambda) : w \mapsto w f_\lambda, \\
\gamma_*(\lambda) &:= \left(\Gamma_0^A \restriction \ker(A^* - \lambda)\right)^{-1} : \mathbb{C} \to \ker(A^* - \lambda) : w \mapsto w g_\lambda,
\end{aligned} \tag{3.13}$$

and the adjoint of $\gamma_* : \mathbb{C} \to L^2(\mathbb{R})$ satisfies

$$\gamma_*^*(\lambda) : L^2(\mathbb{R}) \to \mathbb{C} : g \mapsto (g, g_\lambda)_{L^2(\mathbb{R})}, \quad \lambda \in \mathbb{C}. \tag{3.14}$$

The Weyl-Titchmarsh function $M(\lambda) := \Gamma_1^B \gamma(\lambda)$, $\lambda \in \mathbb{C}$, associated to the boundary triple $\{\mathcal{G}, \Gamma^B, \Gamma^A\}$ is given by

$$M(\lambda) = 2\pi u_-(0;\lambda)u_+(0;\lambda) \tag{3.15}$$

and admits the integral representation

$$M(\lambda) = \frac{1}{2\pi^{\frac{1}{2}}} \int_0^\infty \frac{\exp(\lambda t - \frac{1}{12}t^3)}{t^{\frac{1}{2}}}\,\mathrm{d}t, \quad \lambda \in \mathbb{C}. \tag{3.16}$$

Proof. The formulas (3.13) follow from (3.10), (3.12) and $\rho(B_0) = \rho(A_0) = \mathbb{C}$. The representation of M in (3.16) is a consequence of (A.3) and (A.4)

$$\begin{aligned} M(\lambda) = f_\lambda(0) = 2\pi u_-(0;\lambda)u_+(0;\lambda) &= 2\pi \operatorname{Ai}\left(\lambda e^{-2\pi \mathrm{i}/3}\right) \operatorname{Ai}\left(\lambda e^{2\pi \mathrm{i}/3}\right) \\ = \frac{\pi}{2}(\operatorname{Ai}(\lambda)^2 + \operatorname{Bi}(\lambda)^2) &= \frac{1}{2\pi^{\frac{1}{2}}} \int_0^\infty \frac{\exp(\lambda t - \frac{1}{12}t^3)}{t^{\frac{1}{2}}}\,\mathrm{d}t, \quad \lambda \in \mathbb{C}. \end{aligned}$$

The adjoint of $\gamma_*(\lambda)$ in (3.14) is obtained directly from the definition. □

Finally, to define the extension $A_{\alpha\delta}$, $\alpha \in \mathbb{R}$, of A from (3.7), we need to realize the transmission conditions

$$f(0+) = f(0-), \quad f'(0+) - f'(0-) = \alpha f(0), \quad \alpha \in \mathbb{R}, \tag{3.17}$$

see (3.1). Notice that for $\alpha \neq 0$, the second condition in (3.17) can be expressed as

$$\left(\Gamma_1^B - \left(-\frac{1}{\alpha}\right)\Gamma_0^B\right) f = 0.$$

Lemma 3.5. *Let $\{\mathcal{G}, \Gamma^B, \Gamma^A\}$ be the boundary triple for the dual pair $\{A, B\}$ from Lemma 3.3 and let the associated γ-fields and the Weyl-Titchmarsh function M be as in Lemma 3.4. Then the operator $A_{\alpha\delta}$, $\alpha \in \mathbb{R}\setminus\{0\}$, in* (3.1) *coincides with*

$$A_\Theta = B^* \restriction \ker(\Gamma_1^B - \Theta\Gamma_0^B), \quad \Theta = -\frac{1}{\alpha}. \tag{3.18}$$

Moreover, the resolvent of $A_{\alpha\delta}$ satisfies the Krein-type formula

$$(A_{\alpha\delta} - \lambda)^{-1} = (A^{\mathbb{R}} - \lambda)^{-1} - \frac{\alpha}{1 + \alpha M(\lambda)}(\cdot, g_{\overline{\lambda}})_{L^2(\mathbb{R})} f_\lambda, \tag{3.19}$$

for all $\lambda \in \mathbb{C}$ such that

$$1 + \alpha M(\lambda) \neq 0. \tag{3.20}$$

In the special cases, we have $A_{0\delta} = A^{\mathbb{R}}$ and $A_{\pm\infty\delta} = A_{\mathrm{D}}^{\mathbb{R}^-} \oplus A_{\mathrm{D}}^{\mathbb{R}^+}$. Furthermore,

$$(A_{\mathrm{D}}^{\mathbb{R}^-} \oplus A_{\mathrm{D}}^{\mathbb{R}^+} - \lambda)^{-1} = (A^{\mathbb{R}} - \lambda)^{-1} - \frac{1}{M(\lambda)}(\cdot, g_{\overline{\lambda}})_{L^2(\mathbb{R})} f_\lambda, \tag{3.21}$$

for $\lambda \in \mathbb{C}$ such that $M(\lambda) \neq 0$, i.e. $\lambda \notin \sigma(A_{\mathrm{D}}^{\mathbb{R}^-}) \cup \sigma(A_{\mathrm{D}}^{\mathbb{R}^+})$.

Proof. The claims follow from Lemmas 3.3, 3.4 and the general Krein-type resolvent formula having the form

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)[\Theta - M(\lambda)]^{-1}\gamma_*(\overline{\lambda})^*,$$

for all $\lambda \in \rho(A_0)$ such that $(\Theta - M(\lambda))^{-1} \in \mathcal{B}(\mathcal{G})$; see [29, Prop. 5.2, Thm. 5.5] and [9, Sec. 6]. □

4. Spectrum of $A_{\alpha\delta}$

4.1. **Qualitative properties.** The qualitative properties of the spectrum of $A_{\alpha\delta}$ can be easily obtained from the properties of the function M, see [29, Prop. 5.2], [9, Sec. 6], and the resolvent formula (3.19).

