On Kreĭn's formula

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Abstract

Kreĭn's formula provides a parametrization of the generalized resolvents and Štraus extensions of a closed symmetric operator with equal possibly infinite defect numbers in a Hilbert space in terms of Nevanlinna families in a parameter space. The aim of this note is to give a simple complete analytical proof of Kreĭn's formula.

Key words: Kreĭn's formula, Kreĭn-Naimark formula, generalized resolvent, symmetric operator, selfadjoint extension, boundary triplet

1 Introduction

Let S be a closed symmetric operator in a Hilbert space $\mathfrak{H}$. Then S admits selfadjoint extensions in $\mathfrak{H}$ if and only if the deficiency indices of S coincide. These canonical selfadjoint extensions and their resolvents can be parametrized via the so-called Kreĭn's formula

$$(A_{\mathcal{T}} - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\Big(M(\lambda) - \mathcal{T}\Big)^{-1}\gamma(\bar{\lambda})^* \tag{1.1}$$

with the help of selfadjoint operators and relations $\mathcal{T}$ in a defect subspace of S. Here A_0 is a fixed selfadjoint extension of S, M is a Q-function or Weyl

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function of the pair $\{S, A_0\}$, and $\gamma(\cdot)$ is a defect function. However, in [12,13] M.G. Kreĭn proved the far more general formula

$$P_{\mathfrak{H}}\Big(\widetilde{A}-\lambda\Big)^{-1}\upharpoonright_{\mathfrak{H}}=(A_0-\lambda)^{-1}-\gamma(\lambda)\Big(M(\lambda)+\tau(\lambda)\Big)^{-1}\gamma(\bar{\lambda})^* \tag{1.2}$$

which gives a description of the compressed resolvents of selfadjoint extensions $\widetilde{A}$ of S in larger Hilbert spaces $\widetilde{\mathfrak{H}}$ in terms of Nevanlinna functions and Nevanlinna families $\tau(\lambda)$; see also [23] for the case of infinite defect numbers. Kreĭn's formula has been extended to various settings, see for instance [3,11,14,16,17,24], and [18–20] for a different parametrization due to M.A. Naimark; cf. [1].

Kreĭn's formula has an interpretation in terms of the boundary triplets and Weyl functions due to V.A. Derkach and M.M. Malamud [8,9]. In this setting the Nevanlinna family $\tau(\lambda)$ in (1.2) plays the role of an abstract boundary condition. A geometric interpretation of Kreĭn's formula involving boundary triplets and boundary relations can be found in [5–7].

Kreĭn's formula (1.2) for the generalized resolvents is an important tool in many applications in modern analysis and mathematical physics, see, e.g. [2,15,21,22], and it is the aim of this note to provide a simple analytical proof of (1.2). In Section 3 a variant of Kreĭn's formula for compressed coresolvents of unitary extensions of an isometric operator is proved. Here the parameter functions belong to the Schur class. Cayley transformation leads to Kreĭn's formula for symmetric operators and relations in a special case, see Section 4. In Section 5 the connection with boundary triplets is made and an example from Sturm-Liouville theory is discussed.

2 Preliminaries

2.1 Linear relations

A (closed) *linear relation* in a Hilbert space $\mathfrak{H}$ is a (closed) linear subspace of the Cartesian product $\mathfrak{H}\times\mathfrak{H}$. The elements of a linear relation T will be denoted by $\widehat{f}=\{f,f'\}\in T$, $f,f'\in\mathfrak{H}$. Furthermore, $\operatorname{dom} T$, $\ker T$, $\operatorname{ran} T$, and $\operatorname{mul} T$ stand for the *domain*, *kernel*, *range*, and *multi-valued part* of T, respectively. The *inverse relation* T^{-1} is defined by $T^{-1}=\{\{f',f\}:\{f,f'\}\in T\}$. The *sum* T_1+T_2, the *componentwise sum* $T_1 \,\widehat{+}\, T_2$, and the *product* T_2T_1

of two linear relations T_1 and T_2 are defined by

$$\begin{aligned} T_1 + T_2 &= \Big\{\{f, g+k\} : \{f,g\} \in T_1,\ \{f,k\} \in T_2\Big\}, \\ T_1 \mathbin{\widehat{+}} T_2 &= \Big\{\{f+h, g+k\} : \{f,g\} \in T_1,\ \{h,k\} \in T_2\Big\}, \\ T_2T_1 &= \Big\{\{f,k\} : \{f,g\} \in T_1,\ \{g,k\} \in T_2\Big\}, \end{aligned}$$

respectively. Linear (closed) operators in $\mathfrak{H}$ will be identified with linear (closed) relations via their graphs. The linear space of everywhere defined bounded linear operators from $\mathfrak{H}$ into a Hilbert space $\mathfrak{K}$ will be denoted by $\mathbf{B}(\mathfrak{H},\mathfrak{K})$ and by $\mathbf{B}(\mathfrak{H})$ if $\mathfrak{H}=\mathfrak{K}$. A linear relation T in a Hilbert space $\mathfrak{H}$ is said to have the *eigenvalue* $\nu \in \mathbb{C}$ if $\ker(T-\nu)$ is nontrivial. The following notations will be used: $\mathfrak{N}_\nu(T) = \ker(T-\nu)$ and $\widehat{\mathfrak{N}}_\nu(T) := \{\{f,\nu f\} \in T : f \in \mathfrak{N}_\nu(T)\}$. The *resolvent set* $\rho(T)$ of a closed linear relation T in $\mathfrak{H}$ is the set of all $\lambda \in \mathbb{C}$ such that $(T-\lambda)^{-1} \in \mathbf{B}(\mathfrak{H})$; and $(T-\lambda)^{-1}$ is called the *resolvent operator*. Observe that for each $\lambda \in \rho(T)$:

$$T = \Big\{\,\{(T-\lambda)^{-1}h, (I+\lambda(T-\lambda)^{-1}h\} :\, h \in \mathfrak{H}\,\Big\}. \tag{2.1}$$

For $1/\lambda \in \rho(T)$ the operator $(I-\lambda T)^{-1}$ is called the *coresolvent* of T.

Lemma 2.1 *Let T and Q be linear relations in a Hilbert space $\mathfrak{H}$ and assume* $\ker T = \{0\}$. *Then*

$$(Q+T)^{-1} = T^{-1}\left(QT^{-1}+I\right)^{-1}.$$

If, in addition, Q, T^{-1}, and $(Q+T)^{-1}$ belong to $\mathbf{B}(\mathfrak{H})$, then $(QT^{-1}+I)^{-1} \in \mathbf{B}(\mathfrak{H})$.

2.2 Special relations

The adjoint relation T^* of a linear relation T is defined by

$$T^* := \Big\{\,\{g,g'\} : (f',g) = (f,g') \text{ for all } \{f,f'\} \in T\,\Big\}.$$

A linear relation T is *isometric* if $T^{-1} \subset T^*$ and *unitary* if $T^{-1} = T^*$. Observe that T is isometric if and only if T is an operator with $\|Tf\| = \|f\|$ for all $f \in \operatorname{dom} T$, and that T is unitary if and only if T is isometric with $\operatorname{dom} T = \operatorname{ran} T = \mathfrak{H}$. A linear relation T is *symmetric* if $T \subset T^*$ and *self-adjoint* if $T = T^*$. Observe that T is symmetric if and only if $(f',f) \in \mathbb{R}$ for all $\{f,f'\} \in T$. A linear relation T is *accumulative* if $\operatorname{Im}(f',f) \le 0$ and *dissipative* if $\operatorname{Im}(f',f) \ge 0$ for all $\{f,f'\} \in T$. The relation T is *maximal accumulative* (*maximal dissipative*) if T is accumulative (dissipative) and there exists no proper accumulative (dissipative, respectively) extension of T in $\mathfrak{H}$.

Note that T is maximal accumulative (maximal dissipative) if and only if T is accumulative (dissipative) and $\mathbb{C}_+ \subset \rho(T)$ ($\mathbb{C}_- \subset \rho(T)$, respectively).

2.3 Cayley transforms

Let T be a linear relation in $\mathfrak{H}$ and let $\mu \in \mathbb{C}_+$ be a fixed point in the upper halfplane $\mathbb{C}_+$. The *Cayley transform* $C_\mu(T)$ of a linear relation T in $\mathfrak{H}$ is defined by

$$C_\mu(T) := \Big\{ \{f' - \mu f, f' - \bar{\mu} f\} : \{f, f'\} \in T \Big\}$$

and the corresponding inverse Cayley transform of a linear relation V in $\mathfrak{H}$ is given by $\{\{h' - h, \mu h' - \bar{\mu} h\} : \{h, h'\} \in V\}$. Note that $\operatorname{dom} C_\mu(T) = \operatorname{ran}(T - \mu)$ and $\operatorname{ran} C_\mu(T) = \operatorname{ran}(T - \bar{\mu})$, and that $(\operatorname{dom} C_\mu(T))^\perp = \mathfrak{N}_{\bar{\mu}}(T^*)$ and $(\operatorname{ran} C_\mu(T))^\perp = \mathfrak{N}_\mu(T^*)$. Clearly, T is a symmetric (selfadjoint) relation if and only if $C_\mu(T)$ is an isometric (unitary, respectively) operator. Moreover, T is accumulative (dissipative) if and only if $C_\mu(T)$ ($C_\mu(T)^{-1}$, respectively) is a contractive operator and T is maximal accumulative (maximal dissipative) if and only if $C_\mu(T)$ ($C_\mu(T)^{-1}$, respectively) is an everywhere defined contractive operator. If T is a relation with a nonempty resolvent set, then the resolvent of T and the coresolvent of the Cayley transform $C_\mu(T)$ are connected via

$$\frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}} \Big(I - z C_\mu(T) \Big)^{-1} = I + (\lambda - \mu)(T - \lambda)^{-1}, \quad \lambda \in \rho(T). \tag{2.2}$$

Here the mapping z is defined by

$$z(\lambda) = \frac{\lambda - \mu}{\lambda - \bar{\mu}}, \qquad \lambda \neq \bar{\mu}. \tag{2.3}$$

Clearly, z maps the upper halfplane $\mathbb{C}_+$ onto the unit disk $\mathbb{D}$. The argument λ in the mapping z will often be suppressed.

