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THE DIRICHLET-TO-NEUMANN MAP FOR SCHRÖDINGER OPERATORS WITH COMPLEX POTENTIALS

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Abstract. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let $q\colon \Omega \to \mathbb{C}$ be a bounded complex potential. We study the Dirichlet-to-Neumann graph associated with the operator $-\Delta + q$ and we give an example in which it is *not* m-sectorial.

1. Introduction.

The classical Dirichlet-to-Neumann operator D is a positive self-adjoint operator acting on functions defined on the boundary $\Gamma = \partial\Omega$ of a bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. The operator D is defined as follows. Let $\varphi, \psi \in L_2(\Gamma)$. Then $\varphi \in \operatorname{dom} D$ and $D\varphi = \psi$ if and only if there exists a $u \in H^1(\Omega)$ such that

$$\left[\begin{array}{ll} \operatorname{Tr} u = \varphi, & \\ -\Delta u = 0 & \text{weakly on } \Omega, \\ \partial_\nu u = \psi, & \end{array}\right. \tag{1}$$

where ∂_ν is the (weak) normal derivative. The Dirichlet-to-Neumann operator can also be described by form methods, see, e.g. [4]. Define the form $\mathfrak{a}\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v}. \tag{2}$$

Let $\varphi, \psi \in L_2(\Gamma)$. Then $\varphi \in \operatorname{dom} D$ and $D\varphi = \psi$ if and only if there exists a $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and $\mathfrak{a}(u, v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in H^1(\Omega)$. The Dirichlet-to-Neumann operator plays a central role in direct and inverse spectral problems and has attracted a lot of attention; for a small selection of recent contributions of operator theoretic flavor see [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 23, 24, 25, 26, 27].

There are various extensions of the Dirichlet-to-Neumann operator. The first one is where the operator $-\Delta$ in (1) is replaced by a formally symmetric pure second-order strongly elliptic differential operator in divergence form. Then one again

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obtains a self-adjoint version of the Dirichlet-to-Neumann operator, which enjoys a description with a form by making the obvious changes in (2). Similarly, if one replaces the operator $-\Delta$ in (1) by a pure second-order strongly elliptic differential operator in divergence form (which is possibly not symmetric), then the associated Dirichlet-to-Neumann operator is an m-sectorial operator.

There occurs a significant difference if one replaces the operator $-\Delta$ in (1) by a formally symmetric second-order strongly elliptic differential operator in divergence form, this time with lower-order terms. Then it might happen that D is no longer a self-adjoint operator, because it could be multivalued. Nevertheless, it turns out that D is a self-adjoint graph, which is lower bounded (see [6] Theorems 4.5 and 4.15, or [8] Theorem 5.7).

The aim of this note is to consider the case where the operator $-\Delta$ in (1) is replaced by $-\Delta + q$, where $q \colon \Omega \to \mathbb{C}$ is a bounded measurable *complex* valued function; in a similar way a general second-order strongly elliptic operator in divergence form with lower-order terms could be considered. In Section 2 the form method from [3, 4, 5, 6] will be adapted and applied to the present situation in an abstract form, and in Section 3 the Dirichlet-to-Neumann graph D associated with $-\Delta + q$ will be studied. Although one may expect that D is an m-sectorial graph it turns out in Example 3.7 that this is *not* the case in general.

2. **Forms.** In this section we review and extend the form methods and the theory of self-adjoint graphs.

Let V and H be Hilbert spaces. Let $\mathfrak{a} \colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form. Continuous means that there exists an $M > 0$ such that $|\mathfrak{a}(u,v)| \leq M \, \|u\|_V \, \|v\|_V$ for all $u, v \in V$. Let $j \in \mathcal{L}(V,H)$ be an operator. Define the graph D in $H \times H$ by

$$D = \{(\varphi, \psi) \in H \times H : \text{there exists a } u \in V \text{ such that} \\ j(u) = \varphi \text{ and } \mathfrak{a}(u,v) = (\psi, j(v))_H \text{ for all } v \in V\}.$$

We call D the **graph associated with** $(\mathfrak{a}, j)$.

In general, if A is a graph in H, then the **domain** of A is

$$\operatorname{dom} A = \{x \in H : (x,y) \in A \text{ for some } y \in H\}$$

and the **multivalued part** is

$$\operatorname{mul} A = \{y \in H : (0,y) \in A\}.$$

We say that A is **single valued**, or an **operator**, if $\operatorname{mul} A = \{0\}$. In that case one can identify A with a map from $\operatorname{dom} A$ into H.

Clearly $\operatorname{mul} D \neq \{0\}$ if $j(V)$ is not dense in H. If $(\varphi, \psi) \in D$, then there might be more than one $u \in V$ such that $j(u) = \varphi$ and $\mathfrak{a}(u,v) = (\psi, j(v))_H$ for all $v \in V$. For that reason we introduce the space

$$W_j(\mathfrak{a}) = \{u \in \ker j : \mathfrak{a}(u,v) = 0 \text{ for all } v \in V\}.$$

If $u_0 \in V$ is such that $j(u_0) = \varphi$ and $\mathfrak{a}(u_0,v) = (\psi, j(v))_H$ for all $v \in V$, then

$$\{u \in V : j(u) = \varphi \text{ and } \mathfrak{a}(u,v) = (\psi, j(v))_H \text{ for all } v \in V\} = u_0 + W_j(\mathfrak{a}).$$

Note that $W_j(\mathfrak{a})$ is closed in V.