Proposition 4.1. *Let the operator $A_{\alpha\delta}$, $\alpha \in \mathbb{R} \setminus \{0\}$, be as in* (3.18)*. Then*

i) *the resolvent set of $A_{\alpha\delta}$ is non-empty, the resolvent of $A_{\alpha\delta}$ is compact and in the Schatten-class $\mathcal{S}_p$ with any $p > 3/2$;*

ii) *all eigenvalues of $A_{\alpha\delta}$ can be found as zeros of the entire function*

$$\mathbb{C} \ni \lambda \mapsto 1 + \alpha M(\lambda), \tag{4.1}$$

where M is as in (3.13)*. Moreover,*

$$\lambda \in \sigma_{\mathrm{p}}(A_{\alpha\delta}) \quad \Longleftrightarrow \quad \overline{\lambda} \in \sigma_{\mathrm{p}}(A_{\alpha\delta}) \tag{4.2}$$

and the algebraic multiplicity of every eigenvalue is equal to the multiplicity of the corresponding zero of (4.1)*;*

iii) *as $\alpha \to 0$, every eigenvalue of $A_{\alpha\delta}$ diverges to ∞;*

iv) *as $\alpha \to \pm\infty$, every eigenvalue of $A_{\alpha\delta}$ either diverges to ∞ or converges to a (simple) eigenvalue of $A_{\mathrm{D}}^{\mathbb{R}^-} \oplus A_{\mathrm{D}}^{\mathbb{R}^+}$, moreover, every eigenvalue of the latter is approximated by exactly one simple eigenvalue of $A_{\alpha\delta}$.*

Proof. i) By (3.16) and the dominated convergence we infer that $M(\lambda) \to 0$ as $\lambda \to -\infty$. Thus, for every $\alpha \in \mathbb{R} \setminus \{0\}$, there is $c_\alpha < 0$ such that all $\lambda \in (-\infty, c_\alpha)$ satisfy (3.20). Hence the resolvent set of $A_{\alpha\delta}$ is non-empty and the resolvent formula (3.19) is valid for such λ. The resolvent is moreover compact and in the claimed Schatten-class since it is a rank-one perturbation of the resolvent of $A^{\mathbb{R}}$ with such a property, see Section 2.1.

ii) The first claim follows from [29, Prop. 5.2] or [9, Sec. 6] and $\Theta = -1/\alpha$. The symmetry of the spectrum with respect to $\mathbb{R}$ can be seen from (4.1) and (3.16). For the equality of the multiplicities see [9, Thm. 6.4].

iii) The claim follows by ii) and Hurwitz's theorem, see e.g. [13, Thm. VII.25]. To this end, consider a family of entire functions $p_\alpha = 1 + \alpha M$, $\alpha \in \mathbb{R}$, and observe that $p_0(\lambda) = 1$, $\lambda \in \mathbb{C}$, has no zeros.

iv) As in iii), the claim is a consequence of ii) and Hurwitz's theorem. Here a suitable family of entire functions reads $r_\alpha = \alpha^{-1} + M$, $\alpha \in \mathbb{R} \setminus \{0\}$, and $r_{\pm\infty} = M$. Recall that the zeros of M coincide with the eigenvalues of $A_{\mathrm{D}}^{\mathbb{R}^-} \oplus A_{\mathrm{D}}^{\mathbb{R}^+}$, see also (3.21). □

Remark 4.2. We note that while the eigenvalues of $A_{\alpha\delta}$ are obtained as zeros of the entire function (4.1), i.e. as solutions of the equation

$$2\pi \operatorname{Ai}\left(\lambda e^{-\frac{2\pi i}{3}}\right) \operatorname{Ai}\left(\lambda e^{\frac{2\pi i}{3}}\right) = -\frac{1}{\alpha},$$

the eigenvalues of $A_{\beta\delta'}$, studied in [18], are obtained as solutions of the equation

$$2\pi \,\mathrm{Ai}'\left(\lambda e^{-\frac{2\pi \mathrm{i}}{3}}\right) \mathrm{Ai}'\left(\lambda e^{\frac{2\pi \mathrm{i}}{3}}\right) = -\frac{1}{\beta}.$$

To investigate the spectral properties of $A_{\alpha\delta}$ in more detail, we summarize the asymptotic behavior of M.

Lemma 4.3. *Let the function* M *be as in* (3.13) *and let* $\epsilon > 0$ *be fixed (and small). Then* M *obeys the following asymptotic relations as* $\lambda \to \infty$ *(and the following conditions on* $\operatorname{Arg}\lambda$ *are satisfied):*

i) *for* $\operatorname{Arg}\lambda \in [0, \frac{\pi}{3} - \epsilon]$,

$$M(\lambda) = \frac{e^{\frac{4}{3}\lambda^{\frac{3}{2}}}}{2\lambda^{\frac{1}{2}}}(1 + \mathcal{O}(|\lambda|^{-\frac{3}{2}})); \tag{4.3}$$

ii) *for* $\operatorname{Arg}\lambda \in [\frac{\pi}{3} - \epsilon, \frac{\pi}{3} + \epsilon]$ *and with* $\omega := e^{-\frac{\pi}{3}\mathrm{i}}\lambda$,

$$M(e^{\frac{\pi}{3}\mathrm{i}}\omega) = \frac{e^{-\frac{\pi}{6}\mathrm{i}}}{2\omega^{\frac{1}{2}}}\left(e^{\frac{4}{3}\mathrm{i}\omega^{\frac{3}{2}}} + \mathrm{i}\right)\left(1 + \mathcal{O}(|\omega|^{-\frac{3}{2}})\right); \tag{4.4}$$

iii) *for* $\operatorname{Arg}\lambda \in [\frac{\pi}{3} + \epsilon, \pi]$

$$M(\lambda) = \frac{\mathrm{i}}{2\lambda^{\frac{1}{2}}}\left(1 + \mathcal{O}(|\lambda|^{-\frac{3}{2}})\right). \tag{4.5}$$

Proof. The claims follow from the definition of M in (3.15) and asymptotic properties of $u_\pm$ in Lemma A.1. □

4.2. **The real eigenvalue.** We investigate the real spectrum of $A_{\alpha\delta}$. It turns out that there are no real eigenvalues for $\alpha > 0$ and exactly one simple real eigenvalue $\lambda_0(\alpha)$ for $\alpha < 0$. As expected, $\lambda_0(\alpha)$ diverges as $\alpha \to 0-$ since the spectrum of $A_{0\delta} = A^{\mathbb{R}}$ is empty. Moreover, for $\alpha \to -\infty$, the eigenvalue $\lambda_0(\alpha)$ exhibits the "usual" asymptotic behavior of the eigenvalue of $-\partial_x^2 + \alpha\delta$ in $L^2(\mathbb{R})$; see Figure 1 for illustration.