2.4 Nevanlinna families and Schur functions

Let $\mathcal{G}$ and $\mathcal{G}'$ be Hilbert spaces. A $\mathbf{B}(\mathcal{G}, \mathcal{G}')$-valued function Θ holomorphic on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is called a *Schur function* if $\|\Theta(z)\| \leq 1$, $z \in \mathbb{D}$, and Θ is called a *uniformly contractive Schur function* $\|\Theta(z)\| < 1$, $z \in \mathbb{D}$. The class of $\mathbf{B}(\mathcal{G}, \mathcal{G}')$-valued Schur functions will be denoted by $\mathbf{S}(\mathcal{G}, \mathcal{G}')$ and by $\mathbf{S}(\mathcal{G})$ if $\mathcal{G} = \mathcal{G}'$.

A family of linear relations $\tau(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in the Hilbert space $\mathcal{G}$ is called a *Nevanlinna family* if:

(i) for every $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$) the relation $\tau(\lambda)$ is maximal dissipative (maximal accumulative, respectively);
(ii) $\tau(\bar{\lambda}) = \tau(\lambda)^*$ holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
(iii) for some, and hence for all, $\nu \in \mathbb{C}_+$ ($\nu \in \mathbb{C}_-$) the $\mathbf{B}(\mathcal{G})$-valued function $\lambda \mapsto (\tau(\lambda) + \nu)^{-1}$ is holomorphic on $\mathbb{C}_+$ ($\mathbb{C}_-$, respectively).

If, in addition, $\tau(\lambda) \in \mathbf{B}(\mathcal{G})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then τ is called a *Nevanlinna function.* A Nevanlinna family $\tau(\lambda)$ is said to be *uniformly strict* if τ is a Nevanlinna function and $\operatorname{Im} \tau(\lambda)$ is uniformly positive (uniformly negative) for $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$, respectively). Note $(M(\lambda) + \tau(\lambda))^{-1} \in \mathbf{B}(\mathfrak{H})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, if $\tau(\lambda)$ is a Nevanlinna family and M is a uniformly strict Nevanlinna function.

Schur functions and Nevanlinna families are closely connected. In fact, if $\mu \in \mathbb{C}_+$ and z is as in (2.3), then for $\nu \in \mathbb{C}_+$ the formula

$$\Theta(z) := I - (\nu - \bar{\nu})(\tau(\lambda) + \nu)^{-1}, \qquad \lambda \in \mathbb{C}_+, \tag{2.4}$$

provides a one-to-one correspondence between (uniformly strict) Nevanlinna families τ in $\mathcal{G}$ and (uniformly contractive) Schur functions Θ from $\mathbf{S}(\mathcal{G})$.

3 Unitary extensions of isometric operators

Let V be a closed isometric operator in a Hilbert space $\mathfrak{H}$. The *defect numbers* of V are the dimensions of the spaces $(\operatorname{dom} V)^\perp$ and $(\operatorname{ran} V)^\perp$. The adjoint relation of V has a decomposition in terms of relations in the Cartesian product $\mathfrak{H} \times \mathfrak{H}$:

$$V^* = V^{-1} \,\widehat{+}\, \Big((\operatorname{ran} V)^\perp \times \{0\}\Big) \,\widehat{+}\, \Big(\{0\} \times (\operatorname{dom} V)^\perp\Big), \quad \text{direct sum.}$$

Note that V admits unitary extensions in $\mathfrak{H}$ if and only if the defect numbers of V coincide.

3.1 Unitary extensions in exit spaces

Let $\widetilde{U}$ be a unitary extension of V in a Hilbert space $\mathfrak{H} \oplus \mathfrak{K}$. If $\mathfrak{K}$ is nontrivial, then $\widetilde{U}$ is called an *exit space extension* and the Hilbert space $\mathfrak{K}$ is the *exit space.* Observe that V always admits unitary exit space extensions. The *Štraus extensions* $W(z)$, $z \in \mathbb{D}$, of V corresponding to $\widetilde{U}$ are defined by

$$W(z) = \Big\{\{P_\mathfrak{H} f, P_\mathfrak{H} \widetilde{U} f\} : f \in \mathfrak{H} \oplus \mathfrak{K},\, (I - z\widetilde{U})f \in \mathfrak{H}\Big\}. \tag{3.1}$$

The *generalized coresolvent* $P_{\mathfrak{H}}(I-z\widetilde{U})^{-1}\restriction_{\mathfrak{H}}$ of V satisfies

$$P_{\mathfrak{H}}(I-z\widetilde{U})^{-1}\restriction_{\mathfrak{H}} = \bigl(I-zW(z)\bigr)^{-1}, \quad z\in\mathbb{D}.$$

Let the unitary extension $\widetilde{U}$ in $\mathfrak{H}\oplus\mathfrak{K}$ have the matrix decomposition

$$\widetilde{U} = \begin{pmatrix} T & F & 0 \\ G & H & 0 \\ 0 & 0 & V \end{pmatrix} : \begin{pmatrix} \mathfrak{K} \\ (\operatorname{dom} V)^{\perp} \\ \operatorname{dom} V \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ (\operatorname{ran} V)^{\perp} \\ \operatorname{ran} V \end{pmatrix}, \tag{3.2}$$

where the entries are bounded linear operators. The *transfer function*

$$\Theta(z) := H + zG(1-zT)^{-1}F, \quad z\in\mathbb{D}, \tag{3.3}$$

of the *unitary colligation*

$$\begin{pmatrix} T & F \\ G & H \end{pmatrix} : \begin{pmatrix} \mathfrak{K} \\ (\operatorname{dom} V)^{\perp} \end{pmatrix} \to \begin{pmatrix} \mathfrak{K} \\ (\operatorname{ran} V)^{\perp} \end{pmatrix} \tag{3.4}$$

belongs to the Schur class $\mathbf{S}((\operatorname{dom} V)^{\perp}, (\operatorname{ran} V)^{\perp})$. Note that the exit space $\mathfrak{K}$ is considered as the state space of the colligation. Conversely, every function from the class $\mathbf{S}((\operatorname{dom} V)^{\perp}, (\operatorname{ran} V)^{\perp})$ can be realized as the transfer function (3.3) of a unitary colligation (3.4), see, e.g., [4]. Associate with Θ and V the function

$$\widetilde{\Theta}(z) := \begin{pmatrix} \Theta(z) & 0 \\ 0 & V \end{pmatrix} : \begin{pmatrix} (\operatorname{dom} V)^{\perp} \\ \operatorname{dom} V \end{pmatrix} \to \begin{pmatrix} (\operatorname{ran} V)^{\perp} \\ \operatorname{ran} V \end{pmatrix}, \quad z\in\mathbb{D}. \tag{3.5}$$

Then the generalized coresolvent of V is given by

$$P_{\mathfrak{H}}(I-z\widetilde{U})^{-1}\restriction_{\mathfrak{H}} = (I-z\widetilde{\Theta}(z))^{-1}, \quad z\in\mathbb{D} \tag{3.6}$$

and, clearly, the Štraus extensions $W(z)$ satisfy

$$W(z) = \widetilde{\Theta}(z), \quad z\in\mathbb{D}. \tag{3.7}$$

Theorem 3.1 *Let V be a closed isometric operator in $\mathfrak{H}$. Then* (3.7) *establishes a one-to-one correspondence between the Štraus extensions $W(z)$, $z\in\mathbb{D}$, of V and the Schur functions $\Theta\in\mathbf{S}((\operatorname{dom} V)^{\perp}, (\operatorname{ran} V)^{\perp})$. In particular, if the defect numbers of V coincide, then there is a one-to-one correspondence between the canonical unitary extensions U of V and the (constant) unitary mappings $\Theta\in\mathbf{B}((\operatorname{dom} V)^{\perp}, (\operatorname{ran} V)^{\perp})$.*

Let V be a closed isometric operator in $\mathfrak{H}$ and denote by $\widetilde{V}$ the trivial extension of V to the whole space $\mathfrak{H}$, i.e. $\widetilde{V}h = Vh$ for $h \in \operatorname{dom} V$ and $\widetilde{V}h = 0$ for $h \in (\operatorname{dom} V)^\perp$, so that $\widetilde{V} \in \mathbf{B}(\mathfrak{H})$ is a partial isometry. The Hilbert space $\mathfrak{H}$ admits the direct sum decomposition

$$\mathfrak{H} = \operatorname{ran}(I - zV) + (\operatorname{dom} V)^\perp, \quad \text{direct sum,} \tag{3.8}$$

and the operator $P_{(\operatorname{dom} V)^\perp}(I - z\widetilde{V})^{-1}$ is the projection onto $(\operatorname{dom} V)^\perp$ parallel to $\operatorname{ran}(I - zV)$. Associated with the trivial extension of V is the unitary operator

$$\widehat{U} = \begin{pmatrix} \widetilde{V} & -\restriction_{(\operatorname{ran} V)^\perp} \\ -P_{(\operatorname{dom} V)^\perp} & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ (\operatorname{ran} V)^\perp \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ (\operatorname{dom} V)^\perp \end{pmatrix}.$$

Consider this colligation as a unitary extension of the trivial isometric operator from $(\operatorname{ran} V)^\perp$ to $(\operatorname{dom} V)^\perp$. In this case the Hilbert space $\mathfrak{H}$ serves as the state space and the transfer function X is given by the Schur function

$$X(z) := zP_{(\operatorname{dom} V)^\perp}\left(I - z\widetilde{V}\right)^{-1} \restriction_{(\operatorname{ran} V)^\perp}, \quad z \in \mathbb{D}. \tag{3.9}$$

It follows from the Schwarz lemma that X is a uniformly contractive Schur function.

