Boundary value problems with eigenvalue depending boundary conditions

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We investigate some classes of eigenvalue dependent boundary value problems of the form

$$f' - \lambda f = k, \qquad \tau(\lambda)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0, \qquad \hat{f} = \binom{f}{f'} \in A^+,$$

where $A \subset A^+$ is a symmetric relation in a Krein space $\mathcal{K}$, τ is a matrix function and Γ_0, Γ_1 are abstract boundary mappings. It is assumed that A admits a selfadjoint extension in $\mathcal{K}$ which locally has the same spectral properties as a definitizable relation, and that τ is a matrix function which locally can be represented with the resolvent of a selfadjoint definitizable relation. The strict part of τ is realized as the Weyl function of a symmetric operator T in a Krein space $\mathcal{H}$, a selfadjoint extension $\widetilde{A}$ of $A \times T$ in $\mathcal{K} \times \mathcal{H}$ with the property that the compressed resolvent $P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}}\, k$ yields the unique solution of the boundary value problem is constructed, and the local spectral properties of this so-called linearization $\widetilde{A}$ are studied. The general results are applied to indefinite Sturm-Liouville operators with eigenvalue dependent boundary conditions.

1 Introduction

The main objective of this paper is the investigation of a class of abstract boundary value problems with boundary conditions depending on the eigenvalue parameter. For this let A be a closed symmetric operator or relation of finite defect n in some Krein space $\mathcal{K}$, let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the adjoint A^+, and assume that τ is an $\mathcal{L}(\mathbb{C}^n)$-valued function locally holomorphic in some open subset of the extended complex plane which is symmetric with respect to the real line such that $\tau(\overline{\lambda}) = \tau(\lambda)^*$ holds for all λ belonging to the set $\mathfrak{h}(\tau)$ of points of holomorphy of τ. We study boundary value problems of the following form: For a given $k \in \mathcal{K}$ and $\lambda \in \mathfrak{h}(\tau)$ find a vector $\hat{f} = \binom{f}{f'} \in A^+$ such that

$$f' - \lambda f = k \qquad \text{and} \qquad \tau(\lambda)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0 \tag{1.1}$$

holds. Under additional assumptions on τ and A, a solution of this problem can be obtained with the help of the compressed resolvent of a selfadjoint extension $\widetilde{A}$ of A which acts in a larger Krein space. Such a selfadjoint relation $\widetilde{A}$ is said to be a *linearization* of the boundary value problem (1.1). Based on the idea of a coupling method developed in [15] (see also [29, 30]) we construct a linearization of (1.1) and we study its spectral properties, which are closely connected with the solvability of this boundary value problem.

In the case that A is a symmetric operator or relation in a Hilbert space and τ is a Nevanlinna function or a generalized Nevanlinna function, boundary value problems of the form (1.1) have extensively been studied in a more or less abstract framework in the last decades (see e.g. [1, 11, 15, 20, 21, 23, 26, 45, 46]).

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Problems of the type (1.1) with symmetric operators and relations of defect one in Krein spaces and special classes of scalar functions in the boundary condition were considered in [3, 7] and [10]. In [13] and [14] symmetric operators or relations of infinite defect in Krein spaces and operator functions in the boundary condition were allowed. Very general classes of locally holomorphic functions in the boundary condition can be found in e.g. [14, 20] and [22].

Here we assume - roughly speaking - that A admits a selfadjoint extension in $\mathcal{K}$ which locally has the same spectral properties as a definitizable operator or relation and that τ is a matrix function which locally can be represented with the resolvent of a selfadjoint definitizable relation. More precisely, let Ω be some domain in $\overline{\mathbb{C}}$ symmetric with respect to the real line such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected. We will suppose that the selfadjoint extension $A_0 := \ker \Gamma_0$ of A is *definitizable over* Ω, i.e., for every subdomain Ω' of Ω with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, there exists a selfadjoint projection E which reduces A_0 such that $A_0 \cap (E\mathcal{K})^2$ is definitizable in the Krein space $E\mathcal{K}$ and Ω' belongs to the resolvent set of $A_0 \cap ((1-E)\mathcal{K})^2$. With the help of approximative eigensequences or the local spectral function of A_0 the spectral points of A_0 in $\Omega \cap \overline{\mathbb{R}}$ can be classified in points of positive and negative type and critical points, cf. [32, 36]. Furthermore, we assume that τ is an $\mathcal{L}(\mathbb{C}^n)$-valued *locally definitizable function in* Ω, that is, for every domain Ω', $\overline{\Omega'} \subset \Omega$, τ can be written as the sum of a definitizable function, see [34, 35], and a function holomorphic on Ω'. Similarly to selfadjoint operators and relations definitizable over Ω the points in $\Omega \cap \overline{\mathbb{R}}$ can be classified in points of positive and negative type and critical points of τ. The well-known representation of Nevanlinna functions and generalized Nevanlinna functions with the help of resolvents of selfadjoint operators and relations in Hilbert and Pontryagin spaces (see e.g. [41]) was generalized to locally definitizable functions in [37], i.e., the locally definitizable function τ can be minimally represented with a selfadjoint relation T_0 definitizable over Ω', $\overline{\Omega'} \subset \Omega$, in some Krein space $\mathcal{H}$ such that the sign types of τ and T_0 coincide in $\Omega' \cap \overline{\mathbb{R}}$.

Following the idea of the coupling method for the construction of the linearization $\widetilde{A}$ of (1.1) from [15] we have to realize the function τ in the boundary condition of (1.1) as the Weyl function corresponding to a symmetric operator $T \subset T_0$ in $\mathcal{H}$ and a boundary triplet for T^+. This is possible if the $\mathcal{L}(\mathbb{C}^n)$-valued locally definitizable function τ is *strict*, that is,

$$\bigcap_{\lambda \in \Omega \cap \mathfrak{h}(\tau)} \ker \frac{\tau(\lambda) - \tau(\mu_0)^*}{\lambda - \overline{\mu}_0} = \{0\}$$

holds for some $\mu_0 \in \mathfrak{h}(\tau) \cap \Omega$, see Lemma 3.2 and Theorem 3.3. For matrix-valued generalized Nevanlinna functions this is known from [17] and for scalar and matrix-valued local generalized Nevanlinna functions from [7] and [8]. We emphasize, that the Weyl function corresponding to a boundary triplet of a symmetric operator in a Krein or Pontryagin space is in general not strict, cf. Example 3.8. In the case that τ is a non-strict locally definitizable matrix function we show in Theorem 3.5 that τ can be written in the form