Theorem 4.4. *Let the operator* $A_{\alpha\delta}$, $\alpha \in \mathbb{R} \setminus \{0\}$, *be as in* (3.18). *Then*

i) *for* $\alpha > 0$, *there are no real eigenvalues of* $A_{\alpha\delta}$;

ii) *for* $\alpha < 0$, *there is exactly one simple real eigenvalue* $\lambda_0(\alpha)$ *of* $A_{\alpha\delta}$. *Moreover, this eigenvalues satisfies*

$$\lambda_0(\alpha) = \left(\frac{3}{4}\log\frac{1}{|\alpha|}\right)^{\frac{2}{3}}\left(1 + \mathcal{O}\left(\frac{\log\left(\log\frac{1}{|\alpha|}\right)}{\log\frac{1}{|\alpha|}}\right)\right), \quad \alpha \to 0-, \tag{4.6}$$

$$\lambda_0(\alpha) = -\frac{\alpha^2}{4}\left(1 + \mathcal{O}(|\alpha|^{-3})\right), \qquad \alpha \to -\infty. \tag{4.7}$$

Proof. i) Since $M(\lambda) > 0$ for $\lambda \in \mathbb{R}$, see (3.16), the function (4.1) has no real zero if $\alpha > 0$, i.e. there are no real eigenvalues.

ii) Relying on (3.16), we infer that $M'(\lambda) > 0$ for all $\lambda \in \mathbb{R}$ and

$$\lim_{\lambda \to -\infty} M(\lambda) = 0, \qquad \lim_{\lambda \to +\infty} M(\lambda) = +\infty, \tag{4.8}$$

by the monotone convergence theorem (or directly by (4.3) and (4.5)). Hence the function (4.1) has exactly one real zero $\lambda(\alpha)$ if $\alpha < 0$, which is the sought eigenvalue. It is straightforward from (4.8) that

$$\lim_{\alpha \to -\infty} \lambda_0(\alpha) = -\infty, \qquad \lim_{\alpha \to 0-} \lambda_0(\alpha) = +\infty.$$

Finally, to obtain the asymptotic formula (4.6), we use the asymptotic behavior of M in (4.3) for $\lambda \to +\infty$ leading to the equation

$$\frac{e^{\frac{4}{3}\lambda^{\frac{3}{2}}}}{2\lambda^{\frac{1}{2}}}(1 + \mathcal{O}(|\lambda|^{-\frac{3}{2}})) = \frac{1}{|\alpha|}.$$

By taking the logarithm and using Taylor's theorem, we arrive at

$$\frac{4}{3}\lambda^{\frac{3}{2}} \left(1 + \mathcal{O}(|\lambda|^{-\frac{3}{2}} \log \lambda)\right) = \log \frac{1}{|\alpha|}$$

and hence

$$\lambda = \left(\frac{3}{4} \log \frac{1}{|\alpha|}\right)^{\frac{2}{3}} \left(1 + \mathcal{O}(|\lambda|^{-\frac{3}{2}} \log \lambda)\right). \tag{4.9}$$

Further manipulation and application of Taylor's theorem yield

$$|\lambda|^{-\frac{3}{2}} \approx \left(\log \frac{1}{|\alpha|}\right)^{-1} \quad \text{and} \quad \log \lambda \approx \log \left(\log \frac{1}{|\alpha|}\right),$$

and combining this with (4.9) we conclude (4.6). The formula (4.7) is obtain analogously using the expansion of M in (4.5) for $\lambda \to -\infty$.

□

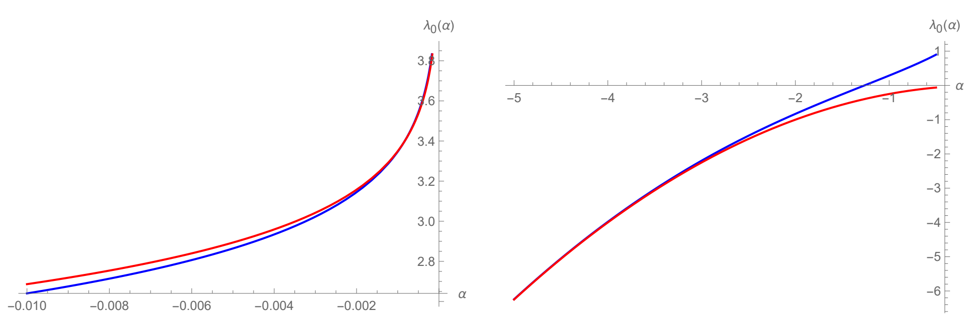

FIGURE 1. A comparison of the numerically computed eigenvalue in blue and the asymptotic expansions (4.6) (left) and (4.7) (right) in red.

4.3. **Non-real eigenvalues.** Besides the real eigenvalue when $\alpha < 0$, the spectrum of $A_{\alpha\delta}$ contains countably many non-real eigenvalues for every $\alpha \neq 0$. For large modulus, these eigenvalues are simple and asymptotically close to the eigenvalues of $A_{\pm\infty\delta} = A_{\mathrm{D}}^{\mathbb{R}^-} \oplus A_{\mathrm{D}}^{\mathbb{R}^+}$, see Figure 2 for illustration.

Theorem 4.5. *Let the operator $A_{\alpha\delta}$, $\alpha \in \mathbb{R} \setminus \{0\}$, be as in* (3.18) *and let $\epsilon > 0$. Then there are countable many non-real eigenvalues $\{\lambda_n^{\pm}(\alpha)\}_{n\in\mathbb{N}}$ of $A_{\alpha\delta}$. With an exception of possibly a finite number $N(\epsilon,\alpha) \in \mathbb{N}$ of them, all are simple and lie in the sectors around the rays $e^{\pm\frac{\pi}{3}\mathrm{i}}\mathbb{R}_+$, i.e.*

$$\left|\operatorname{Arg}(\lambda_n^{\pm}(\alpha)) \mp \frac{\pi}{3}\right| \leq \epsilon, \quad n > N(\epsilon, \alpha).$$

Moreover, they satisfy the asymptotic expansion as $n \to \infty$

$$\begin{aligned} \lambda_n^+(\alpha) &= e^{\frac{\pi}{3}\mathrm{i}}\left(|a_n| - \frac{\mathrm{i}\log\frac{4|a_n|}{\alpha^2} + \frac{2}{3}\pi + 2\operatorname{Arg}(\alpha)}{4|a_n|^{\frac{1}{2}}}\right) + \mathcal{O}(|a_n|^{-1}), \\ \lambda_n^-(\alpha) &= \overline{\lambda_n^+(\alpha)}, \end{aligned} \tag{4.10}$$

where $\{a_n\}_{n\in\mathbb{N}}$ are the zeros of Ai*, see* (A.5)*.*

Proof. The proof is based on (4.1) and the equality of the multiplicities in Proposition 4.1, the asymptotic expansion of M in Lemma 4.3 and Rouché's theorem.