Now assume that V has equal defect numbers. Fix some unitary operator $\Theta_0 \in \mathbf{B}((\operatorname{dom} V)^\perp, (\operatorname{ran} V)^\perp)$ and let U_0 be the canonical unitary extension of V with matrix decomposition

$$U_0 = \begin{pmatrix} \Theta_0 & 0 \\ 0 & V \end{pmatrix} : \begin{pmatrix} (\operatorname{dom} V)^\perp \\ \operatorname{dom} V \end{pmatrix} \to \begin{pmatrix} (\operatorname{ran} V)^\perp \\ \operatorname{ran} V \end{pmatrix}. \tag{3.10}$$

Then the identity

$$P_{(\operatorname{dom} V)^\perp}\left(I - z\widetilde{V}\right)^{-1}(I - zU_0) = \left(I - X(z)\Theta_0\right)P_{(\operatorname{dom} V)^\perp} \tag{3.11}$$

holds for all $z \in \mathbb{D}$. Indeed, (3.11) is clear for $h \in \operatorname{dom} V$. If $h \in (\operatorname{dom} V)^\perp$, then

$$P_{(\operatorname{dom} V)^\perp}\left(I - z\widetilde{V}\right)^{-1}(I - zU_0)h = P_{(\operatorname{dom} V)^\perp}\left(I - z\widetilde{V}\right)^{-1}h - X(z)\Theta_0 h.$$

Now let $k = (I - z\widetilde{V})^{-1}h$ so that $h = (I - z\widetilde{V})k$. With $k = k_0 + k_1$, $k_0 \in \operatorname{dom} V$, $k_1 \in (\operatorname{dom} V)^\perp$, it follows $h = (I - zV)k_0 + k_1$ which together with the direct sum decomposition (3.8) shows $k_1 = h$. Hence $P_{(\operatorname{dom} V)^\perp}(I - z\widetilde{V})^{-1}h = h$ and (3.11) is true. Observe that (3.11) leads to the following identity

$$P_{(\operatorname{dom} V)^\perp}\Big(I - z\widetilde{V}\Big)^{-1} = \Big(I - X(z)\Theta_0\Big)P_{(\operatorname{dom} V)^\perp}\Big(I - zU_0\Big)^{-1}, \quad z \in \mathbb{D}. \quad (3.12)$$

Lemma 3.2 *The Štraus extensions corresponding to the unitary extension*

$$\widehat{U}_{\Theta_0} = \begin{pmatrix} \widetilde{V} & -\upharpoonright_{(\operatorname{ran} V)^\perp} \Theta_0 \\ -P_{(\operatorname{dom} V)^\perp} & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H} \\ (\operatorname{dom} V)^\perp \end{pmatrix} \to \begin{pmatrix} \mathfrak{H} \\ (\operatorname{dom} V)^\perp \end{pmatrix} \quad (3.13)$$

of the trivial isometric operator in $(\operatorname{dom} V)^\perp$ *are given by* $X(z)\Theta_0$, $z \in \mathbb{D}$. *In particular,*

$$P_{(\operatorname{dom} V)^\perp}\Big(I - z\widehat{U}_{\Theta_0}\Big)^{-1} \upharpoonright_{(\operatorname{dom} V)^\perp} = \Big(I - zX(z)\Theta_0\Big)^{-1}. \quad (3.14)$$

3.3 Kreĭn's formula for isometric operators

Assume that the closed isometric operator V has equal defect numbers. The following theorem parallels Theorem 3.1 for the generalized coresolvents of V, cf. [10].

Theorem 3.3 *Let V be a closed isometric operator with equal defect numbers and let U_0 be a fixed canonical unitary extension of V as in* (3.10). *Then*

$$\begin{aligned} P_{\mathfrak{H}}\Big(I - z\widetilde{U}\Big)^{-1} \upharpoonright_{\mathfrak{H}} = \Big(I - zU_0\Big)^{-1} + z\Big(I - zU_0\Big)^{-1} \upharpoonright_{(\operatorname{ran} V)^\perp} \Big(\Theta(z) - \Theta_0\Big) \cdot \\ \cdot \Big(I - X(z)\Theta(z)\Big)^{-1}\Big(I - X(z)\Theta_0\Big)P_{(\operatorname{dom} V)^\perp}\Big(I - zU_0\Big)^{-1} \end{aligned} \quad (3.15)$$

establishes a one-to-one correspondence between the generalized coresolvents of V and the Schur functions $\Theta \in \mathbf{S}((\operatorname{dom} V)^\perp, (\operatorname{ran} V)^\perp)$.

Proof. Let $\widetilde{U}$ be a unitary extension of V in $\mathfrak{H} \oplus \mathfrak{K}$ as in (3.2) and let the function $\widetilde{\Theta}$ be as in (3.5). Due to (3.6) it follows that

$$\begin{aligned} & P_{\mathfrak{H}}\Big(I - z\widetilde{U}\Big)^{-1} \upharpoonright_{\mathfrak{H}} - \Big(I - zU_0\Big)^{-1} \\ & \quad = z\Big(I - zU_0\Big)^{-1}\Big(\widetilde{\Theta}(z) - U_0\Big)\Big(I - z\widetilde{\Theta}(z)\Big)^{-1} \\ & \quad = z\Big(I - zU_0\Big)^{-1} \upharpoonright_{(\operatorname{ran} V)^\perp} \Big(\Theta(z) - \Theta_0\Big)P_{(\operatorname{dom} V)^\perp}\Big(I - z\widetilde{\Theta}(z)\Big)^{-1}. \end{aligned} \quad (3.16)$$

It suffices to rewrite the expression $P_{(\operatorname{dom} V)^\perp}(I - z\widetilde{\Theta}(z))^{-1}$. First observe that

$$P_{(\operatorname{dom} V)^\perp}\Big(I - z\widetilde{\Theta}(z)\Big)^{-1}\Big(I - z\widetilde{V}\Big) = \Big(I - X(z)\Theta(z)\Big)^{-1}P_{(\operatorname{dom} V)^\perp}, \quad z \in \mathbb{D}. \tag{3.17}$$

In fact, (3.17) is clear for $h \in \operatorname{dom} V$. For $h \in (\operatorname{dom} V)^\perp$ and

$$k = \Big(I - z\widetilde{\Theta}(z)\Big)^{-1}\Big(I - z\widetilde{V}\Big)h = \Big(I - z\widetilde{\Theta}(z)\Big)^{-1}h$$

it follows $h = (I - z\widetilde{\Theta}(z))k$. With $k = k_0 + k_1$, $k_0 \in \operatorname{dom} V$, $k_1 \in (\operatorname{dom} V)^\perp$, this gives $h = (I - zV)k_0 + (I - z\Theta(z))k_1$, i.e. $z\Theta(z)k_1 = (I - zV)k_0 + k_1 - h$ which together with (3.8) implies $k_1 - h = X(z)\Theta(z)k_1$ or $k_1 = (I - X(z)\Theta(z))^{-1}h$. Thus (3.17) is valid and this leads to

$$P_{(\operatorname{dom} V)^\perp}\Big(I - z\widetilde{\Theta}(z)\Big)^{-1} = \Big(I - X(z)\Theta(z)\Big)^{-1}P_{(\operatorname{dom} V)^\perp}\Big(I - z\widetilde{V}\Big)^{-1}, \quad z \in \mathbb{D}. \tag{3.18}$$

By means of (3.18) and (3.12) the term $P_{(\operatorname{dom} V)^\perp}(I - z\widetilde{\Theta}(z))^{-1}$ in (3.16) is given by

$$\Big(I - X(z)\Theta(z)\Big)^{-1}\Big(I - X(z)\Theta_0\Big)P_{(\operatorname{dom} V)^\perp}\Big(I - zU_0\Big)^{-1}. \tag{3.19}$$

Substitution of (3.19) in (3.16) leads to (3.15).

Conversely, if Θ belongs to $\mathbf{S}((\operatorname{dom} V)^\perp, (\operatorname{ran} V)^\perp)$, then there exists a Hilbert space $\mathfrak{K}$ and a unitary colligation of the form (3.4) such that Θ is the corresponding transfer function, [4]. Define $\widetilde{U}$ by (3.2); then (3.16) holds and by means of (3.12), (3.18), and (3.19) it follows that the generalized coresolvent of $\widetilde{U}$ satisfies (3.15). $\square$

4 Selfadjoint extensions of symmetric relations

Let S be a closed symmetric relation in a Hilbert space $\mathfrak{H}$ and let $\mu \in \mathbb{C}_+$ be fixed. The defect numbers of S are the dimensions of $\mathfrak{N}_{\bar{\mu}}(S^*)$ and $\mathfrak{N}_\mu(S^*)$. The adjoint relation S^* has the von Neumann decomposition

$$S^* = S \mathbin{\hat{+}} \widehat{\mathfrak{N}}_\mu(S^*) \mathbin{\hat{+}} \widehat{\mathfrak{N}}_{\bar{\mu}}(S^*), \qquad \text{direct sum.} \tag{4.1}$$

Note that S admits selfadjoint extensions in $\mathfrak{H}$ if and only if the defect numbers of S are equal. These and the following observations parallel those for closed isometric operators via the Cayley transform.