We say that the form $\mathfrak{a}$ is j**-elliptic** if there exist $\mu, \omega > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u) + \omega \, \|j(u)\|_H^2 \geq \mu \, \|u\|_V^2 \tag{3}$$

for all $u \in V$. Graphs associated with j-elliptic forms behave well.

Theorem 2.1. *Suppose that $\mathfrak{a}$ is j-elliptic and $j(V)$ is dense in H. Then D is an m-sectorial operator. Also $W_j(\mathfrak{a}) = \{0\}$.*

Proof. See [4] Theorem 2.1 and Proposition 2.3(ii). □

If $\Omega \subset \mathbb{R}^d$ is a bounded open set with Lipschitz boundary, $V = H^1(\Omega)$, $H = L_2(\Gamma)$, $j = \mathrm{Tr}$ and $\mathfrak{a}$ is as in (2), then D is the Dirichlet-to-Neumann operator as in the introduction; cf. Section 3 for more details.

In general the form $\mathfrak{a}$ is not j-elliptic. An example occurs if one replaces $\mathfrak{a}$ in (2) by

$$\mathfrak{a}(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v} - \lambda \int_\Omega u \, \overline{v}$$

with $\lambda \in \sigma(-\Delta_D)$, where Δ_D is the Laplacian on Ω with Dirichlet boundary conditions. Then (3) fails for every $\mu, \omega > 0$ if u is a corresponding eigenfunction and $j = \mathrm{Tr}$. In addition, the graph associated with $(\mathfrak{a}, j)$ is not single valued any more. We emphasize that we are interested in the graph associated with $(\mathfrak{a}, j)$. To get around the problem that the form $\mathfrak{a}$ is not j-elliptic, it is convenient to introduce a different Hilbert space and a different map $\tilde{j}$.

Throughout the remainder of this paper we adopt the following hypothesis.

Hypothesis 2.2. *Let V, H and $\widetilde{H}$ be Hilbert spaces and let $\mathfrak{a} \colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form. Let $j \in \mathcal{L}(V, H)$ and let D be the graph associated with $(\mathfrak{a}, j)$. Furthermore, let $\tilde{j} \in \mathcal{L}(V, \widetilde{H})$ be a* compact *map and assume that the form $\mathfrak{a}$ is $\tilde{j}$-elliptic, that is, there are $\tilde{\mu}, \tilde{\omega} > 0$ such that*

$$\mathrm{Re}\, \mathfrak{a}(u) + \tilde{\omega} \, \|\tilde{j}(u)\|_{\widetilde{H}}^2 \geq \tilde{\mu} \, \|u\|_V^2 \tag{4}$$

for all $u \in V$.

As example, if $\Omega \subset \mathbb{R}^d$ is a bounded open set with Lipschitz boundary as before, then one can choose $V = H^1(\Omega)$, $H = L_2(\Gamma)$, $\widetilde{H} = L_2(\Omega)$, $j = \mathrm{Tr}$ and $\tilde{j}$ is the inclusion map from $H^1(\Omega)$ into $L_2(\Omega)$. For $\mathfrak{a}$ one can choose a continuous sesquilinear form on $H^1(\Omega)$ like in (2). We consider this example in more detail in Section 3.

In general, if A is a graph in H, then A is called **symmetric** if $(x,y)_H \in \mathbb{R}$ for all $(x,y) \in A$. The graph A is called **surjective** if for all $y \in H$ there exists an $x \in H$ such that $(x,y) \in A$. The graph A is called **self-adjoint** if A is symmetric and for all $s \in \mathbb{R} \setminus \{0\}$ the graph $A + i\,s\,I$ is surjective, where for all $\lambda \in \mathbb{C}$ we define the graph $(A + \lambda\, I)$ by

$$(A + \lambda\, I) = \{(x, y + \lambda\, x) : (x,y) \in A\}.$$

A symmetric graph A is called **bounded below** if there exists an $\omega > 0$ such that $(x,y)_H + \omega\, \|x\|_H^2 \geq 0$ for all $(x,y) \in A$.

Under the above main assumptions we can state the following theorem for symmetric forms.

Theorem 2.3. *Adopt Hypothesis 2.2. Suppose $\mathfrak{a}$ is symmetric. Then D is a self-adjoint graph which is bounded below.*

Proof. See [6] Theorems 4.5 and 4.15, or [8] Theorem 5.7. □

We next wish to study the case when $\mathfrak{a}$ is not symmetric.