Due to the symmetry of eigenvalues with respect to the real axis in (4.2), we focus only on the upper complex half plane. First, the asymptotic behavior of M in Lemma 4.3 shows that, for λ with large modulus, the eigenvalue equation $1 + \alpha M(\lambda) = 0$ cannot be satisfied outside a sector around $e^{\frac{\pi}{3}\mathrm{i}}\mathbb{R}_+$.

On the other hand, in the sector, the eigenvalue equation can be rewritten employing the expansion (4.4) and the notation $\omega^{\frac{3}{2}} = \xi$ as

$$e^{\frac{4}{3}\xi\mathrm{i}+\frac{5}{6}\pi\mathrm{i}+\mathrm{i}\operatorname{Arg}\alpha} = \frac{2\xi^{\frac{1}{3}}}{|\alpha|}\left(1 + \mathcal{O}(|\xi|^{-\frac{1}{3}})\right) \tag{4.11}$$

for $|\xi|$ large. Notice that ξ lies in a sector $\Sigma(\epsilon) := \{\xi \in \mathbb{C} : \operatorname{Re}\xi > 0, |\operatorname{Im}\xi| \leq \epsilon \operatorname{Re}\xi\}$ around $\mathbb{R}_+$, where $\epsilon > 0$ can be selected arbitrarily small. We define numbers

$$c_n := \frac{3}{4}\left(2n\pi - \frac{5}{6}\pi - \operatorname{Arg}\alpha\right), \quad n \in \mathbb{N},$$

and split $\Sigma(\epsilon)$ into vertical strips of the equal width around c_n

$$S(n,\epsilon) := \left\{z \in \Sigma(\epsilon) \,:\, |\operatorname{Re} z - c_n| \leq \frac{3}{4}\pi\right\}, \quad n \in \mathbb{N}; \tag{4.12}$$

notice that $(c_{n+1} - c_n)/2 = 3\pi/4$.

For a large $n \in \mathbb{N}$, we consider the equation (4.11) for $\xi \in S(n,\epsilon)$. Comparing the absolute values of the left and right hand side of (4.11), we make an ansatz

$$\xi = c_n - \frac{3}{4}\mathrm{i}\log\frac{2b_n}{|\alpha|} - \frac{3}{4}\mathrm{i}u,$$

where $u \in \mathbb{C}$ and

$$b_n := \left(\frac{3}{2}n\pi\right)^{\frac{1}{3}}.$$

Note that

$$c_n = \frac{3}{2}n\pi\left(1 + \mathcal{O}(n^{-1})\right) \quad \text{and} \quad c_n^{\frac{1}{3}} = b_n\left(1 + \mathcal{O}(n^{-1})\right).$$

Since $\operatorname{Re}\xi = c_n + \frac{3}{4}\operatorname{Im}u$ and as $\xi \in S(n,\epsilon)$ we first conclude $|\operatorname{Im}u| \le \pi$ and hence also

$$\operatorname{Re}\xi = \frac{3}{2}n\pi\left(1 + \mathcal{O}(n^{-1})\right). \tag{4.13}$$

Furthermore, from $\operatorname{Im}\xi = -\frac{3}{4}\log\frac{2b_n}{|\alpha|} - \frac{3}{4}\operatorname{Re}u$ we obtain $\operatorname{Re}u = -\frac{4}{3}\operatorname{Im}\xi - \log\frac{2b_n}{|\alpha|}$. Using $\xi \in S(n,\epsilon)$ we have $|\operatorname{Im}\xi| \le \epsilon\operatorname{Re}\xi$ and together with (4.13) we then find $|\operatorname{Re}u| = \epsilon\,\mathcal{O}(n)$. Recall that $\epsilon > 0$ can be selected sufficiently small, thus Taylor's theorem yields

$$\xi^{\frac{1}{3}} = b_n\left(1 + \epsilon\,\mathcal{O}(1) + \mathcal{O}(n^{-1}\log n)\right),$$

hence (4.11) can be rewritten as

$$e^u = (1 + \epsilon\,\mathcal{O}(1) + \mathcal{O}(n^{-1}\log n))(1 + \mathcal{O}(n^{-\frac{1}{3}})). \tag{4.14}$$

If (4.14) has a solution for a large n, then $|\operatorname{Re}u| = \mathcal{O}(1)$ as $n \to \infty$ since the right hand side of (4.14) remains in a neighborhood of 1 as $n \to \infty$. Nevertheless, then

$$\xi^{\frac{1}{3}} = b_n\left(1 + \mathcal{O}(n^{-1}\log n)\right),$$

and (4.14) can be further rewritten as

$$e^u = 1 + \mathcal{O}(n^{-\frac{1}{3}}). \tag{4.15}$$

Since $|\operatorname{Im}u| \le \pi$ and $|\operatorname{Re}u| = \mathcal{O}(1)$ as $n \to \infty$, the equation (4.15) can have a solution only if $u = o(1)$ as $n \to \infty$ as the right hand side tends to 1.

Finally, to find a solution we employ Rouché's theorem in a neighborhood of 0 and with functions $f(u) = u$ and $g(u) = e^u - 1 + \mathcal{O}(n^{-\frac{1}{3}})$ obtained from (4.15). Since $|f(u) - g(u)| = \mathcal{O}(u^2) + \mathcal{O}(n^{-\frac{1}{3}})$ and $|f(u)| = |u|$, we can select a sufficiently large $C > 0$ so that for all sufficiently large $n \in \mathbb{N}$ and all $|u| = Cn^{-\frac{1}{3}}$, we have $|f(u) - g(u)| < |f(u)| + |g(u)|$. Hence the unique solution $u_n = \mathcal{O}(n^{-\frac{1}{3}})$ exists for each sufficiently large $n \in \mathbb{N}$.