4.1 Selfadjoint extensions in exit spaces

Let $\widetilde{A}$ be a selfadjoint extension of S in the Hilbert space $\mathfrak{H} \oplus \mathfrak{K}$. If $\mathfrak{K}$ is nontrivial, then $\widetilde{A}$ is called an *exit space extension* and the Hilbert space $\mathfrak{K}$ is the *exit space*. Observe that S always admits selfadjoint exit space extensions. The *Štraus extensions* $T(\lambda)$ of S corresponding to $\widetilde{A}$ are defined by

$$T(\lambda) = \Big\{ \{P_{\mathfrak{H}} f, P_{\mathfrak{H}} f'\} : \{f, f'\} \in \widetilde{A},\ f' - \lambda f \in \mathfrak{H} \Big\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

The *generalized resolvent* $P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \restriction_{\mathfrak{H}}$ of S satisfies

$$P_{\mathfrak{H}} \big(\widetilde{A} - \lambda\big)^{-1} \restriction_{\mathfrak{H}} = \big(T(\lambda) - \lambda\big)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{4.2}$$

and hence $T(\lambda)^* = T(\bar{\lambda})$ holds for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Therefore it suffices to consider the Štraus extensions for $\lambda \in \mathbb{C}_+$.

Clearly the Cayley transform $\widetilde{U} = C_\mu(\widetilde{A})$ of $\widetilde{A}$ is a unitary extension of the isometric operator $V = C_\mu(S)$ in $\mathfrak{H}$. Observe that the Štraus extensions $W(z)$ of V corresponding to $\widetilde{U}$ (see (3.1)) and the Štraus extensions $T(\lambda)$ of S corresponding to $\widetilde{A}$ are connected via $W(z) = C_\mu(T(\lambda))$, where z is as in (2.3). This also gives the following translation of Theorem 3.1.

Theorem 4.1 *Let S be a closed symmetric relation in a Hilbert space $\mathfrak{H}$ and let $\mu \in \mathbb{C}_+$. Then there is a one-to-one correspondence between the Štraus extensions $T(\lambda)$, $\lambda \in \mathbb{C}_+$, of S and the Schur functions $\Theta \in \mathbf{S}(\mathfrak{N}_{\bar{\mu}}(S^*), \mathfrak{N}_\mu(S^*))$, via*

$$T(\lambda) = S \,\widehat{+}\, \Big\{ \{(\Theta(z) - I) f_{\bar{\mu}}, (\mu\Theta(z) - \bar{\mu}) f_{\bar{\mu}}\} : f_{\bar{\mu}} \in \mathfrak{N}_{\bar{\mu}}(S^*) \Big\}, \quad \textit{direct sum.} \tag{4.3}$$

In particular, if the defect numbers of S coincide, then there is a one-to-one correspondence between the canonical selfadjoint extensions A of S and the unitary mappings $\Theta \in \mathbf{B}(\mathfrak{N}_{\bar{\mu}}(S^*), \mathfrak{N}_\mu(S^*))$, via

$$A = S \,\widehat{+}\, \Big\{ \{(\Theta - I) f_{\bar{\mu}}, (\mu\Theta - \bar{\mu}) f_{\bar{\mu}}\} : f_{\bar{\mu}} \in \mathfrak{N}_{\bar{\mu}}(S^*) \Big\}, \quad \textit{direct sum.} \tag{4.4}$$

4.2 A special selfadjoint relation

Assume that the defect numbers of the closed symmetric relation S in $\mathfrak{H}$ are equal, let $\mu \in \mathbb{C}_+$, and fix a unitary operator $\Theta_0 \in \mathbf{B}(\mathfrak{N}_{\bar{\mu}}(S^*), \mathfrak{N}_\mu(S^*))$. The elements $\widehat{f} \in S^*$ will be decomposed in

$$\widehat{f} = \{f, f'\} = \{f_0, f_0'\} + \{f_\mu, \mu f_\mu\} + \{f_{\bar{\mu}}, \bar{\mu} f_{\bar{\mu}}\} \in S \,\widehat{+}\, \widehat{\mathfrak{N}}_\mu(S^*) \,\widehat{+}\, \widehat{\mathfrak{N}}_{\bar{\mu}}(S^*) \tag{4.5}$$

according to (4.1). Define the relation $\widehat{A}$ in $\mathfrak{H} \oplus \mathfrak{N}_{\bar\mu}(S^*)$ by

$$\widehat{A} = \left\{ \left\{ \begin{pmatrix} f \\ f_{\bar\mu} + \Theta_0^* f_\mu \end{pmatrix}, \begin{pmatrix} f' \\ \mu f_{\bar\mu} + \bar\mu \Theta_0^* f_\mu \end{pmatrix} \right\} : \widehat{f} = \{f, f'\} \in S^* \right\}, \tag{4.6}$$

with the notational convention as in (4.5). The Cayley transform of $\widehat{A}$ is given by

$$C_\mu(\widehat{A}) = \left\{ \left\{ \begin{pmatrix} f' - \mu f \\ (\bar\mu - \mu)\Theta_0^* f_\mu \end{pmatrix}, \begin{pmatrix} f' - \bar\mu f \\ (\mu - \bar\mu) f_{\bar\mu} \end{pmatrix} \right\} : \widehat{f} = \{f, f'\} \in S^* \right\}.$$

Set $V = C_\mu(S)$, so that $\mathfrak{N}_{\bar\mu}(S^*) = (\operatorname{dom} V)^\perp$ and $\mathfrak{N}_\mu(S^*) = (\operatorname{ran} V)^\perp$, and let $\widetilde{V}$ be the trivial extension of V onto $\mathfrak{H}$ (see Section 3.2). The identities

$$\begin{aligned}
P_{\mathfrak{H}} C_\mu(\widehat{A}) \restriction_{\mathfrak{H}} &= \left\{ \left\{ f' - \mu f, f' - \bar\mu f \right\} : \widehat{f} = \{f, f'\} \in S^*,\ f_\mu = 0 \right\} \\
&= C_\mu(S \,\widehat{+}\, \widehat{\mathfrak{N}}_{\bar\mu}(S^*)) = \widetilde{V}, \\
P_{\mathfrak{N}_{\bar\mu}(S^*)} C_\mu(\widehat{A}) \restriction_{\mathfrak{H}} &= \left\{ \left\{ f' - \mu f, (\mu - \bar\mu) f_{\bar\mu} \right\} : \widehat{f} = \{f, f'\} \in S^*,\ f_\mu = 0 \right\} \\
&= -P_{\mathfrak{N}_{\bar\mu}(S^*)} = -P_{(\operatorname{dom} V)^\perp}, \\
P_{\mathfrak{H}} C_\mu(\widehat{A}) \restriction_{\mathfrak{N}_{\bar\mu}(S^*)} &= \left\{ \left\{ (\bar\mu - \mu)\Theta_0^* f_\mu, f' - \bar\mu f \right\} : \widehat{f} = \{f, \mu f\} \in S^* \right\} \\
&= - \restriction_{\mathfrak{N}_\mu(S^*)} \Theta_0 = - \restriction_{(\operatorname{ran} V)^\perp} \Theta_0, \\
P_{\mathfrak{N}_{\bar\mu}(S^*)} C_\mu(\widehat{A}) \restriction_{\mathfrak{N}_{\bar\mu}(S^*)} &= \left\{ \left\{ (\bar\mu - \mu)\Theta_0^* f_\mu, (\mu - \bar\mu) f_{\bar\mu} \right\} : \widehat{f} = \{f, \mu f\} \in S^* \right\} = 0,
\end{aligned}$$

show that $C_\mu(\widehat{A}) = \widehat{U}_{\Theta_0}$, cf. (3.13), and, in particular, that $\widehat{A}$ is selfadjoint. The following result parallels Lemma 3.2.

Lemma 4.2 *The Štraus extensions of the trivial symmetric operator in $\mathfrak{N}_{\bar\mu}(S^*)$ corresponding to the selfadjoint extension $\widehat{A}$ are given by $-M_{\bar\mu}(\lambda)$, where*

$$M_{\bar\mu}(\lambda) = \left\{ \{ f_{\bar\mu} + \Theta_0^* f_\mu, -\mu f_{\bar\mu} - \bar\mu \Theta_0^* f_\mu \} : \widehat{f} \in \widehat{\mathfrak{N}}_\lambda(S^*) \right\}. \tag{4.7}$$

In particular,

$$P_{\mathfrak{N}_{\bar\mu}(S^*)} \left(\widehat{A} - \lambda \right)^{-1} \restriction_{\mathfrak{N}_{\bar\mu}(S^*)} = -\left(M_{\bar\mu}(\lambda) + \lambda \right)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{4.8}$$

and $M_{\bar\mu}$ is a uniformly strict Nevanlinna function connected with the uniformly contractive Schur function $X(\cdot)\Theta_0$ (see (3.9)) by

$$M_{\bar\mu}(\lambda) = -\mu + (\mu - \bar\mu)\left(I - X(z)\Theta_0 \right)^{-1}, \quad \lambda \in \mathbb{C}_+. \tag{4.9}$$

Proof. It follows from (4.6) that the Štraus extensions of the trivial symmetric operator $\{0,0\}$ in $\mathfrak{N}_{\bar\mu}(S^*)$ have the form (4.7) and then (4.2) coincides with (4.8). From $C_\mu(\widehat{A}) = \widehat{U}_{\Theta_0}$ and (2.2) one concludes that

$$\frac{\mu-\bar\mu}{\lambda-\bar\mu}\, P_{\mathfrak{N}_{\bar\mu}(S^*)}\Big(I - z\widehat{U}_{\Theta_0}\Big)^{-1} \restriction_{\mathfrak{N}_{\bar\mu}(S^*)} = I + (\lambda-\mu)P_{\mathfrak{N}_{\bar\mu}(S^*)}(\widehat{A}-\lambda)^{-1} \restriction_{\mathfrak{N}_{\bar\mu}(S^*)}$$

holds. Now use (3.14), (4.8) and (2.4) with $\nu=\mu$ to obtain (4.9). □

4.3 Kreĭn's formula for symmetric relations

Let the defect numbers of S be equal, fix $\mu \in \mathbb{C}_+$ and a unitary operator $\Theta_0 \in \mathbf{B}(\mathfrak{N}_{\bar\mu}(S^*), \mathfrak{N}_\mu(S^*))$, and let

$$\mathcal{A}_0 := S \mathbin{\widehat{+}} \Big\{\{\,(\Theta_0 - I)f_{\bar\mu}, (\mu\Theta_0 - \bar\mu)f_{\bar\mu}\} : f_{\bar\mu} \in \mathfrak{N}_{\bar\mu}(S^*)\,\Big\} \tag{4.10}$$

be the corresponding selfadjoint extension of S in $\mathfrak{H}$ via (4.4). Furthermore, define the function $\lambda \mapsto \gamma_{\bar\mu}(\lambda) \in \mathbf{B}(\mathfrak{N}_{\bar\mu}(S^*), \mathfrak{H})$ by

$$\gamma_{\bar\mu}(\lambda) = \Big(I + (\lambda-\bar\mu)(\mathcal{A}_0-\lambda)^{-1}\Big)\restriction_{\mathfrak{N}_{\bar\mu}(S^*)} . \tag{4.11}$$