$$\lambda \mapsto \tau(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & \tau_s(\lambda) \end{pmatrix} + S, \quad S = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix},$$

where τ_s is a strict $\mathcal{L}(\mathbb{C}^s)$-valued locally definitizable function which is also minimally represented by the relation T_0, $s < n$, and S is a symmetric matrix constant. With the help of a suitable $(n-s)$-dimensional extension B of A and a boundary triplet $\{\mathbb{C}^s, \Gamma_0^s, \Gamma_1^s\}$ for B^+ we rewrite the boundary value problem (1.1) in the form

$$f' - \lambda f = k, \quad \tau_s(\lambda)\Gamma_0^s \hat{f} + \Gamma_1^s \hat{f} = 0, \quad \hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in B^+. \tag{1.2}$$

The linearization $\widetilde{A}$ of the boundary value problem (1.1), (1.2) will be constructed in Theorem 4.3. Here $\widetilde{A}$ is a finite dimensional perturbation in resolvent sense of the selfadjoint relation $A_0 \times T_0$ in the Krein space $\mathcal{K} \times \mathcal{H}$. If the sign types of A_0 and τ are d-compatible in $\Omega \cap \overline{\mathbb{R}}$ (see Definition 4.1), then it follows that $A_0 \times T_0$ is locally definitizable over Ω' and a recent perturbation result from [4] implies that the linearization $\widetilde{A}$ is also locally definitizable over Ω' and its sign types are d-compatible with the sign types of A_0 and τ in $\Omega' \cap \overline{\mathbb{R}}$.

This paper is organized as follows. In Section 2 we first provide some basic definitions and we recall the definitions of locally definitizable selfadjoint relations and (matrix-valued) locally definitizable functions as well as the concept of boundary triplets and associated Weyl functions. In Section 3 we show how strict matrix-valued locally definitizable functions can be realized as Weyl functions corresponding to symmetric operators of finite defect and suitable boundary triplets. Theorem 3.5 deals with the non-strict case. A simple example of a symmetric operator of defect one in $(\mathbb{C}^2, [\cdot,\cdot])$ and a boundary triplet where the corresponding Weyl function is not strict will be given at the end of Section 3. The λ-dependent boundary value problem (1.1) is studied in Section 4. First we introduce the notion of d-compatibility of sign types of locally definitizable functions and locally definitizable selfadjoint relations in Definition 4.1. The main result in Section 4 is Theorem 4.3. Here we construct a minimal linearization $\widetilde{A}$ of the boundary value problem (1.1), (1.2) such that the compressed resolvent of $\widetilde{A}$ onto the basic space yields the unique solution of (1.1), (1.2). In Theorem 4.5 we show that the eigenvectors corresponding to an eigenvalue μ of $\widetilde{A}$ yield solutions of the "homogeneous" boundary value problem

$$f' - \mu f = 0 \quad \text{and} \quad \tau(\mu)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0, \quad \hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in A^+.$$

We finish Section 4 with some special cases of Theorem 4.3. In Section 5 we formulate the main result from Section 4 for the case that A has a selfadjoint extension which locally has the same spectral properties as a selfadjoint operator or relation in a Pontryagin space and the function τ is a strict local generalized Nevanlinna function. Furthermore, in Theorem 5.2, we consider the "global" case, that is, we assume that A is a densely defined operator in a Pontryagin space and τ is a (not necessarily strict) matrix-valued generalized Nevanlinna function. Finally we show in Section 6 that the general results from Section 4 can be applied to singular indefinite Sturm-Liouville operators with λ-dependent interface conditions.

2 Locally definitizable relations and locally definitizable functions

2.1 Notations and definitions

Let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space. We study linear relations in $\mathcal{K}$, that is, linear subspaces of $\mathcal{K} \times \mathcal{K}$. For the elements in a linear relation we use a vector notation. The set of all closed linear relations in $\mathcal{K}$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{K})$. Linear operators in $\mathcal{K}$ are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations, the inverse etc., we refer to [24]. The direct sum of subspaces will be denoted by $\widehat{+}$. The linear space of bounded linear operators defined on a Krein space $\mathcal{K}$ with values in a separable Krein space $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{K},\mathcal{H})$. If $\mathcal{K} = \mathcal{H}$ we simply write $\mathcal{L}(\mathcal{K})$. The elements of $\mathcal{K} \times \mathcal{H}$ will be denoted in the form $\{k, h\}$, $k \in \mathcal{K}$, $h \in \mathcal{H}$. $\mathcal{K} \times \mathcal{H}$ equipped with the inner product $[\{k,h\},\{k',h'\}] := [k,k'] + [h,h']$, $k,k' \in \mathcal{K}$, $h,h' \in \mathcal{H}$, is also a Krein space. If S is a relation in $\mathcal{K}$ and T is a relation in $\mathcal{H}$ we shall write $S \times T$ for the direct product of S and T which is a relation in $\mathcal{K} \times \mathcal{H}$,

$$S \times T = \left\{ \begin{pmatrix} \{s,t\} \\ \{s',t'\} \end{pmatrix} \;\middle|\; \begin{pmatrix} s \\ s' \end{pmatrix} \in S,\ \begin{pmatrix} t \\ t' \end{pmatrix} \in T \right\}. \tag{2.1}$$

For the pair $\binom{\{s,t\}}{\{s',t'\}}$ on the right hand side of (2.1) we shall also write $\{\hat{s},\hat{t}\}$, where $\hat{s} = \binom{s}{s'}$, $\hat{t} = \binom{t}{t'}$.