Returning back to the variable ξ, we obtain the solutions

$$\xi_n = b_n^3 - \frac{3}{4}\mathrm{i}\log\frac{2b_n}{|\alpha|} - \frac{5}{8}\pi - \frac{3}{4}\operatorname{Arg}\alpha + \mathcal{O}(b_n^{-1}).$$

The asymptotic expansions of Airy zeros a_n, see (A.5), yields that

$$b_n^3 = |a_n|^{\frac{3}{2}} + \frac{3}{8}\pi + \mathcal{O}(b_n^{-3}) = |a_n|^{\frac{3}{2}}(1 + \mathcal{O}(b_n^{-3}))$$

hence

$$\begin{aligned}\xi_n &= |a_n|^{\frac{3}{2}} - \frac{3}{8}\mathrm{i}\log\frac{4|a_n|}{\alpha^2} - \frac{\pi}{4} - \frac{3}{4}\operatorname{Arg}\alpha + \mathcal{O}(|a_n|^{-\frac{1}{2}}) \\ &= |a_n|^{\frac{3}{2}}\left(1 - \frac{3}{8|a_n|^{\frac{3}{2}}}\left(\mathrm{i}\log\frac{4|a_n|}{\alpha^2} + \frac{2}{3}\pi + 2\operatorname{Arg}\alpha\right) + \mathcal{O}(|a_n|^{-2})\right).\end{aligned}$$

Finally, expressing the solution in terms of $\omega = \xi^{\frac{2}{3}}$ and multiplying by the phase $e^{\frac{\pi}{3}\mathrm{i}}$, we arrive at (4.10). □

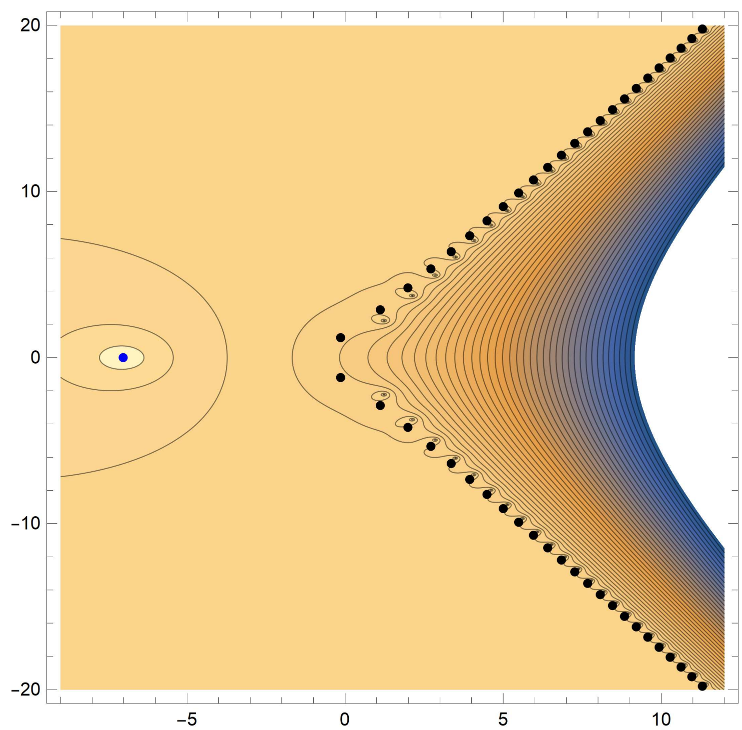

FIGURE 2. Comparison of the contour plot of $-\log|1 + \alpha M(\lambda)|$ for $\alpha = -5.3$, the main asymptotic terms of non-real eigenvalues in (4.10) (black balls) and the negative eigenvalue in (4.7) (blue ball).

5. Resolvent norm

We investigate the resolvent norm for $\lambda \to +\infty$. The main observation, similar to the case of δ', see [18, Sec. 8], is that the analysis can be reduced to the decoupled case $A_{\mathrm{D}}^{\mathbb{R}^-} \oplus A_{\mathrm{D}}^{\mathbb{R}^+}$ with decaying resolvent norm and a power-bounded rank-one perturbation.

Theorem 5.1. *Let the operator $A_{\alpha\delta}$, $\alpha \in \mathbb{R}\setminus\{0\}$, be as in* (3.18). *Then*

$$\|(A_{\alpha\delta}-\lambda)^{-1}\| = \frac{(2\pi)^{\frac{1}{2}}}{|\alpha|}\lambda^{\frac{1}{4}} + \mathcal{O}(\lambda^{-\frac{1}{4}}), \quad \lambda\to+\infty. \tag{5.1}$$

Proof. Let $\lambda>0$ be sufficiently large. By Lemma 3.5, the resolvent of $A_{\alpha\delta}$ can be expressed as

$$\begin{aligned}
(A_{\alpha\delta}-\lambda)^{-1} &= (A^{\mathbb{R}}-\lambda)^{-1} - \frac{\alpha}{1+\alpha M(\lambda)}(\cdot, g_\lambda)_{L^2(\mathbb{R})} f_\lambda \\
&= (A^{\mathbb{R}}-\lambda)^{-1} - \left(\frac{1}{M(\lambda)} - \frac{1}{M(\lambda)(1+\alpha M(\lambda))}\right)(\cdot, g_\lambda)_{L^2(\mathbb{R})} f_\lambda \\
&= (A_{\mathrm{D}}^{\mathbb{R}^-}\oplus A_{\mathrm{D}}^{\mathbb{R}^+}-\lambda)^{-1} + \frac{1}{M(\lambda)(1+\alpha M(\lambda))}(\cdot, g_\lambda)_{L^2(\mathbb{R})} f_\lambda \\
&=: R_1(\lambda)+R_2(\lambda).
\end{aligned}$$

Since $\|f_\lambda\|_{L^2(\mathbb{R})} = \|g_\lambda\|_{L^2(\mathbb{R})}$, we have

$$\|R_2(\lambda)\| = \frac{\|f_\lambda\|^2_{L^2(\mathbb{R})}}{|\alpha||M(\lambda)|^2|1+(\alpha M(\lambda))^{-1}|}$$

and Lemmas A.1 and A.2 yield

$$\begin{aligned}
\|f_\lambda\|^2_{L^2(\mathbb{R})} &= 4\pi^2\left(|u_+(0;\lambda)|^2\|u_-(\cdot;\lambda)\|^2_{L^2(\mathbb{R}_-)} + |u_-(0;\lambda)|^2\|u_+(\cdot;\lambda)\|^2_{L^2(\mathbb{R}_+)}\right) \\
&= \left(\frac{\pi}{8}\right)^{\frac{1}{2}}\frac{e^{\frac{8}{3}\lambda^{\frac{3}{2}}}}{\lambda^{\frac{3}{4}}}(1+\mathcal{O}(\lambda^{-\frac{3}{2}})), \quad \lambda\to+\infty.
\end{aligned}$$

Hence

$$\|R_2(\lambda)\| = \frac{(2\pi)^{\frac{1}{2}}}{|\alpha|}\lambda^{\frac{1}{4}}(1+\mathcal{O}(\lambda^{-\frac{3}{2}})), \qquad \lambda\to+\infty$$

and the claim follows since (2.6) implies that $\|R_1(\lambda)\| = \mathcal{O}(\lambda^{-\frac{1}{4}})$, $\lambda\to+\infty$. □

As corollary of the improved resolvent behavior at $\mathbb{R}_+$ comparing to $A^{\mathbb{R}}$, we obtain the completeness of the eigensystem.