Proposition 2.4. *Adopt Hypothesis 2.2. Then the graph D is closed.*

Proof. Let $((\varphi_n,\psi_n))_{n\in\mathbb{N}}$ be a sequence in D, let $(\varphi,\psi)\in H\times H$ and suppose that $\lim_{n\to\infty}(\varphi_n,\psi_n)=(\varphi,\psi)$ in $H\times H$. For all $n\in\mathbb{N}$ there exists a unique $u_n\in W_j(\mathfrak{a})^\perp$ such that $j(u_n)=\varphi_n$ and

$$\mathfrak{a}(u_n,v)=(\psi_n,j(v))_H \tag{5}$$

for all $v\in V$, where the orthogonal complement is in V.

We first show that $(\tilde{j}(u_n))_{n\in\mathbb{N}}$ is bounded in $\widetilde{H}$. Suppose not. Set $\tau_n=\|\tilde{j}(u_n)\|_{\widetilde{H}}$ for all $n\in\mathbb{N}$. Passing to a subsequence if necessary, we may assume that $\tau_n>0$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}\frac{1}{\tau_n}=0$. Define $w_n=\frac{1}{\tau_n}\,u_n$ for all $n\in\mathbb{N}$. Then

$$\mathfrak{a}(w_n,v)=(\tfrac{1}{\tau_n}\,\psi_n,j(v))_H \tag{6}$$

for all $v\in V$. Choose $v=w_n$. Then

$$\operatorname{Re}\mathfrak{a}(w_n)\le\frac{\|\psi_n\|_H}{\tau_n}\,\|j\|\,\|w_n\|_V$$

for all $n\in\mathbb{N}$. Let $\tilde{\mu},\tilde{\omega}>0$ be as in (4). Then

$$\|w_n\|_V\le\tfrac12\,\tilde{\mu}\,\|w_n\|_V^2+\tfrac{1}{2\tilde{\mu}}\le\tfrac{1}{2\tilde{\mu}}+\tfrac12\,\tilde{\omega}+\tfrac12\operatorname{Re}\mathfrak{a}(w_n).$$

So

$$|\operatorname{Re}\mathfrak{a}(w_n)|\le\frac{\|\psi_n\|_H\,\|j\|}{\tau_n}\Big(\tfrac{1}{2\tilde{\mu}}+\tfrac12\,\tilde{\omega}+\tfrac12|\operatorname{Re}\mathfrak{a}(w_n)|\Big)$$

for all $n\in\mathbb{N}$. Since $(\|\psi_n\|_H)_{n\in\mathbb{N}}$ is bounded and $\frac{\|\psi_n\|_H\,\|j\|}{\tau_n}<1$ for all large $n\in\mathbb{N}$, it follows that $(\operatorname{Re}\mathfrak{a}(w_n))_{n\in\mathbb{N}}$ is bounded. Together with (4) it then follows that $(w_n)_{n\in\mathbb{N}}$ is bounded in V. Passing to a subsequence if necessary there exists a $w\in W_j(\mathfrak{a})^\perp$ such that $\lim_{n\to\infty}w_n=w$ weakly in V. Then $\tilde{j}(w)=\lim_{n\to\infty}\tilde{j}(w_n)$ in $\widetilde{H}$ since $\tilde{j}$ is compact. So $\|\tilde{j}(w)\|_{\widetilde{H}}=1$ and in particular $w\neq0$. Alternatively, for all $v\in V$ it follows from (6) that

$$\mathfrak{a}(w,v)=\lim_{n\to\infty}\mathfrak{a}(w_n,v)=\lim_{n\to\infty}\tfrac{1}{\tau_n}\,(\psi_n,j(v))_H=0.$$

Moreover, $j(w)=\lim_{n\to\infty}\frac{1}{\tau_n}\,j(u_n)=\lim_{n\to\infty}\frac{1}{\tau_n}\,\varphi_n=0$, where the limits are in the weak topology on H. So $w\in W_j(\mathfrak{a})$. Therefore $w\in W_j(\mathfrak{a})\cap W_j(\mathfrak{a})^\perp=\{0\}$ and $w=0$. This is a contradiction. So $(\tilde{j}(u_n))_{n\in\mathbb{N}}$ is bounded in $\widetilde{H}$.

Let $n\in\mathbb{N}$. Then with $v=u_n$ in (5) one deduces that

$$\begin{aligned}|\operatorname{Re}\mathfrak{a}(u_n)|&=|\operatorname{Re}(\psi_n,j(u_n))_H|\\&\le\|\psi_n\|_H\,\|j\|\,\|u_n\|_V\\&\le\tfrac12\,\tilde{\mu}\,\|u_n\|_V^2+\frac{\|\psi_n\|_H^2\,\|j\|^2}{2\tilde{\mu}}\\&\le\tfrac12\operatorname{Re}\mathfrak{a}(u_n)+\tfrac12\,\tilde{\omega}\,\|\tilde{j}(u_n)\|_{\widetilde{H}}^2+\frac{\|\psi_n\|_H^2\,\|j\|^2}{2\tilde{\mu}},\end{aligned}$$

where we used (4) in the last step. Hence $(\operatorname{Re}\mathfrak{a}(u_n))_{n\in\mathbb{N}}$ is bounded. Using again (4) one establishes that $(u_n)_{n\in\mathbb{N}}$ is bounded in V. Passing to a subsequence if necessary, there exists a $u\in V$ such that $\lim u_n=u$ weakly in V. Then $j(u)=\lim j(u_n)=\lim\varphi_n=\varphi$ weakly in H. Finally let $v\in V$. Then (5) gives

$$\mathfrak{a}(u,v)=\lim_{n\to\infty}\mathfrak{a}(u_n,v)=\lim_{n\to\infty}(\psi_n,j(v))_H=(\psi,j(v))_H.$$