For a linear relation S in $\mathcal{K}$ the adjoint relation $S^+ \in \widetilde{\mathcal{C}}(\mathcal{K})$ is defined as

$$S^+ := \left\{ \begin{pmatrix} g \\ g' \end{pmatrix} \;\middle|\; [f',g] = [f,g'] \text{ for all } \begin{pmatrix} f \\ f' \end{pmatrix} \in S \right\}. \tag{2.2}$$

Note that (2.2) extends the definition of the adjoint operator. A linear relation S is called *symmetric* (*selfadjoint*) if $S \subset S^+$ (resp. $S = S^+$). The resolvent set $\rho(S)$ of a closed linear relation $S \in \widetilde{\mathcal{C}}(\mathcal{K})$ is the set of all $\lambda \in \mathbb{C}$ such that $(S-\lambda)^{-1} \in \mathcal{L}(\mathcal{K})$, the spectrum $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. The extended spectrum $\widetilde{\sigma}(S)$ of S is defined by $\widetilde{\sigma}(S) = \sigma(S)$ if $S \in \mathcal{L}(\mathcal{K})$ and $\widetilde{\sigma}(S) = \sigma(S) \cup \{\infty\}$ otherwise. A point $\lambda \in \mathbb{C}$ is called a *point of regular type* of $S \in \widetilde{\mathcal{C}}(\mathcal{K})$, $\lambda \in r(S)$, if $(S-\lambda)^{-1}$ is a bounded operator. We say that $\lambda \in \mathbb{C}$ belongs to the *approximate point spectrum of* S, denoted by $\sigma_{ap}(S)$, if there exists a sequence $\binom{x_n}{y_n} \in S$, $n = 1, 2, \ldots$, such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|y_n - \lambda x_n\| = 0$. The *extended approximate point spectrum* $\widetilde{\sigma}_{ap}(S)$ *of* S is defined by

$$\widetilde{\sigma}_{ap}(S) := \begin{cases} \sigma_{ap}(S) \cup \{\infty\} & \text{if} \quad 0 \in \sigma_{ap}(S^{-1}) \\ \sigma_{ap}(S) & \text{if} \quad 0 \notin \sigma_{ap}(S^{-1}) \end{cases}.$$

We remark, that the boundary points of $\widetilde{\sigma}(S)$ in $\overline{\mathbb{C}}$ belong to $\widetilde{\sigma}_{ap}(S)$.

2.2 Locally definitizable selfadjoint relations

In the following we briefly recall the definition and some basic properties of locally definitizable selfadjoint relations and a perturbation result on the stability of such operators and relations under finite rank perturbations, see [4, 33, 36].

For this we first remind the reader on the notion of spectral points of positive and negative type with respect to a selfadjoint relation A_0 in the separable Krein space $(\mathcal{K}, [\cdot,\cdot])$, cf. [36, 44]. A point $\lambda \in \sigma_{ap}(A_0)$ is said to be of *positive type* (*negative type*) with respect to A_0, if for every sequence $\binom{x_n}{y_n} \in A_0$, $n = 1, 2\ldots$, with $\|x_n\| = 1$, $\lim_{n\to\infty} \|y_n - \lambda x_n\| = 0$ we have

$$\liminf_{n\to\infty}\, [x_n, x_n] > 0 \quad \big(\text{resp.} \ \limsup_{n\to\infty}\, [x_n, x_n] < 0\big).$$

If $\infty \in \widetilde{\sigma}_{ap}(A_0)$, ∞ is said to be of *positive type* (*negative type*) with respect to A_0 if the point 0 is of positive type (resp. negative type) with respect to A_0^{-1}. The set of all spectral points of positive type (negative type) with respect to A_0 will be denoted by $\sigma_{++}(A_0)$ (resp. $\sigma_{--}(A_0)$). An open subset Δ of $\overline{\mathbb{R}}$ is said to be of *positive type* (*negative type*) with respect to A_0 if each point $\lambda \in \Delta \cap \widetilde{\sigma}(A_0)$ is of positive type (resp. negative type) with respect to A_0. An open subset Δ of $\overline{\mathbb{R}}$ is called of *definite type* with respect to A_0 if Δ is either of positive or of negative type with respect to A_0.

Let in the following Ω be a domain in $\overline{\mathbb{C}}$ symmetric with respect to the real line such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper open half plane $\mathbb{C}^+$ and lower open half plane $\mathbb{C}^-$ are simply connected. The next definition can be found in [33], see also [36].

Definition 2.1 A selfadjoint relation A_0 in the Krein space $\mathcal{K}$ is said to be *definitizable over* Ω if $\sigma(A_0) \cap (\Omega \backslash \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A_0, no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the nonreal spectrum of A_0 in Ω and the following holds.

(i) Every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighborhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu \backslash \{\mu\}$ are of definite type with respect to A_0.

(ii) For every finite union Δ, $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, of open connected subsets there exists $m \geq 1$, $M > 0$ and an open neighborhood $\mathcal{U}$ of $\overline{\Delta}$ in Ω such that

$$\|(A_0 - \lambda)^{-1}\| \leq M(1 + |\lambda|)^{2m-2}\, |\text{Im}\ \lambda|^{-m}$$

holds for all $\lambda \in \mathcal{U} \backslash \overline{\mathbb{R}}$.

By [36, Theorem 4.7] a selfadjoint relation A_0 in $\mathcal{K}$ is definitizable over $\overline{\mathbb{C}}$ if and only if A_0 is *definitizable*, that is, the resolvent set of A_0 is non-empty and there exists a rational function $r \neq 0$ with poles only in $\rho(A_0)$ such that $[r(A_0)x, x] \geq 0$ holds for all $x \in \mathcal{K}$. We refer to [43] for a detailed investigation of definitizable operators, see also [25, §4 and §5].

It is also important to note, that a selfadjoint relation A_0 in $\mathcal{K}$ is definitizable over Ω if and only if for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, there exists a selfadjoint projection E in $\mathcal{K}$ such that A_0 can be decomposed in

$$A_0 = \big(A_0 \cap (E\mathcal{K})^2\big) \mathbin{\widehat{+}} \big(A_0 \cap ((1-E)\mathcal{K})^2\big), \tag{2.3}$$

where $A_0 \cap (E\mathcal{K})^2$ is a definitizable relation in the Krein space $E\mathcal{K}$ and $\widetilde{\sigma}(A_0 \cap ((1-E)\mathcal{K})^2) \cap \Omega' = \emptyset$ holds, cf. [36, Theorem 4.8]. We shall say that a selfadjoint relation A_0 which is definitizable over Ω is of *type* π_+ (*type* π_-) *over* Ω, if for every domain Ω', $\overline{\Omega'} \subset \Omega$, in the decomposition (2.3) the space $(E\mathcal{K}, [\cdot,\cdot])$ is a Pontryagin space with finite rank of negativity (resp. finite rank of positivity) and the resolvent set of the selfadjoint relation $A_0 \cap (E\mathcal{K})^2$ is nonempty.