Corollary 5.2. *The eigensystem of the operator $A_{\alpha\delta}$, $\alpha\in\mathbb{R}\setminus\{0\}$, is complete in $L^2(\mathbb{R})$.*

Proof. The claim follows by the classical sufficient condition on the completeness of eigensystem, see [16, Cor. XI.31]. To this end, recall that the resolvent of $A_{\alpha\delta}$ is in $\mathcal{S}_p$ for every $p>3/2$, see Proposition 4.1, and we have the resolvent bounds (3.2) and (5.1). □

Appendix A. Airy functions

Following [15, Chap. 9], the standard solutions of the Airy equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = zw$$

are $\mathrm{Ai}(z)$, $\mathrm{Bi}(z)$ and $\mathrm{Ai}(ze^{\mp 2\pi\mathrm{i}/3})$; all entire functions of z. For our purposes, the most suitable choice are functions $\mathrm{Ai}(ze^{\mp 2\pi\mathrm{i}/3})$ which satisfy

$$W\left[\mathrm{Ai}\left(ze^{-2\pi\mathrm{i}/3}\right), \mathrm{Ai}\left(ze^{2\pi\mathrm{i}/3}\right)\right] = \frac{1}{2\pi\mathrm{i}}.$$

In the following, let $\zeta := \frac{2}{3}z^{3/2}$ and $\epsilon > 0$ is fixed and sufficiently small. For $z \in \mathbb{C}$ satisfying $|\operatorname{Arg} z| \leq \pi - \epsilon$, the functions Ai and Ai$'$ have the uniform asymptotic expansions as $z \to \infty$

$$\mathrm{Ai}(z) \sim \frac{e^{-\zeta}}{2\pi^{\frac{1}{2}} z^{\frac{1}{4}}} \sum_{k=0}^{\infty} (-1)^k \frac{u_k}{\zeta^k}, \qquad \mathrm{Ai}'(z) \sim -\frac{z^{\frac{1}{4}} e^{-\zeta}}{2\pi^{\frac{1}{2}}} \sum_{k=0}^{\infty} (-1)^k \frac{v_k}{\zeta^k}; \tag{A.1}$$

the numbers u_k, v_k, $k \in \mathbb{N}_0$, are explicit and $u_0 = v_0 = 1$, see [15, Eq. 9.7.2, 9.7.5, 9.7.6]. For $z \in \mathbb{C}$ satisfying $|\operatorname{Arg} z| \leq \frac{2}{3}\pi - \epsilon$, we have as $z \to \infty$

(A.2)

$$\mathrm{Ai}(-z) \sim \frac{1}{\pi^{\frac{1}{2}} z^{\frac{1}{4}}} \left(\cos(\zeta - \tfrac{\pi}{4}) \sum_{k=0}^{\infty} (-1)^k \frac{u_{2k}}{\zeta^{2k}} + \sin(\zeta - \tfrac{\pi}{4}) \sum_{k=0}^{\infty} (-1)^k \frac{u_{2k+1}}{\zeta^{2k+1}} \right),$$

$$\mathrm{Ai}'(-z) \sim \frac{z^{\frac{1}{4}}}{\pi^{\frac{1}{2}}} \left(\sin(\zeta - \tfrac{\pi}{4}) \sum_{k=0}^{\infty} (-1)^k \frac{v_{2k}}{\zeta^{2k}} - \cos(\zeta - \tfrac{\pi}{4}) \sum_{k=0}^{\infty} (-1)^k \frac{v_{2k+1}}{\zeta^{2k+1}} \right),$$

see [15, Eq. 9.7.9, 9.7.10].

The Airy functions satisfy the connection formulas (see [15, Eq. 9.2.11])

$$\mathrm{Ai}(ze^{\mp 2\pi\mathrm{i}/3}) = \frac{1}{2} e^{\mp \pi\mathrm{i}/3} (\mathrm{Ai}(z) \pm \mathrm{i}\,\mathrm{Bi}(z)), \quad z \in \mathbb{C}, \tag{A.3}$$

and we have the integral representation (see [15, Eq. 9.11.4])

$$\mathrm{Ai}(z)^2 + \mathrm{Bi}(z)^2 = \frac{1}{\pi^{\frac{3}{2}}} \int_0^{\infty} \frac{\exp(zt - \frac{1}{12}t^3)}{t^{\frac{1}{2}}} \,\mathrm{d}t, \quad z \in \mathbb{C}. \tag{A.4}$$

The function Ai, Ai$'$ have infinitely many zeros denoted by a_n, a_n', $n \in \mathbb{N}$, respectively. All of the zeros are negative, they are arranged in ascending order of absolute value and obey the asymptotic relations, see [15, §9.9(iv)],

$$\begin{aligned} a_n &= -\left(\frac{3}{8}\pi(4n-1)\right)^{\frac{2}{3}} (1 + \mathcal{O}(n^{-2})), \\ a_n' &= -\left(\frac{3}{8}\pi(4n-3)\right)^{\frac{2}{3}} (1 + \mathcal{O}(n^{-2})), \qquad n \to +\infty. \end{aligned} \tag{A.5}$$

In the following we shall make use of the functions

$$u_{\pm}(x;\lambda) := \mathrm{Ai}\left(e^{\pm 2\pi\mathrm{i}/3}(-\mathrm{i}x + \lambda)\right), \quad x \in \mathbb{R}, \quad \lambda \in \mathbb{C}. \tag{A.6}$$

Observe that

$$-u_\pm'' + \mathrm{i}x\, u_\pm = \lambda u_\pm, \quad \lambda \in \mathbb{C},$$

and from (A.1) we conclude $u_\pm(\cdot;\lambda) \in L^2(\mathbb{R}_\pm)$ for every $\lambda \in \mathbb{C}$. It is also clear that the functions $x \mapsto u_\mp(-x;\lambda) \in L^2(\mathbb{R}_\pm)$ solve the equations

$$-u_\mp'' - \mathrm{i}x\, u_\mp = \lambda u_\mp, \quad \lambda \in \mathbb{C}.$$