So $(\varphi, \psi) \in D$ and D is closed. □

Proposition 2.5. *Adopt Hypothesis 2.2. Suppose j is compact. Then the map $(\varphi, \psi) \mapsto \varphi$ from D into H is compact.*

Proof. Define $Z : D \to W_j(\mathfrak{a})^\perp$ by

$$Z(\varphi, \psi) = u,$$

where $u \in W_j(\mathfrak{a})^\perp$ is the unique element such that $j(u) = \varphi$ and $\mathfrak{a}(u, v) = (\psi, j(v))_H$ for all $v \in V$. We first show that the graph of Z is closed. Let $((\varphi_n, \psi_n))_{n\in\mathbb{N}}$ be a sequence in D, let $(\varphi, \psi) \in H \times H$ and $u \in V$. Suppose that $\lim \varphi_n = \varphi$, $\lim \psi_n = \psi$ in H and $\lim u_n = u$ in V, where $u_n = Z(\varphi_n, \psi_n)$ for all $n \in \mathbb{N}$. Since D is closed by Proposition 2.4 it follows that $(\varphi, \psi) \in D$. Moreover, $j(u) = \lim j(u_n) = \lim \varphi_n = \varphi$ and

$$\mathfrak{a}(u, v) = \lim \mathfrak{a}(u_n, v) = \lim(\psi_n, j(v))_H = (\psi, j(v))_H$$

for all $v \in V$. Since $u_n \in W_j(\mathfrak{a})^\perp$ for all $n \in \mathbb{N}$, it is clear that also $u \in W_j(\mathfrak{a})^\perp$. Hence $Z(\varphi, \psi) = u$ and Z has closed graph.

The closed graph theorem, together with Proposition 2.4 implies that Z is continuous. Since j is compact, the composition $j \circ Z$ is compact. But $(j \circ Z)(\varphi, \psi) = \varphi$ for all $(\varphi, \psi) \in D$. □

In general, if A is a graph in H, then A is called **invertible** if it is surjective, closed and the reflected graph $\{(y, x) : (x, y) \in A\}$ is single-valued. If the *graph* A is invertible then we define the *operator* $A^{-1} : H \to H$ by $A^{-1}y = x$ if $(x, y) \in A$. The **resolvent set** $\rho(A)$ of A is the set of all $\lambda \in \mathbb{C}$ such that $(A - \lambda I)$ is invertible. We say that A has **compact resolvent** if $(A - \lambda I)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$.

Corollary 2.6. *Adopt Hypothesis 2.2. Suppose j is compact. Then the graph D has compact resolvent.*

For the sequel it is convenient to introduce the space

$$V_j(\mathfrak{a}) = \{u \in V : \mathfrak{a}(u, v) = 0 \text{ for all } v \in \ker j\}.$$

Theorem 2.7. *Adopt Hypothesis 2.2. If $V_j(\mathfrak{a}) \cap \ker j = \{0\}$ and $\operatorname{ran} j$ is dense in H, then D is an m-sectorial operator.*

Proof. See [2] Theorem 8.11. □

Note that the operator A_D in the next lemma is the Dirichlet Laplacian if $\mathfrak{a}$ is as in (2) and $\tilde{j}$ is the inclusion map from $H^1(\Omega)$ into $L_2(\Omega)$.

Lemma 2.8. *Adopt Hypothesis 2.2. Suppose that $\tilde{j}(\ker j)$ is dense in $\widetilde{H}$ and $\tilde{j}$ is injective. Then the graph A_D associated with $(\mathfrak{a}|_{\ker j \times \ker j}, \tilde{j}|_{\ker j})$ is an operator and one has the following.*

(a) $\ker A_D = \tilde{j}(V_j(\mathfrak{a}) \cap \ker j)$.
(b) $0 \notin \sigma(A_D)$ *if and only if* $V_j(\mathfrak{a}) \cap \ker j = \{0\}$.
(c) *If* $\ker A_D = \{0\}$ *and* $\operatorname{ran} j$ *is dense in* H, *then* $\operatorname{mul} D = \{0\}$.