The following theorem from [4] on finite rank perturbations in resolvent sense of locally definitizable relations will be used frequently in Section 4. It is well known for selfadjoint relations which are definitizable (over $\overline{\mathbb{C}}$), cf. [38].

Theorem 2.2 *Let Ω be a domain as above, let A_0 and B_0 be selfadjoint relations in the Krein space $\mathcal{K}$ such that $\rho(A_0) \cap \rho(B_0) \cap \Omega \neq \emptyset$, and assume that*

$$\dim\big(\operatorname{ran}\big((B_0-\lambda)^{-1} - (A_0-\lambda)^{-1}\big)\big) < \infty$$

holds for some $\lambda \in \rho(A_0) \cap \rho(B_0)$. Then A_0 is definitizable over Ω if and only if B_0 is definitizable over Ω. Moreover, if A_0 is definitizable over Ω and $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is an open interval with endpoint $\mu \in \Omega \cap \overline{\mathbb{R}}$ and Δ is of positive type (negative type) with respect to A_0, then there exists an open interval Δ', $\Delta' \subset \Delta$, with endpoint μ such that Δ' is of positive type (resp. negative type) with respect to B_0.

2.3 Matrix-valued locally definitizable functions

In this section we recall the definition of matrix-valued locally definitizable functions from [37]. For this, let Ω be a domain as in Section 2.2 and let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued piecewise meromorphic function in $\Omega\backslash\overline{\mathbb{R}}$ which is symmetric with respect to the real axis, that is $\tau(\overline{\lambda}) = \tau(\lambda)^*$ for all points λ of holomorphy of τ. If, in addition, no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of nonreal poles of τ we write $\tau \in M^{n\times n}(\Omega)$. The set of the points of holomorphy of τ in $\Omega\backslash\overline{\mathbb{R}}$ and all points $\mu \in \Omega \cap \mathbb{R}$ such that τ can be analytically continued to μ and the continuations from $\Omega \cap \mathbb{C}^+$ and $\Omega \cap \mathbb{C}^-$ coincide, is denoted by $\mathfrak{h}(\tau)$.

The following definition of sets of positive and negative type with respect to matrix functions and Definition 2.4 below of locally definitizable matrix functions can be found in [37].

Definition 2.3 Let $\tau \in M^{n\times n}(\Omega)$. An open subset $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is said to be of *positive type* with respect to τ if for every $x \in \mathbb{C}^n$ and every sequence (μ_k) of points in $\Omega \cap \mathbb{C}^+ \cap \mathfrak{h}(\tau)$ which converges in $\overline{\mathbb{C}}$ to a point of Δ we have

$$\liminf_{k\to\infty} \operatorname{Im}\big(\tau(\mu_k)x, x\big) \geq 0.$$

An open subset $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is said to be of *negative type* with respect to τ if Δ is of positive type with respect to $-\tau$. Δ is said to be of *definite type* with respect to τ if Δ is either of positive or of negative type with respect to τ.

Definition 2.4 A function $\tau \in M^{n\times n}(\Omega)$ is called *definitizable in* Ω if the following holds.

(i) Every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighborhood I_μ in $\overline{\mathbb{R}}$ such that both components of $I_\mu\backslash\{\mu\}$ are of definite type with respect to τ.

(ii) For every finite union Δ of open connected subsets in $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, there exist $m \geq 1$, $M > 0$ and an open neighborhood $\mathcal{U}$ of $\overline{\Delta}$ in $\overline{\mathbb{C}}$ such that

$$\|\tau(\lambda)\| \leq M(1+|\lambda|)^{2m}\,|\operatorname{Im}\lambda|^{-m}$$

holds for all $\lambda \in \mathcal{U}\backslash\overline{\mathbb{R}}$.

A function $\tau \in M^{n\times n}(\overline{\mathbb{C}})$ which is definitizable in $\overline{\mathbb{C}}$ is called *definitizable*, see [36]. We note that $\tau \in M^{n\times n}(\overline{\mathbb{C}})$ is definitizable if and only if there exists a rational function g symmetric with respect to the real axis such that the poles of g belong to $\mathfrak{h}(\tau) \cup \{\infty\}$ and $g\tau$ is the sum of a Nevanlinna function and a meromorphic function in $\overline{\mathbb{C}}$, cf. [36]. For a comprehensive study of definitizable functions we refer to the papers [34, 35] of P. Jonas. We mention only that the generalized Nevanlinna class is a subclass of the definitizable functions. Recall that a function $\tau \in M^{n\times n}(\overline{\mathbb{C}})$ is called a *generalized Nevanlinna function* if the kernel K_τ,

$$K_\tau(\lambda,\mu) = \frac{\tau(\lambda) - \tau(\overline{\mu})}{\lambda - \overline{\mu}},$$

has finitely many negative squares, see, e.g., [41, 42].

In [37] it is shown that a function $\tau \in M^{n\times n}(\Omega)$ is definitizable in Ω if and only if for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, τ can be written as the sum

$$\tau = \tau_0 + \tau_{(0)} \tag{2.4}$$

of an $\mathcal{L}(\mathbb{C}^n)$-valued definitizable function τ_0 and an $\mathcal{L}(\mathbb{C}^n)$-valued function $\tau_{(0)}$ which is locally holomorphic on $\overline{\Omega'}$. We shall say that a locally definitizable function $\tau \in M^{n\times n}(\Omega)$ is a *local generalized Nevanlinna function in* Ω if for every domain Ω', $\overline{\Omega'} \subset \Omega$, the function τ_0 in the decomposition (2.4) is an $\mathcal{L}(\mathbb{C}^n)$-valued generalized Nevanlinna function.

The following theorem from [37, §3.1] establishes a connection between selfadjoint relations which are locally definitizable (selfadjoint relations which are locally of type π_+) and $\mathcal{L}(\mathbb{C}^n)$-valued locally definitizable functions (resp. $\mathcal{L}(\mathbb{C}^n)$-valued local generalized Nevanlinna functions).