Note also that

$$W[u_-(x;\lambda), u_+(x;\lambda)] = -\frac{1}{2\pi}. \tag{A.7}$$

Lemma A.1. *Let functions $u_\pm$ be as in* (A.6) *and let $\epsilon > 0$ be fixed and sufficiently small. Then we have the uniform expansion as $\lambda \to \infty$ (and conditions and x and $\operatorname{Arg}\lambda$ are satisfied)*

i) *for $x \le 0$ and $\operatorname{Arg}\lambda \in [0,\pi]$,*

$$u_-(x;\lambda) = \frac{e^{\frac{\pi}{6}\mathrm{i}}}{2\pi^{\frac12}} \frac{e^{\frac23(\lambda-\mathrm{i}x)^{\frac32}}}{(\lambda-\mathrm{i}x)^{\frac14}} \left(1 + \mathcal{O}(|\lambda|^{-\frac32})\right); \tag{A.8}$$

ii) *for $x \ge 0$ and $\operatorname{Arg}\lambda \in [0, \frac{\pi}{3} - \epsilon]$,*

$$u_+(x;\lambda) = \frac{e^{-\frac{\pi}{6}\mathrm{i}}}{2\pi^{\frac12}} \frac{e^{\frac23(\lambda-\mathrm{i}x)^{\frac32}}}{(\lambda-\mathrm{i}x)^{\frac14}} \left(1 + \mathcal{O}(|\lambda|^{-\frac32})\right); \tag{A.9}$$

iii) *for $\operatorname{Arg}\lambda \in [\frac{\pi}{3} - \epsilon, \frac{\pi}{3} + \epsilon]$ and $\omega := e^{-\frac{\pi}{3}\mathrm{i}}\lambda$,*

$$\begin{aligned} u_+(0; e^{\frac{\pi}{3}\mathrm{i}}\omega) = \frac{1}{\pi^{\frac12}\omega^{\frac14}} \Big(&\cos\left(\tfrac23\omega^{\frac32} - \tfrac{\pi}{4}\right)\left(1 + \mathcal{O}(|\omega|^{-3})\right) \\ &+ \sin\left(\tfrac23\omega^{\frac32} - \tfrac{\pi}{4}\right)\mathcal{O}(|\omega|^{-\frac32})\Big); \end{aligned}$$

iv) *for $\operatorname{Arg}\lambda \in [\frac{\pi}{3} + \epsilon, \pi]$,*

$$u_+(0;\lambda) = \frac{e^{-\frac23(e^{-\frac43\pi\mathrm{i}}\lambda)^{\frac32}}}{2\pi^{\frac12}(e^{-\frac43\pi\mathrm{i}}\lambda)^{\frac14}} \left(1 + \mathcal{O}(|\lambda|^{-\frac32})\right).$$

Proof. The claim i) follows from (A.1) and the estimate $|\lambda - \mathrm{i}x| \ge |\lambda|$ for $x \le 0$ and $\operatorname{Arg}\lambda \in [0,\pi]$. For ii) observe first that $|\lambda| \le C\mathbb{R}e\lambda$ holds for some $C > 0$ and all λ with $\operatorname{Arg}\lambda \in [0, \frac{\pi}{3} - \epsilon]$, and hence $|\lambda - \mathrm{i}x| \ge C^{-1}|\lambda|$ for $x \ge 0$ and $\operatorname{Arg}\lambda \in [0, \frac{\pi}{3} - \epsilon]$. Now ii) is a consequence of (A.1). Claim iii) follows easily from (A.2) and iv) is again a consequence of (A.1), where $e^{\frac23\pi\mathrm{i}}\lambda = e^{-\frac43\pi\mathrm{i}}\lambda$ was used. □

Lemma A.2. *Let functions $u_\pm$ be as in* (A.6)*. Then as $\lambda \to +\infty$*

$$\begin{aligned} \|u_-(\cdot;\lambda)\|^2_{L^2(\mathbb{R}_-)} &= \frac{1}{4(2\pi)^{\frac12}\lambda^{\frac14}} e^{\frac43\lambda^{\frac32}}(1 + \mathcal{O}(\lambda^{-\frac32})), \\ \|u_+(\cdot;\lambda)\|^2_{L^2(\mathbb{R}_+)} &= \frac{1}{4(2\pi)^{\frac12}\lambda^{\frac14}} e^{\frac43\lambda^{\frac32}}(1 + \mathcal{O}(\lambda^{-\frac32})). \end{aligned} \tag{A.10}$$

Proof. From the expansion (A.8) and the change of integration variable $x = -\lambda t$ we obtain

$$\|u_-(\cdot;\lambda)\|^2_{L^2(\mathbb{R}_-)} = \frac{\lambda^{\frac{1}{2}}}{4\pi} \int_0^\infty \frac{e^{\frac{4}{3}\lambda^{\frac{3}{2}} p(t)}}{|1+\mathrm{i}t|^{\frac{1}{2}}} \mathrm{d}t \, (1 + \mathcal{O}(\lambda^{-\frac{3}{2}})), \tag{A.11}$$

where $p(t) := \mathbb{R}e\big((1+\mathrm{i}t)^{\frac{3}{2}}\big)$, $t \in \mathbb{R}$. Since p is even, simple manipulations and the expansion (A.9) reveal that the asymptotic behavior of $\|u_+(\cdot;\lambda)\|^2_{L^2(\mathbb{R}_+)}$ is determined by the same integral.