It follows from the resolvent identity and (4.11) that

$$\gamma_{\bar\mu}(\lambda) = (I + (\lambda-\mu)(\mathcal{A}_0-\lambda)^{-1})\gamma_{\bar\mu}(\mu), \quad \lambda \in \rho(\mathcal{A}_0). \tag{4.12}$$

Theorem 4.3 *Let S be a closed symmetric relation with equal defect numbers in the Hilbert space $\mathfrak{H}$, let $\mathcal{A}_0$ be the canonical selfadjoint extension of S in* (4.10)*, and let the functions $\gamma_{\bar\mu}$ and $M_{\bar\mu}$ be as in* (4.11) *and* (4.7)*, respectively. Then*

$$P_{\mathfrak{H}}\Big(\widetilde{A}-\lambda\Big)^{-1} \restriction \mathfrak{H} = (\mathcal{A}_0-\lambda)^{-1} - \gamma_{\bar\mu}(\lambda)\Big(M_{\bar\mu}(\lambda) + \tau(\lambda)\Big)^{-1}\gamma_{\bar\mu}(\bar\lambda)^*, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{4.13}$$

establishes a one-to-one correspondence between the generalized resolvents of S and the Nevanlinna families τ in $\mathfrak{N}_{\bar\mu}(S^*)$. Moreover, the Nevanlinna family τ in (4.13) *and the Schur function Θ in* (4.3) *are connected via*

$$\Theta(z) = \Theta_0\left(I - (\mu-\bar\mu)(\tau(\lambda)-\bar\mu)^{-1}\right), \quad \lambda \in \mathbb{C}_+. \tag{4.14}$$

Proof. Let $\widetilde{A}$ be a selfadjoint extension of S in $\mathfrak{H} \oplus \mathfrak{K}$, let $\widetilde{U} = C_\mu(\widetilde{A})$ and $U_0 = C_\mu(\mathcal{A}_0)$. Then it follows from (2.2) that

$$\begin{aligned} &P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright_{\mathfrak{H}} -(\mathcal{A}_0 - \lambda)^{-1} \\ &\qquad = \frac{\mu - \bar{\mu}}{(\lambda - \mu)(\lambda - \bar{\mu})} \Big\{ P_{\mathfrak{H}}\Big(I - z\widetilde{U}\Big)^{-1} \upharpoonright_{\mathfrak{H}} - \Big(I - zU_0\Big)^{-1}\Big\}. \end{aligned} \tag{4.15}$$

According to Theorem 3.3 there exists a Schur function $\Theta \in \mathbf{S}(\mathfrak{N}_{\bar{\mu}}(S^*), \mathfrak{N}_\mu(S^*))$ such that the term $P_{\mathfrak{H}}(I - z\widetilde{U})^{-1} \upharpoonright_{\mathfrak{H}} -(I - zU_0)^{-1}$, $\lambda \in \mathbb{C}_+$, coincides with

$$\begin{aligned} &z\Big(I - zU_0\Big)^{-1} \upharpoonright_{\mathfrak{N}_\mu(S^*)} \Big(\Theta(z) - \Theta_0\Big) \cdot \\ &\qquad \cdot \Big(I - X(z)\Theta(z)\Big)^{-1}\Big(I - X(z)\Theta_0\Big)P_{\mathfrak{N}_{\bar{\mu}}(S^*)}\Big(I - zU_0\Big)^{-1}. \end{aligned} \tag{4.16}$$

It follows from (2.2), (4.12), $\gamma_{\bar{\mu}}(\mu) = \upharpoonright_{\mathfrak{N}_\mu(S^*)} \Theta_0$, and $P_{\mathfrak{N}_{\bar{\mu}}(S^*)} = \gamma_{\bar{\mu}}(\bar{\mu})^*$, that

$$z\Big(I - zU_0\Big)^{-1} \upharpoonright_{\mathfrak{N}_\mu(S^*)} = \frac{\lambda - \mu}{\mu - \bar{\mu}} \gamma_{\bar{\mu}}(\lambda)\Theta_0^*, \quad P_{\mathfrak{N}_{\bar{\mu}}(S^*)}\Big(I - zU_0\Big)^{-1} = \frac{\lambda - \bar{\mu}}{\mu - \bar{\mu}} \gamma_{\bar{\mu}}(\bar{\lambda})^*. \tag{4.17}$$

Insertion of (4.17) into (4.16) shows that the lefthand side of (4.15) is given by

$$-\gamma_{\bar{\mu}}(\lambda)\Big\{\frac{1}{\bar{\mu} - \mu} \Big(\Theta_0^*\Theta(z) - I\Big)\Big(I - X(z)\Theta(z)\Big)^{-1}\Big(I - X(z)\Theta_0\Big)\Big\}\gamma_{\bar{\mu}}(\bar{\lambda})^*. \tag{4.18}$$

It will be shown that the term $\{\cdots\}$ in (4.18) is equal to $(M_{\bar{\mu}}(\lambda) + \tau(\lambda))^{-1}$, where

$$\tau(\lambda) := \Big\{ \{(\Theta_0^*\Theta(z) - I)h, (\bar{\mu}\Theta_0^*\Theta(z) - \mu)h\} : h \in \mathfrak{N}_{\bar{\mu}}(S^*) \Big\}.$$

Observe first that since $\Theta_0^*\Theta(\cdot)$ is a Schur function in $\mathfrak{N}_{\bar{\mu}}(S^*)$ (2.4) with $\nu = -\bar{\mu}$ implies that $\tau(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is a Nevanlinna family in $\mathfrak{N}_{\bar{\mu}}(S^*)$ and that (4.14) holds. Note in particular

$$(\tau(\lambda) - \bar{\mu})^{-1} = \frac{1}{\bar{\mu} - \mu}\Big(\Theta_0^*\Theta(z) - I\Big). \tag{4.19}$$

Recall that the relation $M_{\bar{\mu}}(\lambda) + \tau(\lambda)$ is boundedly invertible and apply Lemma 2.1 to $M_{\bar{\mu}}(\lambda) + \tau(\lambda) = M_{\bar{\mu}}(\lambda) + \bar{\mu} + \tau(\lambda) - \bar{\mu}$ to obtain

$$\Big(M_{\bar{\mu}}(\lambda) + \tau(\lambda)\Big)^{-1} = \frac{1}{\bar{\mu} - \mu}\Big(\Theta_0^*\Theta(z) - I\Big)\Big((M_{\bar{\mu}}(\lambda) + \bar{\mu})(\tau(\lambda) - \bar{\mu})^{-1} + I\Big)^{-1}. \tag{4.20}$$

According to Lemma 4.2 $M_{\bar\mu}(\lambda)+\bar\mu=(\mu-\bar\mu)(I-X(z)\Theta_0)^{-1}X(z)\Theta_0$, $\lambda\in\mathbb{C}_+$, and this together with (4.19) implies

$$(M_{\bar\mu}(\lambda)+\bar\mu)(\tau(\lambda)-\bar\mu)^{-1}+I=\Bigl(I-X(z)\Theta_0\Bigr)^{-1}\Bigl(I-X(z)\Theta(z)\Bigr).$$

Therefore (4.20) can be written as

$$\Bigl(M_{\bar\mu}(\lambda)+\tau(\lambda)\Bigr)^{-1}=\frac{1}{\bar\mu-\mu}\Bigl(\Theta_0^*\Theta(z)-I\Bigr)\Bigl(I-X(z)\Theta(z)\Bigr)^{-1}\Bigl(I-X(z)\Theta_0\Bigr). \tag{4.21}$$

Now (4.21) and (4.18) give rise to (4.13).