Theorem 2.5 *Let Ω be a domain as above and let A_0 be a selfadjoint relation in the Krein space $\mathcal{K}$ which is definitizable over Ω (of type π_+ over Ω). Let $\gamma \in \mathcal{L}(\mathbb{C}^n,\mathcal{K})$ and $S = S^* \in \mathcal{L}(\mathbb{C}^n)$, fix some point $\lambda_0 \in \rho(A_0)$ and define*

$$\tau(\lambda) := S + \gamma^+\big((\lambda - \operatorname{Re}\lambda_0) + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(A_0-\lambda)^{-1}\big)\gamma$$

for $\lambda \in \rho(A_0)$. Then the function τ is definitizable in Ω (resp. a local generalized Nevanlinna function in Ω). Moreover, if an open subset $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is of positive type (negative type) with respect to A_0, then Δ is of positive type (resp. negative type) with respect to τ.

The next theorem states that a locally definitizable function (a local generalized Nevanlinna function) can be minimally represented with a locally definitizable selfadjoint relation (resp. a selfadjoint relation which is locally of type π_+). A proof can be found in [37].

Theorem 2.6 *Let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued function which is definitizable in Ω (an $\mathcal{L}(\mathbb{C}^n)$-valued local generalized Nevanlinna function in Ω) and let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$. Then there exists a Krein space $\mathcal{H}$, a selfadjoint relation T_0 in $\mathcal{H}$ which is definitizable over Ω' (resp. of type π_+ over Ω') and a mapping $\gamma' \in \mathcal{L}(\mathbb{C}^n,\mathcal{H})$ such that* (a)-(d) *hold.*

(a) $\rho(T_0) \cap \Omega' = \mathfrak{h}(\tau) \cap \Omega'$.

(b) *For a fixed $\lambda_0 \in \rho(T_0) \cap \Omega'$ and all $\lambda \in \rho(T_0) \cap \Omega'$*

$$\tau(\lambda) = \operatorname{Re}\tau(\lambda_0) + \gamma'^+\big((\lambda - \operatorname{Re}\lambda_0) + (\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)(T_0-\lambda)^{-1}\big)\gamma' \tag{2.5}$$

holds.

(c) *The minimality condition*

$$\mathcal{H} = \operatorname{clsp}\big\{\big(1 + (\lambda-\lambda_0)(T_0-\lambda)^{-1}\big)\gamma' x \mid \lambda \in \rho(T_0)\cap\Omega',\, x \in \mathbb{C}^n\big\} \tag{2.6}$$

is fulfilled.

(d) *An open set Δ of $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \Omega' \cap \overline{\mathbb{R}}$, is of positive type (negative type) with respect to τ if and only if Δ is of positive type (resp. negative type) with respect to T_0.*

If τ and T_0 are as in Theorem 2.6 we shall say that T_0 is a *minimal representing relation* for τ.

2.4 Boundary triplets and Weyl functions

Let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space with corresponding fundamental symmetry J and let $A \in \widetilde{\mathcal{C}}(\mathcal{K})$ be a closed symmetric relation. We say that A is of *defect* $m \in \mathbb{N}_0 \cup \{\infty\}$, if the deficiency indices

$$n_\pm(JA) = \dim\ker\big((JA)^* \mp i\big)$$

of the symmetric relation JA in the Hilbert space $(\mathcal{K}, [J\cdot,\cdot])$ are equal to m. We note that this is equivalent to the fact that there exists a selfadjoint extension of A in $\mathcal{K}$ and that each selfadjoint extension $\hat{A}$ of A in $\mathcal{K}$ satisfies $\dim\big(\hat{A}/A\big) = m$.

We shall use the concept of boundary triplets for the description of the symmetric and selfadjoint extensions of closed symmetric relations in Krein spaces. The following definition is taken from [14].

Definition 2.7 Let A be a closed symmetric relation in the Krein space $\mathcal{K}$. We say that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a *boundary triplet for* A^+ if $\mathcal{G}$ is a Hilbert space and there exist mappings $\Gamma_0, \Gamma_1 : A^+ \to \mathcal{G}$ such that $\Gamma := \binom{\Gamma_0}{\Gamma_1} : A^+ \to \mathcal{G}\times\mathcal{G}$ is surjective, and the relation

$$[f', g] - [f, g'] = \big(\Gamma_1\hat{f}, \Gamma_0\hat{g}\big) - \big(\Gamma_0\hat{f}, \Gamma_1\hat{g}\big) \tag{2.7}$$

holds for all $\hat{f} = \binom{f}{f'}, \hat{g} = \binom{g}{g'} \in A^+$.

In the following we briefly recall some basic facts on boundary triplets which can be found in e.g. [13] and [14]. For the Hilbert space case we refer to [18, 19, 28]. Let $A \in \widetilde{\mathcal{C}}(\mathcal{K})$ be a closed symmetric relation in $\mathcal{K}$ and let $\lambda \in r(A)$ be a point of regular type of A. Then the defect subspace of A at λ is

$$\mathcal{N}_{\lambda,A^+} := \ker(A^+ - \lambda) = \operatorname{ran}\,(A - \overline{\lambda})^{[\perp]}$$

and we define

$$\hat{\mathcal{N}}_{\lambda,A^+} := \big\{\big(\begin{smallmatrix} f_\lambda \\ \lambda f_\lambda\end{smallmatrix}\big) \,|\, f_\lambda \in \mathcal{N}_{\lambda,A^+}\big\}. \tag{2.8}$$

When no confusion can arise we will simply write $\mathcal{N}_\lambda$ and $\hat{\mathcal{N}}_\lambda$ instead of $\mathcal{N}_{\lambda,A^+}$ and $\hat{\mathcal{N}}_{\lambda,A^+}$. If there exists a selfadjoint extension $\hat{A}$ of A in $\mathcal{K}$ such that $\rho(\hat{A}) \neq \emptyset$, then we have

$$A^+ = \hat{A} \,\hat{+}\, \hat{\mathcal{N}}_\lambda \tag{2.9}$$

for all $\lambda \in \rho(\hat{A})$ and there exists a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for A^+ such that $\ker\Gamma_0 = \hat{A}$, cf. [14].