Since

$$p'(t) = \frac{3}{4}\mathrm{i}\left((1+\mathrm{i}t)^{\frac{1}{2}} - (1-\mathrm{i}t)^{\frac{1}{2}}\right), \quad p''(t) = -\frac{3}{8}\left(\frac{1}{(1+\mathrm{i}t)^{\frac{1}{2}}} + \frac{1}{(1-\mathrm{i}t)^{\frac{1}{2}}}\right),$$

we infer that $p'(t) = 0$ if and only if $t = 0$. As $p''(0) < 0$, the function p has the unique global maximum at $t = 0$. The Laplace's method with the maximum at the endpoint, $p'(0) = 0$, $p''(0) = -3/4$ and $p(0) = 1$, see [23, Chap. 2, Thm. 3], yields

$$\int_0^\infty \frac{e^{\frac{4}{3}\lambda^{\frac{3}{2}} p(t)}}{|1+\mathrm{i}t|^{\frac{1}{2}}} \mathrm{d}t = \frac{(2\pi)^{\frac{1}{2}}}{2\lambda^{\frac{3}{4}}} e^{\frac{4}{3}\lambda^{\frac{3}{2}}} (1 + \mathcal{O}(\lambda^{-\frac{3}{2}})), \qquad \lambda \to +\infty.$$

Finally, returning to (A.11), we arrive at (A.10). □

References

[1] S. Albeverio, S. M. Fei, and P. Kurasov, *Point Interactions: PT-Hermiticity and Reality of the Spectrum*, Lett. Math. Phys. **59** (2002), 227–242.

[2] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, Springer, New York, 1988.

[3] S. Albeverio and S. Kuzhel, *One-dimensional Schrödinger operators with $\mathcal{P}$-symmetric zero-range potentials*, J. Phys. A: Math. Gen. **38** (2005), 4975–4988.

[4] Y. Almog, *The Stability of the Normal State of Superconductors in the Presence of Electric Currents*, SIAM J. Math. Anal. **40** (2008), 824–850.

[5] Y. Almog and B. Helffer, *On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent*, Comm. Partial Differential Equations **40** (2015), 1441–1466.

[6] A. Arnal and P. Siegl, *Generalised Airy Operators*, arXiv:2208.14389, 2022.

[7] A. Arnal and P. Siegl, *Resolvent estimates for one-dimensional Schrödinger operators with complex potentials*, J. Funct. Anal. **284** (2023), 109856.

[8] C. Baker and B. Mityagin, *Non-real eigenvalues of the harmonic oscillator perturbed by an odd, two-point interaction*, J. Math. Phys. **61** (2020), 043505.

[9] J. Behrndt, F. Gesztesy, H. Holden, and R. Nichols, *On the index of meromorphic operator-valued functions and some applications*, Functional analysis and operator theory for quantum physics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2017, pp. 95–127.

[10] F. A. Berezin and L. D. Faddeev, *Remark on the Schrödinger equation with singular potential*, Dokl. Akad. Nauk SSSR **137** (1961), 1011–1014.

[11] S. Bögli, P. Siegl, and C. Tretter, *Approximations of spectra of Schrödinger operators with complex potential on $\mathbb{R}^d$*, Comm. Partial Differential Equations **42** (2017), 1001–1041.

[12] W. Bordeaux Montrieux, *Estimation de résolvante et construction de quasimode près du bord du pseudospectre*, arXiv:1301.3102, 2013.

[13] John B. Conway, *Functions of one complex variable*, second ed., Springer-Verlag, New York-Berlin, 1978.

[14] N. Dencker, J. Sjöstrand, and M. Zworski, *Pseudospectra of semiclassical (pseudo-) differential operators*, Commun. Pure Appl. Math. **57** (2004), 384–415.

[15] NIST Digital Library of Mathematical Functions, `http://dlmf.nist.gov/`, Release 1.0.17 of 2017-12-22, F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[16] N. Dunford and J. T. Schwartz, *Linear Operators, Part 2*, John Wiley & Sons, Inc., New York, 1988.

[17] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Oxford University Press, New York, 1987.

[18] D. S. Grebenkov, B. Helffer, and R. Henry, *The Complex Airy Operator on the Line with a Semipermeable Barrier*, SIAM J. Math. Anal. **49** (2017), 1844–1894.

[19] B. Helffer, *Spectral theory and its applications*, Cambridge University Press, 2013.

[20] R. Henry and D. Krejčiřík, *Pseudospectra of the Schrödinger operator with a discontinuous complex potential*, J. Spectr. Theory **7** (2017), 659–697. MR 3713022

[21] I. W. Herbst, *Dilation analyticity in constant electric field. I. The two body problem*, Comm. Math. Phys. **64** (1979), 279–298. MR 520094 (81a:81020)

[22] A. Hussein, D. Krejčiřík, and P. Siegl, *Non-self-adjoint Graphs*, Trans. Amer. Math. Soc. **367** (2015), 2921–2957.

[23] V. Ivrii, *Asymptotic and Perturbational Methods*, `http://www.math.toronto.edu/courses/apm441h1/20161/`, 2016, online textbook.

[24] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1995. MR 1335452 (96a:47025)

[25] D. Krejčiřík, N. Raymond, J. Royer, and P. Siegl, *Non-accretive Schrödinger operators and exponential decay of their eigenfunctions*, Israel J. Math. **221** (2017), 779–802.

[26] David Krejčiřík, H Bíla, and M Znojil, *Closed formula for the metric in the Hilbert space of a $\mathcal{PT}$-symmetric model*, J. Phys. A: Math. Gen. **39** (2006), 10143–10153.

[27] D. Krejčiřík and P. Siegl, *Pseudomodes for Schrödinger operators with complex potentials*, J. Funct. Anal. **276** (2019), 2856–2900.

[28] V.E. Lyantze and O.G. Storozh, *Methods of the Theory of Unbounded Operators (Russian)*, Naukova Dumka, Kiev, 1983.

[29] M. M. Malamud and V. I. Mogilevskii, *Kreĭn type formula for canonical resolvents of dual pairs of linear relations*, Methods Funct. Anal. Topology **8** (2002), 72–100.

[30] M. M. Malamud and V. I. Mogilevskii, *Generalized Resolvents of Symmetric Operators*, Math. Notes **73** (2003), 429–435.

[31] M. M. Malamud, V. I. Mogilevskii, and S. Hassi, *On the unitary equivalence of the proper extensions of a Hermitian operator and the Weyl function*, Math. Notes **91** (2012), 302–307.

[32] B. Mityagin, *The spectrum of a harmonic oscillator operator perturbed by δ-interactions*, Integral Equations Operator Theory **85** (2016), 451–495.

[33] Miloslav Znojil and Vit Jakubský, *Solvability and PT -symmetry in a double-well model with point interactions*, J. Phys. A: Math. Gen. **38** (2005), 5041–5056.

(J. Behrndt and P. Siegl) Institute of Applied Mathematics, Graz University of Technology, Steyrergasse 30, 8010 Graz, Austria

Email address: `behrndt@tugraz.at, siegl@tugraz.at`

(I. Semorádová) Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 12000 Prague 2, Czech Republic

Email address: semorive@cvut.cz