For the converse statement, let τ be a Nevanlinna family in $\mathfrak{N}_{\bar\mu}(S^*)$ and define $\Theta\in\mathbf{S}(\mathfrak{N}_{\bar\mu}(S^*),\mathfrak{N}_\mu(S^*))$ by (4.14). Define $\widetilde\Theta$ by the right hand side of (3.5) and let $\widetilde U$ be a unitary colligation in $\mathfrak{H}\oplus\mathfrak{K}$ whose transfer function is $\widetilde\Theta$. The above calculations show that the inverse Cayley transform $\widetilde A$ of $\widetilde U$ is a selfadjoint extension of S in $\mathfrak{H}\oplus\mathfrak{K}$ giving rise to the generalized resolvent in the right hand side of (4.13). □

5 Boundary triplets and Kreĭn's formula

5.1 Boundary triplets and their Weyl functions

Let S be a closed symmetric relation with equal defect numbers in the Hilbert space $\mathfrak{H}$. A triplet $\{\mathcal{G},\Gamma_0,\Gamma_1\}$ is said to be a *boundary triplet* for S^*, if $\mathcal{G}$ is a Hilbert space and $\Gamma_0,\Gamma_1:S^*\to\mathcal{G}$ are linear mappings such that the mapping $\Gamma:=(\Gamma_0,\Gamma_1)^\top:S^*\to\mathcal{G}\times\mathcal{G}$ is surjective, and the abstract Green's identity

$$(f',g)-(f,g')=(\Gamma_1\widehat f,\Gamma_0\widehat g)-(\Gamma_0\widehat f,\Gamma_1\widehat g) \tag{5.1}$$

holds for all $\widehat f=\{f,f'\}$, $\widehat g=\{g,g'\}\in S^*$. The surjectivity condition and the identity (5.1) imply that the mappings $\Gamma_0,\Gamma_1:S^*\to\mathcal{G}$ are closed and therefore continuous. Note that $S=\ker\Gamma$ and that $\dim\mathcal{G}$ coincides with the defect numbers of S. The mapping

$$\mathcal{T}\mapsto A_{\mathcal{T}}:=\Gamma^{(-1)}\mathcal{T}=\Bigl\{\widehat f\in S^*:\{\Gamma_0\widehat f,\Gamma_1\widehat f\}\in\mathcal{T}\Bigr\}$$

establishes a one-to-one correspondence between the closed linear relations $\mathcal{T}$ in $\mathcal{G}$ and the closed extensions $A_{\mathcal{T}}$ with $S\subset A_{\mathcal{T}}\subset S^*$. Furthermore, $A_{\mathcal{T}}$ is symmetric (selfadjoint, (maximal) accumulative, (maximal) dissipative) if and only if $\mathcal{T}$ is symmetric (selfadjoint, (maximal) accumulative, (maximal) dissipative, respectively) in $\mathcal{G}$. In particular, $A_0 := \ker \Gamma_0$ and $A_1 := \ker \Gamma_1$ are selfadjoint extensions of S. It is not difficult to see that

$$S^* = A_i \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(S^*), \quad \lambda \in \rho(A_i), \quad i = 0, 1, \quad \text{direct sum.} \tag{5.2}$$

In particular, $\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*)$ is a one-to-one mapping onto $\mathcal{G}$. Denote the orthogonal projection in $\mathfrak{H} \oplus \mathfrak{H}$ onto the first component by π_1. The *γ-field* $\lambda \mapsto \gamma(\lambda)$ of $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is defined by

$$\gamma(\lambda) = \pi_1\Big(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*)\Big)^{-1} = \Big\{ \{\Gamma_0 \widehat{f}_\lambda, f_\lambda\} \,:\, \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*)\Big\}, \quad \lambda \in \rho(A_0), \tag{5.3}$$

and the *Weyl function* $\lambda \mapsto M(\lambda)$ is defined by

$$M(\lambda) = \Gamma_1\Big(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(S^*)\Big)^{-1} = \Big\{ \{\Gamma_0 \widehat{f}_\lambda, \Gamma_1 \widehat{f}_\lambda\} \,:\, \widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*)\Big\}, \quad \lambda \in \rho(A_0). \tag{5.4}$$

Since Γ_0 and Γ_1 are bounded and surjective it follows that $\gamma(\lambda) \in \mathbf{B}(\mathcal{G}, \mathfrak{H})$ and $M(\lambda) \in \mathbf{B}(\mathcal{G})$ for all $\lambda \in \rho(A_0)$. It is not difficult to see that

$$\Gamma_0\{\gamma(\lambda)h, \lambda\gamma(\lambda)h\} = h, \quad \Gamma_1\{\gamma(\lambda)h, \lambda\gamma(\lambda)h\} = M(\lambda)h, \quad h \in \mathfrak{H}, \tag{5.5}$$

holds. Furthermore, for all $h \in \mathfrak{H}$:

$$\begin{aligned} \Gamma_0\Big\{(A_0 - \lambda)^{-1}h, (I + \lambda(A_0 - \lambda)^{-1})h\Big\} &= 0, \\ \Gamma_1\Big\{(A_0 - \lambda)^{-1}h, (I + \lambda(A_0 - \lambda)^{-1})h\Big\} &= \gamma(\bar{\lambda})^* h. \end{aligned} \tag{5.6}$$

The first identity is clear, cf. (2.1). The second identity follows from (5.1), (5.2), (5.3), and (2.1), with $\widehat{f} = \{(A_0 - \lambda)^{-1}h, (I + \lambda(A_0 - \lambda)^{-1})h\Big\}$ and $\widehat{g} = \{g, \bar{\lambda} g\} \in \widehat{\mathfrak{N}}_{\bar{\lambda}}(S^*)$. The identities

$$\gamma(\lambda) = \Big(I + (\lambda - \nu)(A_0 - \lambda)^{-1}\Big)\gamma(\nu), \quad M(\lambda) - M(\nu)^* = (\lambda - \bar{\nu})\gamma(\nu)^*\gamma(\lambda) \tag{5.7}$$

hold for all $\lambda, \nu \in \rho(A_0)$, and, hence, γ and M are holomorphic on $\rho(A_0)$. Therefore M is a uniformly strict $\mathbf{B}(\mathcal{G})$-valued Nevanlinna function and (5.7) implies

$$M(\lambda) = \operatorname{Re} M(\nu) + \gamma(\nu)^*\Big((\lambda - \operatorname{Re}\nu) + (\lambda - \nu)(\lambda - \bar{\nu})(A_0 - \lambda)^{-1}\Big)\gamma(\nu) \tag{5.8}$$

for all $\lambda, \nu \in \rho(A_0)$, cf. [16]. For further details, see [8] and [9].

Let $\mathcal{G}$ and $\mathcal{G}'$ be Hilbert spaces and let $W = (W_{ij})_{i,j=1}^2 \in \mathbf{B}(\mathcal{G} \oplus \mathcal{G}, \mathcal{G}' \oplus \mathcal{G}')$ be boundedly invertible and satisfy

$$W^* \begin{pmatrix} 0 & -iI_{\mathcal{G}'} \\ iI_{\mathcal{G}'} & 0 \end{pmatrix} W = \begin{pmatrix} 0 & -iI_{\mathcal{G}} \\ iI_{\mathcal{G}} & 0 \end{pmatrix}. \tag{5.9}$$

Then it is not difficult to see that $W[\tau(\lambda)]$ defined by

$$W[\tau(\lambda)] := \Big\{ \{W_{11}h + W_{12}k, W_{21}h + W_{22}k\} : \{h,k\} \in \tau(\lambda) \Big\} \tag{5.10}$$

is a Nevanlinna family in $\mathcal{G}'$ if and only if τ is a Nevanlinna family in $\mathcal{G}$.

Let S be a closed symmetric relation with equal defect numbers in a Hilbert space $\mathfrak{H}$. Let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}', \Gamma_0', \Gamma_1'\}$ be boundary triplets for S^*. Let $A_0 = \ker \Gamma_0$ and $A_0' = \ker \Gamma_0'$, and let γ and γ' be the corresponding γ-fields and let M and M' be the corresponding Weyl functions, respectively. Then there exists a boundedly invertible operator $W = (W_{ij})_{i,j=1}^2 \in \mathbf{B}\big(\mathcal{G} \oplus \mathcal{G}, \mathcal{G}' \oplus \mathcal{G}'\big)$ with the property (5.9) such that

$$\begin{pmatrix} \Gamma_0' \\ \Gamma_1' \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}. \tag{5.11}$$

To see this, consider $\Gamma = (\Gamma_0, \Gamma_1)^\top$ and $\Gamma' = (\Gamma_0', \Gamma_1')^\top$ on the quotient space S^*/S so that they are bijective. For $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(S^*)$ (5.11) and (5.4) yield $\Gamma_0' \widehat{f}_\lambda = (W_{11} + W_{12}M(\lambda))\Gamma_0 \widehat{f}_\lambda$. Now the restrictions of Γ_0 and Γ_0' to $\widehat{\mathfrak{N}}_\lambda(S^*)$ are one-to-one mappings onto $\mathcal{G}$ and $\mathcal{G}'$, respectively. Hence $(W_{11} + W_{12}M(\lambda))^{-1} \in \mathbf{B}(\mathcal{G}', \mathcal{G})$ for all $\lambda \in \rho(A_0) \cap \rho(A_0')$. This implies that

$$\begin{aligned} \gamma'(\lambda) &= \gamma(\lambda)\Big(W_{11} + W_{12}M(\lambda)\Big)^{-1}, \\ M'(\lambda) &= \Big(W_{21} + W_{22}M(\lambda)\Big)\Big(W_{11} + W_{12}M(\lambda)\Big)^{-1}. \end{aligned}$$

5.3 A special boundary triplet

Let S be a closed symmetric relation with equal defect numbers in the Hilbert space $\mathfrak{H}$. In the following decompose the elements $\widehat{f} = \{f, f'\} \in S^*$ according to von Neumann's formula (4.1) and (4.5). In the next proposition a boundary triplet is constructed where the γ-field and Weyl functions appear in Kreĭn's formula in Theorem 4.3.