Let in the following A, $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and Γ be as in Definition 2.7. It follows that the mapping Γ and the mappings Γ_0 and Γ_1 are continuous, $A = \ker\Gamma$ and the extensions $A_0 := \ker\Gamma_0$ and $A_1 := \ker\Gamma_1$ of A are selfadjoint. The mapping Γ induces, via

$$A_\Theta := \Gamma^{(-1)}\Theta = \big\{\hat{f} \in A^+ \,|\, \Gamma\hat{f} \in \Theta\big\}, \quad \Theta \in \widetilde{\mathcal{C}}(\mathcal{G}), \tag{2.10}$$

a bijective correspondence $\Theta \mapsto A_\Theta$ between the set of all closed linear relations $\widetilde{\mathcal{C}}(\mathcal{G})$ in $\mathcal{G}$ and the set of closed extensions $A_\Theta \subset A^+$ of A. In particular (2.10) gives a one-to-one correspondence between the closed symmetric (selfadjoint) extensions of A and the closed symmetric (resp. selfadjoint) relations in $\mathcal{G}$. In the special case that Θ is a closed operator in $\mathcal{G}$ the corresponding extension A_Θ of A is determined by

$$A_\Theta = \ker\big(\Gamma_1 - \Theta\Gamma_0\big). \tag{2.11}$$

Assume now that $\rho(A_0)$ is nonempty and denote by π_1 the orthogonal projection onto the first component of $\mathcal{K}\times\mathcal{K}$. Then $A^+ = A_0 \,\hat{+}\, \hat{\mathcal{N}}_\lambda$ for every $\lambda \in \rho(A_0)$ and the operators

$$\gamma(\lambda) = \pi_1\big(\Gamma_0 \,|\, \hat{\mathcal{N}}_\lambda\big)^{-1} \in \mathcal{L}(\mathcal{G}, \mathcal{K}) \qquad \text{and} \qquad M(\lambda) = \Gamma_1\big(\Gamma_0 \,|\, \hat{\mathcal{N}}_\lambda\big)^{-1} \in \mathcal{L}(\mathcal{G})$$

are well defined. The functions $\lambda \mapsto \gamma(\lambda)$ and $\lambda \mapsto M(\lambda)$ are called the *γ-field* and the *Weyl function* corresponding to the boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. γ and M are holomorphic on $\rho(A_0)$ and the relations

$$\gamma(\zeta) = (1 + (\zeta - \lambda)(A_0 - \zeta)^{-1})\gamma(\lambda) \quad \text{and} \quad M(\lambda) - M(\zeta)^* = (\lambda - \overline{\zeta})\gamma(\zeta)^+\gamma(\lambda) \tag{2.12}$$

hold for all $\lambda, \zeta \in \rho(A_0)$. With the help of the Weyl function the spectral properties of the closed extensions $A_\Theta \subset A^+$ of A can be described. For a proof of the next theorem see, e.g., [14].

Theorem 2.8 *Let A be a closed symmetric relation in the Krein space $\mathcal{K}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^+. Assume that $A_0 = \ker \Gamma_0$ has a nonempty resolvent set and let γ and M be the γ-field and Weyl function corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. If $\Theta \in \widetilde{\mathcal{C}}(\mathcal{G})$ and A_Θ is the corresponding extension of A via* (2.10)*, then a point $\lambda \in \rho(A_0)$ belongs to $\rho(A_\Theta)$ ($\sigma_i(A_\Theta)$, $i = p, c, r$) if and only if the point* 0 *belongs to $\rho(\Theta - M(\lambda))$ (resp. $\sigma_i(\Theta - M(\lambda))$, $i = p, c, r$) and the formula*

$$(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\big(\Theta - M(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ \tag{2.13}$$

holds for all $\lambda \in \rho(A_0) \cap \rho(A_\Theta)$

Let again $A \in \widetilde{\mathcal{C}}(\mathcal{K})$ be a symmetric relation and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^+, assume that $A_0 = \ker \Gamma_0$ has a nonempty resolvent set and let γ and M be the corresponding γ-field and Weyl function, respectively. For a fixed $\lambda_0 \in \rho(A_0)$ it follows from (2.12) that

$$M(\lambda) = \text{Re}\, M(\lambda_0) + \gamma(\lambda_0)^+\big((\lambda - \text{Re}\,\lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(A_0 - \lambda)^{-1}\big)\gamma(\lambda_0) \tag{2.14}$$

holds for all $\lambda \in \rho(A_0)$. If, in addition, the symmetric relation A has the property

$$\mathcal{K} = \text{clsp}\,\big\{\mathcal{N}_\lambda \,|\, \lambda \in \rho(A_0)\big\},$$

then A_0 fulfils the minimality condition

$$\mathcal{K} = \text{clsp}\,\big\{(1 + (\lambda - \lambda_0)(A_0 - \lambda)^{-1})\gamma(\lambda_0)x \,|\, \lambda \in \rho(A_0),\, x \in \mathcal{G}\big\}.$$

Note that in this case A is automatically an operator without eigenvalues.

3 Realization of matrix-valued locally definitizable functions as Weyl functions

Let A be a closed symmetric relation of finite defect n in the Krein space $\mathcal{K}$ and assume that A has a selfadjoint extension A_0 in $\mathcal{K}$ which is definitizable (of type π_+) over some domain Ω as in Section 2.2. Then there exists a boundary triplet $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ for A^+ such that $A_0 = \ker \Gamma_0$ and by relation (2.14) and Theorem 2.5 the corresponding Weyl function is an $\mathcal{L}(\mathbb{C}^n)$-valued function which is definitizable in Ω (resp. an $\mathcal{L}(\mathbb{C}^n)$-valued local generalized Nevanlinna function over Ω).

In this section we shall show that each $\mathcal{L}(\mathbb{C}^n)$-valued locally definitizable function can be written as the sum of a symmetric matrix constant and a "smaller" $\mathcal{L}(\mathbb{C}^s)$-valued locally definitizable function, $s \le n$, which has an additional property such that it can be represented as the Weyl function corresponding to a suitable boundary triplet of some symmetric operator of defect s. The following considerations hold in particular for (local) generalized Nevanlinna functions. For brevity we formulate most results in this section only for locally definitizable functions.

Definition 3.1 Let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued function which is definitizable in Ω. We shall say that τ is *strict* if

$$\bigcap_{\lambda \in \mathfrak{h}(\tau) \cap \Omega} \ker \frac{\tau(\lambda) - \tau(\mu_0)^*}{\lambda - \overline{\mu}_0} = \{0\} \tag{3.1}$$

holds for some $\mu_0 \in \mathfrak{h}(\tau) \cap \Omega$.