Proof. The graph A_D in $\widetilde{H} \times \widetilde{H}$ associated with $(\mathfrak{a}|_{\ker j \times \ker j}, \tilde{j}|_{\ker j})$ is given by

$$A_D = \{(h, k) \in \widetilde{H} \times \widetilde{H} : \text{there exists a } u \in \ker j \text{ such that } \tilde{j}(u) = h \text{ and } \mathfrak{a}(u, v) = (k, \tilde{j}(v))_{\widetilde{H}} \text{ for all } v \in \ker j\}.$$

Now suppose that $k \in \operatorname{mul} A_D$. Let $u \in \ker j$ be such that $\tilde{j}(u) = 0$ and $\mathfrak{a}(u,v) = (k, \tilde{j}(v))_{\widetilde{H}}$ for all $v \in \ker j$. The assumption that $\tilde{j}$ is injective yields $u = 0$ and hence $0 = \mathfrak{a}(u,v) = (k, \tilde{j}(v))_{\widetilde{H}}$ for all $v \in \ker j$. Since $\tilde{j}(\ker j)$ is dense in $\widetilde{H}$ it follows that $k = 0$. Therefore $\operatorname{mul} A_D = \{0\}$ and A_D is an operator.

'(a)'. '$\supset$'. Let $u \in V_j(\mathfrak{a}) \cap \ker j$. Then $u \in \ker j$. Moreover, $\mathfrak{a}(u,v) = 0$ for all $v \in \ker j$. So $\tilde{j}(u) \in \operatorname{dom} A_D$ and $A_D \tilde{j}(u) = 0$. Therefore $\tilde{j}(u) \in \ker A_D$.

The converse inclusion can be proved similarly.

'(b)'. Since A_D has compact resolvent, this statement follows from part (a) and the injectivity of $\tilde{j}$.

'(c)'. If $\ker A_D = \{0\}$ then $V_j(\mathfrak{a}) \cap \ker j = \{0\}$ by (a). Now Theorem 2.7 yields $\operatorname{mul} D = \{0\}$. □

In Corollary 3.4 we give a class of forms such that the converse of Lemma 2.8(c) is valid.

We conclude this section with some facts on graphs. In general, let A be a graph in H. In the following definitions we use the conventions as in the book [22] of Kato. The **numerical range** of A is the set

$$W(A) = \{(x,y)_H : (x,y) \in A \text{ and } \|x\|_H = 1\}.$$

The graph A is called **sectorial** if there exist $\gamma \in \mathbb{R}$ and $\theta \in [0, \frac{\pi}{2})$ such that $(x,y)_H \in \Sigma_\theta$ for all $(x,y) \in A - \gamma I$. So A is sectorial if and only if there exist $\gamma \in \mathbb{R}$ and $\theta \in [0, \frac{\pi}{2})$ such that $W(A - \gamma I) \subset \Sigma_\theta$. The graph A is called m**-sectorial** if there are $\gamma \in \mathbb{R}$ and $\theta \in [0, \frac{\pi}{2})$ such that $(x,y)_H \in \Sigma_\theta$ for all $(x,y) \in A - \gamma I$ and $A - (\gamma - 1)I$ is invertible. The graph A is called **quasi-accretive** if there exists a $\gamma \in \mathbb{R}$ such that $\operatorname{Re}(x,y)_H \geq 0$ for all $(x,y) \in A - \gamma I$. The graph A is called **quasi** m**-accretive** if there exists a $\gamma \in \mathbb{R}$ such that $\operatorname{Re}(x,y)_H \geq 0$ for all $(x,y) \in A - \gamma I$ and $A - (\gamma - 1)I$ is invertible. Clearly every m-sectorial graph is sectorial and quasi m-accretive. Moreover, every sectorial graph is quasi-accretive.

Lemma 2.9. *Let A be a graph.*

(a) *If not* $\operatorname{dom} A \perp \operatorname{mul} A$*, then the numerical range of A is the full complex plane.*
(b) *If A is a quasi-accretive graph, then* $\operatorname{dom} A \perp \operatorname{mul} A$.
(c) *If A is a quasi m-accretive graph, then* $\operatorname{mul} A = (\operatorname{dom} A)^\perp$.

Proof. '(a)'. There are $x \in \operatorname{dom} A$ and $y' \in \operatorname{mul} A$ such that $(x, y')_H \neq 0$. Without loss of generality we may assume that $\|x\|_H = 1$. There exists a $y \in H$ such that $(x,y) \in A$. Then $(x, y + \tau y') \in A$ for all $\tau \in \mathbb{C}$. So $(x, y + \tau y')_H \in W(A)$ for all $\tau \in \mathbb{C}$.

'(b)'. This follows from Statement (a).

'(c)'. By Statement (b) it remains to show that $(\operatorname{dom} A)^\perp \subset \operatorname{mul} A$. By assumption there exists a $\gamma \in \mathbb{R}$ such that $\operatorname{Re}(x,y)_H \geq 0$ for all $(x,y) \in A - \gamma I$ and $A - (\gamma - 1)I$ is invertible. Without loss of generality we may assume that $\gamma = 0$. Let $y \in (\operatorname{dom} A)^\perp$. Define $x = (A + I)^{-1} y$. Then $x \in \operatorname{dom} A$ and $(x, y - x) \in A$. So $-\|x\|_H^2 = \operatorname{Re}(x, y - x)_H \geq 0$ and $x = 0$. Then $(0, y) \in A$ and $y \in \operatorname{mul} A$ as required. □

3. Complex potentials.

In this section we consider the Dirichlet-to-Neumann map with respect to the operator $-\Delta + q$, where q is a bounded *complex* valued potential on a Lipschitz domain.