Proposition 5.1 *Let S be a closed symmetric relation with equal defect numbers in the Hilbert space $\mathfrak{H}$ and let $\Theta_0 \in \mathbf{B}(\mathfrak{N}_{\bar\mu}(S^*), \mathfrak{N}_\mu(S^*))$ be a unitary mapping. Then $\{\mathfrak{N}_{\bar\mu}(S^*), \Gamma_{0,\bar\mu}, \Gamma_{1,\bar\mu}\}$, where*

$$\Gamma_{0,\bar\mu}\widehat{f} := f_{\bar\mu} + \Theta_0^* f_\mu, \quad \Gamma_{1,\bar\mu}\widehat{f} := -\mu f_{\bar\mu} - \bar\mu\Theta_0^* f_\mu, \quad \widehat{f} \in S^*, \tag{5.12}$$

is a boundary triplet for S^* and the selfadjoint extension $\ker \Gamma_{0,\bar\mu}$ *coincides with the relation $\mathcal{A}_0$ in* (4.10). *The corresponding γ-field and Weyl function are given by $\gamma_{\bar\mu}$ and $M_{\bar\mu}$ in* (4.11) *and* (4.7)*, respectively.*

Proof. Since $\widehat{A}$ in (4.6) is selfadjoint the abstract Green's identity (5.1) follows. Now let $h, k \in \mathfrak{N}_{\bar\mu}(S^*)$ and define $\widehat{f} \in S^*$ by (4.5) with arbitrary $\{f_0, f_0'\} \in S$ and

$$f_{\bar\mu} = -(\mu - \bar\mu)^{-1}(k + \bar\mu h), \quad f_\mu = (\mu - \bar\mu)^{-1}\Theta_0(k + \mu h).$$

Then $\{\Gamma_{0,\bar\mu}\widehat{f}, \Gamma_{1,\bar\mu}\widehat{f}\} = \{h, k\}$ and the mapping $(\Gamma_{0,\bar\mu}, \Gamma_{1,\bar\mu})^\top$ is onto. Therefore $\{\mathfrak{N}_{\bar\mu}(S^*), \Gamma_{0,\bar\mu}, \Gamma_{1,\bar\mu}\}$ is a boundary triplet for S^*. It follows from (4.10) that $\ker \Gamma_{0,\bar\mu} = \mathcal{A}_0$.

From the definition (5.4) one obtains that $M_{\bar\mu}$ in (4.7) is the Weyl function of the boundary triplet in (5.12). Since the value of the γ-field of the boundary triplet $\{\mathfrak{N}_{\bar\mu}(S^*), \Gamma_{0,\bar\mu}, \Gamma_{1,\bar\mu}\}$ at $\bar\mu$ is $\upharpoonright_{\mathfrak{N}_{\bar\mu}(S^*)}$ the first relation in (5.7) with $\nu = \bar\mu$ implies that the γ-field is given by (4.11). □

Proposition 5.2 *The Štraus extension in* (4.2) *corresponding to the generalized resolvent in* (4.13) *is given by*

$$T(\lambda) = \Big\{ \widehat{f} \in S^* : \Big\{ \Gamma_{0,\bar\mu}\widehat{f}, \Gamma_{1,\bar\mu}\widehat{f} \Big\} \in -\tau(\lambda) \Big\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{5.13}$$

Proof. Recall that $S^* = \mathcal{A}_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(S^*)$ and observe that according to Theorem 4.3 an element $\widehat{f} = \{f, f'\} \in \mathfrak{H} \times \mathfrak{H}$ belongs to the Štraus extension $T(\lambda)$ in (4.2) if and only if there exists an element $g \in \mathfrak{H}$ such that

$$\begin{aligned} \{f, f'\} &= \{(\mathcal{A}_0 - \lambda)^{-1} g, g + \lambda(\mathcal{A}_0 - \lambda)^{-1} g\} \\ &- \Big\{ \gamma_{\bar\mu}(\lambda)\Big(M_{\bar\mu}(\lambda) + \tau(\lambda)\Big)^{-1}\gamma_{\bar\mu}(\bar\lambda)^* g \,,\, \lambda\gamma_{\bar\mu}(\lambda)\Big(M_{\bar\mu}(\lambda) + \tau(\lambda)\Big)^{-1}\gamma_{\bar\mu}(\bar\lambda)^* g \Big\}. \end{aligned} \tag{5.14}$$

Note that the first element in the righthand side belongs to $\mathcal{A}_0$.

First the inclusion $T(\lambda) \subset \{\widehat{f} \in S^* : \{\Gamma_{0,\bar\mu}\widehat{f}, \Gamma_{1,\bar\mu}\widehat{f}\} \in -\tau(\lambda)\}$ will be verified. Let $\widehat{f} = \{f, f'\} \in T(\lambda)$ be as in (5.14). Then $\mathcal{A}_0 = \ker \Gamma_{0,\bar\mu}$, (5.6), and (5.5) imply that the element $\{\Gamma_{0,\bar\mu}\widehat{f}, \Gamma_{1,\bar\mu}\widehat{f}\}$ is given by

$$\Big\{-\Big(M_{\bar\mu}(\lambda)+\tau(\lambda)\Big)^{-1}\gamma_{\bar\mu}(\bar\lambda)^*g, \gamma_{\bar\mu}(\bar\lambda)^*g - M_{\bar\mu}(\lambda)\Big(M_{\bar\mu}(\lambda)+\tau(\lambda)\Big)^{-1}\gamma_{\bar\mu}(\bar\lambda)^*g\Big\},$$

which shows that $\{\Gamma_{0,\bar\mu}\widehat{f}, \Gamma_{1,\bar\mu}\widehat{f}\} \in -(M_{\bar\mu}(\lambda)+\tau(\lambda)) + M_{\bar\mu}(\lambda) = -\tau(\lambda)$.

Now the converse inclusion in (5.13) will be shown. Let $\widehat{f} = \{f, f'\} \in S^*$ satisfy $\{\Gamma_{0,\bar\mu}\widehat{f}, \Gamma_{1,\bar\mu}\widehat{f}\} \in -\tau(\lambda)$. Decompose $\widehat{f}$ as $\widehat{f} = \widehat{f}_0 + \widehat{f}_\lambda$ with $\widehat{f}_0 \in \mathcal{A}_0$ and $\widehat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \widehat{\mathfrak{N}}_\lambda(S^*)$. Choose an element $g \in \mathfrak{H}$ such that $\widehat{f}_0 = \{(\mathcal{A}_0-\lambda)^{-1}g, g+\lambda(\mathcal{A}_0-\lambda)^{-1}g\}$. Then it follows from (5.6) that

$$\Big\{\Gamma_{0,\bar\mu}\widehat{f}_\lambda, \gamma_{\bar\mu}(\bar\lambda)^*g + M_{\bar\mu}(\lambda)\Gamma_{0,\bar\mu}\widehat{f}_\lambda\Big\} = \Big\{\Gamma_{0,\bar\mu}\widehat{f}, \Gamma_{1,\bar\mu}(\widehat{f}_0+\widehat{f}_\lambda)\Big\} \in -\tau(\lambda). \quad (5.15)$$

Since $M_{\bar\mu}$ is a uniformly strict Nevanlinna function and τ is a Nevanlinna family $(M_{\bar\mu}(\lambda)+\tau(\lambda))^{-1} \in \mathbf{B}(\mathcal{G})$ and (5.15) implies $\Gamma_{0,\bar\mu}\widehat{f}_\lambda = -(M_{\bar\mu}(\lambda) + \tau(\lambda))^{-1}\gamma_{\bar\mu}(\bar\lambda)^*g$. By $f_\lambda = \gamma_{\bar\mu}(\lambda)\Gamma_{0,\bar\mu}\widehat{f}_\lambda$, so that

$$f_\lambda = -\gamma_{\bar\mu}(\lambda)(M_{\bar\mu}(\lambda)+\tau(\lambda))^{-1}\gamma_{\bar\mu}(\bar\lambda)^*g.$$

Therefore $\widehat{f} = \{f, f'\} = \widehat{f}_0 + \widehat{f}_\lambda$ is of the form (5.14) and hence $\widehat{f} \in T(\lambda)$. $\square$

5.4 Kreĭn's formula

The following theorem is a reformulation of Theorem 4.3 and Proposition 5.2 in terms of general boundary triplets and associated Weyl functions.

Theorem 5.3 *Let S be a closed symmetric relation with equal defect numbers in the Hilbert space $\mathfrak{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for S^*. Let $A_0 = \ker\Gamma_0$ and denote the γ-field and Weyl function of $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ by γ and M, respectively. Then the formula*

$$P_{\mathfrak{H}}\Big(\widetilde{A}-\lambda\Big)^{-1}\restriction_{\mathfrak{H}} = (A_0-\lambda)^{-1} - \gamma(\lambda)\Big(M(\lambda)+\widetilde{\tau}(\lambda)\Big)^{-1}\gamma(\bar\lambda)^*, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{5.16}$$

establishes a one-to-one correspondence between the generalized resolvents of S and the Nevanlinna families $\widetilde{\tau}$ in $\mathcal{G}$. Furthermore, the Štraus extension corresponding to the generalized resolvent in (5.16) *via* (4.2) *is*

$$T(\lambda) = \Big\{\, \widehat{f} \in S^* : \{\Gamma_0\widehat{f}, \Gamma_1\widehat{f}\} \in -\widetilde{\tau}(\lambda)\,\Big\}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{5.17}$$

Proof. For the special boundary triplet $\{\mathfrak{N}_{\bar\mu}(S^*), \Gamma_{0,\bar\mu}, \Gamma_{1,\bar\mu}\}$ from Proposition 5.1 this has been shown in Theorem 4.3 and Proposition 5.2. Now let