If $\lambda_1 \in \mathfrak{h}(\tau) \cap \Omega$ and $\mathcal{U}_{\lambda_1}$ is an open neighborhood of λ_1, $\mathcal{U}_{\lambda_1} \subset \mathfrak{h}(\tau) \cap \Omega$, then it follows from the holomorphy of τ in $\mathfrak{h}(\tau) \cap \Omega$ that in the intersection in (3.1) the set $\mathfrak{h}(\tau) \cap \Omega$ can be replaced by the smaller set $\mathcal{U}_{\lambda_1} \cup \mathcal{U}^*_{\lambda_1}$, where $\mathcal{U}^*_{\lambda_1} = \{\lambda \in \mathbb{C} \,|\, \overline{\lambda} \in \mathcal{U}_{\lambda_1}\}$. Moreover, if (3.1) holds for a point $\mu_0 \in \mathfrak{h}(\tau) \cap \Omega$, then (3.1) holds also for μ_0 replaced by an arbitrary point $\mu_1 \in \mathfrak{h}(\tau) \cap \Omega$.

For a Nevanlinna function τ the property (3.1) is equivalent to the invertibility of $\operatorname{Im} \tau(\lambda)$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$. It is well known that every Nevanlinna function τ (which can be operator-valued) with the property $0 \in \rho(\operatorname{Im} \tau(\lambda_1))$ for some (and hence for all) $\lambda_1 \in \mathbb{C}\backslash\mathbb{R}$ can be realized as the Weyl function of some boundary triplet, see e.g. [18, Theorem 1]. In order to realize an operator-valued Nevanlinna function τ with the property $0 \notin \sigma_p(\operatorname{Im} \tau(\lambda_1))$ as a "Weyl function", so-called generalized boundary triplets were introduced in [19]. Recently V.A. Derkach, S. Hassi, M.M. Malamud and H.S.V. de Snoo developed in [16] the more general concept of a *boundary relation* for a symmetric relation in a Hilbert space. In this framework it is possible to realize every Nevanlinna family as a so-called Weyl family corresponding to a boundary relation.

For generalized Nevanlinna functions the notion strict defined above can be found in [17]. By [17, Proposition 3.1] a matrix-valued generalized Nevanlinna function with this additional property is a Weyl function corresponding to a symmetric operator in a Pontryagin space and a suitable boundary triplet for its adjoint. Here we obtain this result as a special case of Theorem 3.3 in Corollary 3.4.

In the next lemma we show that in a minimal representation (see Theorem 2.6) of a strict locally definitizable function the mapping γ' is always injective and that a strict function does not "contain" a nontrivial constant part.

Lemma 3.2 *Let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued function which is definitizable in Ω. The following assertions* (i)-(iv) *are equivalent.*

(i) *The function τ is strict.*

(ii) *There exists a minimal representation of τ of the form* (2.5)–(2.6) *such that the mapping γ' is injective.*

(iii) *In every representation of the form*

$$\tau(\lambda) = \operatorname{Re} \tau(\lambda_0) + \widetilde{\gamma}^+\big((\lambda - \operatorname{Re} \lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(S_0 - \lambda)^{-1}\big)\widetilde{\gamma},$$

where S_0 is a selfadjoint relation in some Krein space $\widetilde{\mathcal{H}}$, $\rho(S_0) \cap \Omega \neq \emptyset$ and $\lambda_0, \lambda \in \rho(S_0) \cap \Omega$, the mapping $\widetilde{\gamma} \in \mathcal{L}(\mathbb{C}^n, \widetilde{\mathcal{H}})$ is injective.

(iv) *If S' is an operator in $\mathbb{C}^n$ such that $S' \subset \tau(\lambda)$ holds for all $\lambda \in \mathfrak{h}(\tau) \cap \Omega$, then* $\operatorname{dom} S' = \{0\}$.

P r o o f. (i) $\Rightarrow$ (iii) : Assume that condition (3.1) is fulfilled for some $\mu_0 \in \mathfrak{h}(\tau) \cap \Omega$. It is no restriction to assume that μ_0 belongs to $\rho(S_0)$. A straightforward calculation shows that

$$\frac{\tau(\lambda) - \tau(\mu_0)^*}{\lambda - \overline{\mu}_0} = \widetilde{\gamma}^+\big(1 + (\overline{\mu}_0 - \overline{\lambda}_0)(S_0 - \overline{\mu}_0)^{-1}\big)\big(1 + (\lambda - \lambda_0)(S_0 - \lambda)^{-1}\big)\widetilde{\gamma}$$

holds for all $\lambda \in \mathfrak{h}(\tau) \cap \rho(S_0) \cap \Omega$. Hence $\widetilde{\gamma}$ is injective.

The implication (iii) $\Rightarrow$ (ii) evidently holds.

(ii) $\Rightarrow$ (i) : Assume that τ is not strict, i.e. for all $\mu_0 \in \mathfrak{h}(\tau) \cap \Omega'$ there exists some $x_{\mu_0} \neq 0$ such that

$$x_{\mu_0} \in \ker\left(\frac{\tau(\lambda) - \tau(\mu_0)^*}{\lambda - \overline{\mu}_0}\right)$$

holds for all $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$. Let Ω' be a domain as Ω, $\overline{\Omega'} \subset \Omega$, and let τ be represented in the form (2.5)–(2.6) with $\gamma' \in \mathcal{L}(\mathbb{C}^n, \mathcal{H})$ and a selfadjoint relation T_0 in $\mathcal{H}$ which is definitizable over Ω'. Setting $\mu_0 = \overline{\lambda}_0$ we obtain from this representation

$$\begin{aligned} 0 &= \left(\frac{\tau(\lambda) - \tau(\lambda_0)}{\lambda - \lambda_0} x_{\overline{\lambda}_0}, y\right) = \big(\gamma'^+\big(1 + (\lambda - \overline{\lambda}_0)(T_0 - \lambda)^{-1}\big)\gamma' x_{\overline{\lambda}_0}, y\big) \\ &= \big[\gamma' x_{\overline{\lambda}_0}, (1 + (\overline{\lambda} - \lambda_0)(T_0 - \overline{\lambda})^{-1})\gamma' y\big] \end{aligned}$$

for all $y \in \mathbb{C}^n$ and all $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$. The minimality condition (2.6) implies $\gamma' x_{\overline{\lambda}_0} = 0$ and therefore γ' is not injective.