Throughout this section fix a bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary Γ. Let $q\colon \Omega \to \mathbb{C}$ be a bounded measurable function. Choose $V = H^1(\Omega)$, $H = L_2(\Gamma)$, $j = \mathrm{Tr} : H^1(\Omega) \to L_2(\Gamma)$, $\widetilde{H} = L_2(\Omega)$ and $\tilde{j}$ the inclusion of V into $\widetilde{H}$. Then j and $\tilde{j}$ are compact. Moreover, $\operatorname{ran} j$ is dense in H by the Stone–Weierstraß theorem. Define $\mathfrak{a}\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega q\, u\, \overline{v}.$$

Then $\mathfrak{a}$ is a sesquilinear form and it is $\tilde{j}$-elliptic. Let D be the graph associated with $(\mathfrak{a}, j)$. Note that all assumptions in Hypothesis 2.2 are satisfied. In order to describe D, we need the notion of a weak normal derivative.

Let $u \in H^1(\Omega)$ and suppose that there exists an $f \in L_2(\Omega)$ such that $\Delta u = f$ as distribution. Let $\psi \in L_2(\Gamma)$. Then we say that u has **weak normal derivative** ψ if

$$\int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega f\, \overline{v} = \int_\Gamma \psi\, \overline{\mathrm{Tr}\, v}$$

for all $v \in H^1(\Omega)$. Since $\operatorname{ran} j$ is dense in H it follows that ψ is unique and we write $\partial_\nu u = \psi$.

The alluded description of the graph D is as follows.

Lemma 3.1. *Let $\varphi, \psi \in L_2(\Gamma)$. Then the following are equivalent.*

(i) $(\varphi, \psi) \in D$.

(ii) *There exists a $u \in H^1(\Omega)$ such that $\mathrm{Tr}\, u = \varphi$, $(-\Delta + q)u = 0$ as distribution and $\partial_\nu u = \psi$.*

Proof. The easy proof is left to the reader. □

Let $A_D = -\Delta_D + q$, where Δ_D is the Laplacian on Ω with Dirichlet boundary conditions. Then A_D is as in Lemma 2.8. Moreover, $(A_D)^* = -\Delta_D + \overline{q}$.

Proposition 3.2. *Let $u \in \ker A_D$. Then u has a weak normal derivative, that is, $\partial_\nu u \in L_2(\Gamma)$ is defined. Similarly, if $u \in \ker(A_D)^*$, then u has a weak normal derivative.*

Proof. It follows from [21] Theorem B.2 that $u \in H^{3/2}(\Omega)$. Hence $\partial_\nu u \in L_2(\Gamma)$ by [16] Lemma 2.4.

The claim for $(A_D)^*$ follows by replacing q by $\overline{q}$. □

Corollary 3.3. $\operatorname{mul} D = \{\partial_\nu u : u \in \ker A_D\}$.

Note that the right hand side is indeed defined and it is a subspace of $L_2(\Gamma)$ by Proposition 3.2.

Corollary 3.4. *The following are equivalent.*

(i) *D is an m-sectorial operator.*

(ii) $\ker A_D = \{0\}$.

(iii) $\operatorname{mul} D = \{0\}$.

Proof. '(i)⇒(iii)'. An operator has trivial multivalued part.

'(iii)⇒(ii)'. Let $u \in \ker A_D$. Then $\partial_\nu u \in \operatorname{mul} D = \{0\}$ by Corollary 3.3 and $\partial_\nu u = 0$. By the unique continuation property one deduces that $u = 0$.

'(ii)⇒(i)'. It follows from Lemma 2.8(a) that $V_j(\mathfrak{a}) \cap \ker j = \{0\}$. Then use Theorem 2.7. □

We next determine the domain of the Dirichlet-to-Neumann graph D. The proof is a variation of Theorem 5.2 in [8], in which the potential q was real valued.

Theorem 3.5. $\operatorname{dom} D = \{\varphi \in H^1(\Gamma) : (\varphi, \partial_\nu w)_{L_2(\Gamma)} = 0$ *for all* $w \in \ker(A_D)^*\}$.

Proof. '$\subset$'. Let $\varphi \in \operatorname{dom} D$. Let $\psi \in L_2(\Gamma)$ be such that $(\varphi, \psi) \in D$. Then there exists a $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and $\mathfrak{a}(u, v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in H^1(\Omega)$. Note that $(-\Delta + q)u = 0$ as distribution, so $\Delta u = q\, u \in L_2(\Omega)$ as distribution. By [16] Lemma 2.4 there exists a $w \in H^{3/2}(\Omega)$ such that $\Delta w \in L_2(\Omega)$ and $\partial_\nu w = \psi$. Then $u - w \in H^1(\Omega)$ and $\Delta(u - w) \in L_2(\Omega)$. Hence $u - w \in \operatorname{dom} \Delta_N$, where Δ_N is the Laplacian with Neumann boundary conditions. Therefore $u - w \in H^{3/2}(\Omega)$ by [16] Lemma 4.8. Since $w \in H^{3/2}(\Omega)$, also $u \in H^{3/2}(\Omega)$. Because $\Delta u = q\, u \in L_2(\Omega)$ one deduces from [16] (2.11) in Lemma 2.3 that $\varphi = \operatorname{Tr} u \in H^1(\Gamma)$.