$\{\mathcal{G},\Gamma_0,\Gamma_1\}$ be an arbitrary boundary triplet for S^*. Then there exists a boundedly invertible operator $W=(W_{ij})_{i,j=1}^2\in\mathbf{B}(\mathfrak{N}_{\bar\mu}(S^*)\oplus\mathfrak{N}_{\bar\mu}(S^*),\mathcal{G}\oplus\mathcal{G})$ which satisfies

$$W^*\begin{pmatrix}0 & -iI_{\mathcal{G}}\\ iI_{\mathcal{G}} & 0\end{pmatrix}W=\begin{pmatrix}0 & -iI_{\mathfrak{N}_{\bar\mu}(S^*)}\\ iI_{\mathfrak{N}_{\bar\mu}(S^*)} & 0\end{pmatrix},\quad \begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix}=W\begin{pmatrix}\Gamma_{0,\bar\mu}\\ \Gamma_{1,\bar\mu}\end{pmatrix},\tag{5.18}$$

cf. (5.9) and (5.11). Observe that the Štraus extension $T(\lambda)$ in (5.13) with respect to the boundary triplet $\{\mathfrak{N}_{\bar\mu}(S^*),\Gamma_{0,\bar\mu},\Gamma_{1,\bar\mu}\}$ coincides with the extension

$$\Big\{\widehat f\in S^*:\{\Gamma_0\widehat f,\Gamma_1\widehat f\}\in-\widetilde\tau(\lambda)\Big\},\quad \widetilde\tau(\lambda):=W[\tau(\lambda)]$$

with respect to $\{\mathcal{G},\Gamma_0,\Gamma_1\}$, cf. (5.10). Thus it remains to show that $(T(\lambda)-\lambda)^{-1}$ coincides with the righthand side of (5.16). For this let $\widehat f=\{f,f'\}\in T(\lambda)$, and decompose $\widehat f$ as $\widehat f=\widehat f_0+\widehat f_\lambda$ with $\widehat f_0\in A_0$ and $\widehat f_\lambda=\{f_\lambda,\lambda f_\lambda\}\in\widehat{\mathfrak{N}}_\lambda(S^*)$. Choose an element $g\in\mathfrak{H}$ such that $\widehat f_0=\{(A_0-\lambda)^{-1}g,g+\lambda(A_0-\lambda)^{-1}g\}$. Then it follows from (5.6)

$$\Big\{\Gamma_0\widehat f_\lambda,\gamma(\bar\lambda)^*g+M(\lambda)\Gamma_0\widehat f_\lambda\Big\}=\Big\{\Gamma_0\widehat f,\Gamma_1(\widehat f_0+\widehat f_\lambda)\Big\}\in-\widetilde\tau(\lambda)$$

and the same arguments as in the proof of Proposition 5.2 yield

$$f_\lambda=-\gamma(\lambda)\Big(M(\lambda)+\widetilde\tau(\lambda)\Big)^{-1}\gamma(\bar\lambda)^*g.$$

Therefore $\widehat f=\{f,f'\}=\widehat f_0+\widehat f_\lambda$ is of the form

$$\begin{aligned}\{f,f'\}&=\{(A_0-\lambda)^{-1}g,g+\lambda(A_0-\lambda)^{-1}g\}\\ &\quad-\Big\{\gamma(\lambda)\Big(M(\lambda)+\widetilde\tau(\lambda)\Big)^{-1}\gamma(\bar\lambda)^*g,\lambda\gamma(\lambda)\Big(M(\lambda)+\widetilde\tau(\lambda)\Big)^{-1}\gamma(\bar\lambda)^*g\Big\}\end{aligned}$$

and $f'-\lambda f=g$ holds. Hence $f=(T(\lambda)-\lambda)^{-1}g$ is given by the righthand side of (5.16). □

Of particular importance in many applications is the following special case of Kreĭn's formula for canonical extensions.

Corollary 5.4 *Let the assumptions be as in Theorem 5.3. Then the formula*

$$(A_{\mathcal{T}}-\lambda)^{-1}=(A_0-\lambda)^{-1}+\gamma(\lambda)\Big(\mathcal{T}-M(\lambda)\Big)^{-1}\gamma(\bar\lambda)^*,\quad\lambda\in\mathbb{C}\setminus\mathbb{R},$$

establishes a one-to-one correspondence between the canonical selfadjoint extensions $A_{\mathcal{T}}=\{\widehat f\in S^*:\{\Gamma_0\widehat f,\Gamma_1\widehat f\}\in\mathcal{T}\}$ of S and the selfadjoint relations $\mathcal{T}$ in $\mathcal{G}$.

The relation between the parameters in Kreĭn's formula (5.16) and the von Neumann formula (4.3) is given via $\widetilde\tau(\lambda):=W[\tau(\lambda)]$ and (4.14); see also [11].

Let $q_+ \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $q_- \in L^1_{\mathrm{loc}}(\mathbb{R}_-)$ be real functions and suppose that the differential expressions $-\frac{d^2}{dx^2}+q_+$ and $-\frac{d^2}{dx^2}+q_-$ are regular at the endpoint 0 and in the limit point case at the singular endpoints $+\infty$ and $-\infty$, respectively. Denote by $\mathcal{D}^\pm_{\max}$ the linear space of all functions $f_\pm \in L^2(\mathbb{R}_\pm)$ such that $f_\pm$ and $f'_\pm$ are absolutely continuous and $-f''_\pm + q_\pm f_\pm$ belongs to $L^2(\mathbb{R}_\pm)$. Functions in $L^2(\mathbb{R})$ will be written in the form $f = \{f_+, f_-\} \in L^2(\mathbb{R})$, $f_\pm = f \restriction_{\mathbb{R}_\pm} \in L^2(\mathbb{R}_\pm)$.

It is well known that

$$Sf_+ = -f''_+ + q_+ f_+, \qquad \operatorname{dom} S = \Big\{ f_+ \in \mathcal{D}^+_{\max} : f_+(0) = f'_+(0) = 0 \Big\},$$

is a densely defined closed symmetric operator in $L^2(\mathbb{R}_+)$ with defect numbers $(1,1)$. The adjoint

$$S^* f_+ = -f''_+ + q_+ f_+, \qquad \operatorname{dom} S^* = \mathcal{D}^+_{\max},$$

is the usual maximal operator and $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, where $\Gamma_0 f_+ = f_+(0)$ and $\Gamma_1 f_+ = f'_+(0)$, $f_+ \in \mathcal{D}^+_{\max}$, is a boundary triplet for S^*. The Weyl function M coincides with the usual Titchmarsh-Weyl function associated to the differential expression $-\frac{d^2}{dx^2}+q_+$, i.e., if $\varphi_{\lambda,+}$ and $\psi_{\lambda,+}$ are solutions of $-u''_+ + q_+ u_+ = \lambda u_+$ on $\mathbb{R}_+$ satisfying

$$\varphi_{\lambda,+}(0) = 1,\ \varphi'_{\lambda,+}(0) = 0 \quad \text{and} \quad \psi_{\lambda,+}(0) = 0,\ \psi'_{\lambda,+}(0) = 1, \tag{5.19}$$

then for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the function $x \mapsto \varphi_{\lambda,+}(x) + M(\lambda)\psi_{\lambda,+}(x)$ belongs to $L^2(\mathbb{R}_+)$. Similarly, if $\varphi_{\lambda,-}$ and $\psi_{\lambda,-}$ are solutions of $-u''_- + q_- u_- = \lambda u_-$ on $\mathbb{R}_-$ satisfying the same boundary conditions as in (5.19), then the Titchmarsh-Weyl function of the differential expression $-\frac{d^2}{dx^2}+q_-$ is defined as the unique Nevanlinna function τ such that $x \mapsto \varphi_{\lambda,-}(x) - \tau(\lambda)\psi_{\lambda,-}(x)$ belongs to $L^2(\mathbb{R}_-)$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

The next well-known statement shows how the Titchmarsh-Weyl function τ is connected with Štraus extensions of S.

Proposition 5.5 *The maximal differential operator*

$$\widetilde{A} f = -f'' + qf, \qquad q(x) := \begin{cases} q_+(x) & \text{for } x > 0, \\ q_-(x) & \text{for } x < 0, \end{cases}$$
$$\operatorname{dom} \widetilde{A} = \Big\{ f = \{f_+, f_-\} : f_\pm \in \mathcal{D}^\pm_{\max},\ f_+(0) = f_-(0),\ f'_+(0) = f'_-(0) \Big\},$$

in $L^2(\mathbb{R})$ is a selfadjoint extension of S where the exit space is $L^2(\mathbb{R}_-)$. The

Štraus extensions $T(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *of* S *corresponding to* $\widetilde{A}$ *are given by*

$$T(\lambda)f_+ = -f_+'' + q_+ f_+, \quad \operatorname{dom} T(\lambda) = \Big\{ f_+ \in \mathcal{D}_{\max}^+ : \tau(\lambda) f_+(0) + f_+'(0) = 0 \Big\}.$$

Proof. It is clear that $\operatorname{dom} \widetilde{A}$ coincides with the usual maximal domain consisting of all functions $f \in L^2(\mathbb{R})$ such that f and f' are absolutely continuous and $-f'' + qf$ belongs to $L^2(\mathbb{R})$. Furthermore, the differential expression $-\frac{d^2}{dx^2} + q$ is in the limit point case at $\pm\infty$ and hence $\widetilde{A}$ is a selfadjoint extension in $L^2(\mathbb{R})$ of the symmetric operator S in $L^2(\mathbb{R}_+)$.

In order to calculate the Štraus extensions $T(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, of S corresponding to $\widetilde{A}$ observe that $-f'' + qf - \lambda f$, $f = \{f_+, f_-\} \in \operatorname{dom} \widetilde{A}$, can be identified with an element in $L^2(\mathbb{R}_+)$ if and only if $-f_-'' + q_- f_- = \lambda f_-$ holds. Hence $f_+ = P_{L^2(\mathbb{R}_+)} f$ belongs to $\operatorname{dom} T(\lambda)$ if and only if f_- is an $L^2(\mathbb{R}_-)$-solution of $-u_-'' + q_- u_- = \lambda u_-$. Observe that

$$f_-(x) = f_-(0)\Big(\varphi_{\lambda,-}(x) - \tau(\lambda)\psi_{\lambda,-}(x)\Big), \qquad x \in \mathbb{R}_-,$$

and hence $f_-'(0) = -\tau(\lambda) f_-(0)$. Therefore, if $f = \{f_+, f_-\} \in \operatorname{dom} \widetilde{A}$ and $f_+ \in \operatorname{dom} T(\lambda)$, then the function $f_+ \in \mathcal{D}_{\max}^+$ satisfies the boundary condition

$$\tau(\lambda) f_+(0) = \tau(\lambda) f_-(0) = -f_-'(0) = -f_+'(0)$$

and $T(\lambda)f_+ = -f_+'' + q_+ f_+$ holds. $\square$

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