Next let $w \in \ker(A_D)^*$. Then $\operatorname{Tr} w = 0$ and $\Delta w = \overline{q}\, w$ as distribution. Hence

$$\begin{aligned}(\partial_\nu w, \varphi)_{L_2(\Gamma)} &= \int_\Omega \nabla w \cdot \overline{\nabla u} + \int_\Omega (\Delta w)\, \overline{u} \\ &= \int_\Omega \nabla w \cdot \overline{\nabla u} + \int_\Omega \overline{q}\, w\, \overline{u} \\ &= \overline{\int_\Omega \nabla u \cdot \overline{\nabla w} + \int_\Omega q\, u\, \overline{w}} = \overline{\mathfrak{a}(u, w)} = \overline{(\psi, \operatorname{Tr} w)_{L_2(\Gamma)}} = 0,\end{aligned}$$

since $\operatorname{Tr} w = 0$.

'$\supset$'. Let $\varphi \in H^1(\Gamma)$ and suppose that $(\varphi, \partial_\nu w)_{L_2(\Gamma)} = 0$ for all $w \in \ker(A_D)^*$. We first show that there exists a $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and $(-\Delta + q)u = 0$ as distribution.

Let $\mathfrak{a}_D = \mathfrak{a}|_{H^1_0(\Omega) \times H^1_0(\Omega)}$. Then $\mathfrak{a}_D$ is a continuous sesquilinear form. Hence there exists a unique $T \in \mathcal{L}(H^1_0(\Omega))$ such that $\mathfrak{a}_D(u, v) = (Tu, v)_{H^1_0(\Omega)}$ for all $u, v \in H^1_0(\Omega)$. Let $\tilde{\mu}, \tilde{\omega} > 0$ be as in (4). Set $K = \tilde{\omega}\, \tilde{j}_0^* \, \tilde{j}_0 \in \mathcal{L}(H^1_0(\Omega))$, where $\tilde{j}_0 = \tilde{j}|_{H^1_0(\Omega)}$ is the inclusion of $H^1_0(\Omega)$ into $L_2(\Omega)$. Then K is compact and

$$\tilde{\mu}\, \|u\|^2_{H^1_0(\Omega)} \le \operatorname{Re} \mathfrak{a}_D(u) + (Ku, u)_{H^1_0(\Omega)} = \operatorname{Re}((T + K)u, u)_{H^1_0(\Omega)}$$

for all $u \in H^1_0(\Omega)$. So $\tilde{\mu}\, \|u\|_{H^1_0(\Omega)} \le \|(T + K)u\|_{H^1_0(\Omega)}$ for all $u \in H^1_0(\Omega)$. Hence $(T + K)$ is injective and has closed range. Similarly $(T + K)^*$ is injective. So $(T + K)$ is invertible. Since K is compact, one concludes that T is a Fredholm operator. In particular, the range $\operatorname{ran} T$ of T is closed.

It is easy to verify that $\ker T^* = \ker(A_D)^*$. Therefore $\operatorname{ran} T = (\ker T^*)^\perp = (\ker(A_D)^*)^\perp$. Since $\varphi \in H^{1/2}(\Gamma)$ there exists a $\Phi \in H^1(\Omega)$ such that $\operatorname{Tr} \Phi = \varphi$. Because $v \mapsto \mathfrak{a}(\Phi, v)$ is continuous on $H^1_0(\Omega)$, there exists a unique $u_1 \in H^1_0(\Omega)$ such that $(u_1, v)_{H^1_0(\Omega)} = \mathfrak{a}(\Phi, v)$ for all $v \in H^1_0(\Omega)$. If $w \in \ker(A_D)^*$, then the Green theorem implies that

$$(u_1, w)_{H^1_0(\Omega)} = \mathfrak{a}(\Phi, w) = (\operatorname{Tr} \Phi, \partial_\nu w)_{L_2(\Gamma)} = (\varphi, \partial_\nu w)_{L_2(\Gamma)} = 0.$$

So $u_1 \in \operatorname{ran} T$. Hence there exists a $u_2 \in H^1_0(\Omega)$ such that $u_1 = T u_2$. Then $\mathfrak{a}(u_2, v) = \mathfrak{a}(\Phi, v)$ for all $v \in H^1_0(\Omega)$. Define $u = \Phi - u_2 \in H^1(\Omega)$. Then $\operatorname{Tr} u = \operatorname{Tr} \Phi = \varphi$ and $\mathfrak{a}(u, v) = 0$ for all $v \in H^1_0(\Omega)$. So $(-\Delta + q)u = 0$ weakly on Ω.

By [16] (2.11) in Lemma 2.3 there exists a $w \in H^{3/2}(\Omega)$ such that $\Delta w \in L_2(\Omega)$ and $\operatorname{Tr} w = \varphi$. Then $u - w \in H^1(\Omega)$, $\Delta(u - w) \in L_2(\Omega)$ and $\operatorname{Tr}(u - w) = 0$. So $u - w \in \operatorname{dom} \Delta_D$. Therefore $u - w \in H^{3/2}(\Omega)$ by [21] Theorem B.2. Thus

$u \in H^{3/2}(\Omega)$ and hence $\partial_\nu u \in L_2(\Gamma)$ by [16] Lemma 2.4. So $(\varphi, \partial_\nu u) \in D$ by Lemma 3.1 and $\varphi \in \operatorname{dom} D$. □

Corollary 3.6. $(\operatorname{dom} D)^\perp = \{\partial_\nu w : w \in \ker(A_D)^*\}$.

Proof. Let $E = \{\partial_\nu w : w \in \ker(A_D)^*\}$. Since $\dim E < \infty$ and $H^1(\Gamma)$ is dense in $L_2(\Gamma)$ it follows that $H^1(\Gamma) \cap E^\perp$ is dense in $E^\perp$. Observe that $\operatorname{dom} D = H^1(\Gamma) \cap E^\perp$ by Theorem 3.5. Therefore $\overline{\operatorname{dom} D} = E^\perp$ and hence $(\operatorname{dom} D)^\perp = E$. □

Theorem 2.3 states that D is a self-adjoint graph whenever $\mathfrak{a}$ is symmetric, that is whenever the potential q is real valued. If q is complex valued and $\operatorname{mul} D \neq \{0\}$, then in general D is not an m-sectorial graph. A counterexample is as follows.

Example 3.7. Let $\Omega = (0, \pi) \times (0, \pi)$. Let $\tau \in \mathbb{R}$. We will choose τ appropriate below. Define $q \colon \Omega \to \mathbb{C}$ by

$$q(x, y) = \frac{-8i\,\tau(\cos 2x + 2\cos^2 x)}{1 + i\,\tau(\cos 2x + 2\cos^2 x)}.$$

Then $q \in L_\infty(\Omega)$. Consider the operator $-\Delta + (q-2)\,I$, so choose $V = H^1(\Omega)$ and

$$\mathfrak{a}(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Omega (q-2)\,u\,\overline{v}.$$

Define $u \colon \Omega \to \mathbb{C}$ by

$$u(x, y) = (\sin x + i\,\tau \sin 3x) \sin y.$$

Then $u \in H^1_0(\Omega)$. Since $\sin 3x = \sin x(\cos 2x + 2\cos^2 x)$ it follows that $(-\Delta + q\,I)u = 2u$. Hence $u \in \operatorname{dom} A_D$ and $A_D u = 0$. Since $\dim\ker A_D = 1$ if $\tau = 0$, it follows by perturbation, [22] Theorem VII.1.7, that there exists a $\tau_0 > 0$ such that $\dim\ker A_D = 1$ for all $\tau \in (-\tau_0, \tau_0)$. Moreover, if $\tau = 0$, then the operator A_D is self-adjoint and, in particular, $\dim\ker(A_D)^* = \dim\ker A_D = 1$. It is clear that [22] Theorem VII.1.7 applies in the same way to $(A_D)^*$ and hence it is no restriction to assume that $\tau_0 > 0$ above is chosen such that also $\dim\ker(A_D)^* = 1$ for all $\tau \in (-\tau_0, \tau_0)$. Hence it follows that $\ker A_D = \operatorname{span} u$ and $\ker(A_D)^* = \operatorname{span} \overline{u}$ for all $\tau \in (-\tau_0, \tau_0)$.

Note that

$$(\partial_\nu u)(x, 0) = (\partial_\nu u)(x, \pi) = -(\sin x + i\,\tau \sin 3x),$$
$$(\partial_\nu \overline{u})(x, 0) = (\partial_\nu \overline{u})(x, \pi) = -(\sin x - i\,\tau \sin 3x),$$
$$(\partial_\nu u)(0, y) = (\partial_\nu u)(\pi, y) = -(1 + 3i\,\tau) \sin y \quad \text{and}$$
$$(\partial_\nu \overline{u})(0, y) = (\partial_\nu \overline{u})(\pi, y) = -(1 - 3i\,\tau) \sin y$$

for all $x, y \in (0, \pi)$. In the present situation Corollary 3.3 and Corollary 3.6 imply

$$\operatorname{mul} D = \operatorname{span} \partial_\nu u \qquad \text{and} \qquad (\operatorname{dom} D)^\perp = \operatorname{span} \partial_\nu \overline{u}. \tag{7}$$

We assume from now on that $\tau \in (0, \tau_0)$. Then $\partial_\nu u$ and $\partial_\nu \overline{u}$ are linearly independent. Thus $\operatorname{mul} D \not\subset (\operatorname{dom} D)^\perp$ by (7), so not $\operatorname{mul} D \perp \operatorname{dom} D$. Hence D is not a quasi-accretive graph by Lemma 2.9(b). Moreover, the numerical range of D is the full complex plane by Lemma 2.9(a). In particular, D is not an m-sectorial graph.

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