(i) $\Leftrightarrow$ (iv) : If τ is not strict there exist $u, v \in \mathbb{C}^n$, $u \neq 0$, such that $\tau(\lambda)u = v$ holds for all $\lambda \in \mathfrak{h}(\tau) \cap \Omega$. Setting $\mathrm{dom}\, S' = \mathrm{sp}\,\{u\}$ and $S'(\alpha u) = \alpha\tau(\lambda)u$, $\alpha \in \mathbb{C}$, we find that (iv) does not hold. Conversely, if there exists an operator S' such that $\mathrm{dom}\, S' \neq \{0\}$ and $S'x = \tau(\lambda)x$ for all $x \in \mathrm{dom}\, S'$ and all $\lambda \in \mathfrak{h}(\tau) \cap \Omega$, then

$$x \in \bigcap_{\lambda \in \mathfrak{h}(\tau) \cap \Omega} \ker \frac{\tau(\lambda) - \tau(\mu_0)^*}{\lambda - \overline{\mu}_0}$$

holds for every $\mu_0 \in \mathfrak{h}(\tau) \cap \Omega$. Hence τ is not strict. □

The next theorem states that a strict matrix-valued locally definitizable function τ can be realized as the Weyl function corresponding to a symmetric operator and suitable boundary triplet. In the special case that τ is a scalar function which is not equal to a constant Theorem 3.3 was proved in [3], see also [7], and a proof for matrix-valued local generalized Nevanlinna functions was given in [8]. Although the proof here is completely analogous we give a short sketch for the convenience of the reader.

Theorem 3.3 *Let τ be a strict $\mathcal{L}(\mathbb{C}^n)$-valued function which is definitizable in Ω, let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let τ be represented as in* (2.5)–(2.6) *by a selfadjoint relation T_0 which is definitizable over Ω'. Then there exists a closed symmetric operator $T \subset T_0$ of defect n and a boundary triplet $\{\mathbb{C}^n, \Gamma_0', \Gamma_1'\}$ for T^+, $T_0 = \ker \Gamma_0'$, such that τ coincides with the corresponding Weyl function on Ω'.*

For $\mathcal{L}(\mathbb{C}^n)$-valued generalized Nevanlinna functions Theorem 3.3 yields the following corollary (cf. [17, Proposition 3.1]).

Corollary 3.4 *Let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued strict generalized Nevanlinna function. Then there exists a closed symmetric operator T of defect n in a Pontryagin space with finite rank of negativity and a boundary triplet $\{\mathbb{C}^n, \Gamma_0', \Gamma_1'\}$ for T^+ such that τ is the corresponding Weyl function.*

Proof of Theorem 3.3. Let τ be represented by a selfadjoint relation T_0 in a Krein space $\mathcal{H}$ as in (2.5)–(2.6). For all $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$ and a fixed $\lambda_0 \in \mathfrak{h}(\tau) \cap \Omega'$ we define the mapping

$$\gamma'(\lambda) := \big(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1}\big)\gamma' \ \in \mathcal{L}(\mathbb{C}^n, \mathcal{H}). \tag{3.2}$$

Then we have $\gamma'(\lambda_0) = \gamma'$ and $\gamma'(\zeta) = (1 + (\zeta - \lambda)(T_0 - \zeta)^{-1})\gamma'(\lambda)$ for all $\lambda, \zeta \in \mathfrak{h}(\tau) \cap \Omega'$. For some $\mu \in \mathfrak{h}(\tau) \cap \Omega'$ we define the closed symmetric relation

$$T := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in T_0 \,\middle|\, [g - \overline{\mu} f, \gamma'(\mu)h] = 0 \ \text{ for all } \ h \in \mathbb{C}^n \right\} \tag{3.3}$$

in $\mathcal{H}$. Note that T has defect n and does not depend on the choice of $\mu \in \mathfrak{h}(\tau) \cap \Omega'$. We have $\mathcal{N}_{\lambda,T^+} = \mathrm{ran}\,\gamma'(\lambda)$ and the minimality condition (2.6) implies $\mathcal{H} = \mathrm{clsp}\,\{\mathcal{N}_{\lambda,T^+} \mid \lambda \in \rho(T_0) \cap \Omega'\}$. As τ is strict and T_0 is a minimal representing relation for τ by Lemma 3.2 $\gamma' \in \mathcal{L}(\mathbb{C}^n, \mathcal{H})$ is injective. Since the operator $1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1}$ is an isomorphism of $\mathcal{N}_{\lambda_0,T^+}$ onto $\mathcal{N}_{\lambda,T^+}$ the mapping $\gamma'(\lambda)$ in (3.2) is an isomorphism of $\mathbb{C}^n$ onto $\mathcal{N}_{\lambda,T^+}$. The inverse of this mapping is denoted by $\gamma'(\lambda)^{(-1)}$.

We write the elements $\hat{f} \in T^+$, for some fixed $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$, in the form

$$\hat{f} = \begin{pmatrix} f_0 \\ f_0' \end{pmatrix} + \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix},$$

where $\left(\begin{smallmatrix} f_0 \\ f_0' \end{smallmatrix}\right) \in T_0$ and $f_\lambda \in \mathcal{N}_{\lambda,T^+}$ (see (2.9)). Let $\Gamma_0', \Gamma_1' : T^+ \to \mathbb{C}^n$ be the linear mappings defined by

$$\Gamma_0' \hat{f} := \gamma'(\lambda)^{(-1)} f_\lambda \quad \text{and} \quad \Gamma_1' \hat{f} := \gamma'(\lambda)^+ (f_0' - \overline{\lambda} f_0) + \tau(\lambda)\gamma'(\lambda)^{(-1)} f_\lambda.$$

One verifies that the definition of Γ_0 and Γ_1 does not depend on the choice of $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$, and the same calculation as in [7, Proof of Theorem 3.3] shows that $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ is a boundary triplet for T^+. Then it is easy to see that the corresponding Weyl function coincides with τ on $\mathfrak{h}(\tau) \cap \Omega'$. □

In the next theorem we show that a non-strict locally definitizable matrix function can be written as the sum of a "smaller" strict function and a symmetric matrix constant. Note, that by Lemma 3.2 a non-strict function τ contains a nontrivial symmetric operator S'.

Theorem 3.5 *Let τ be a non-strict $\mathcal{L}(\mathbb{C}^n)$-valued function which is definitizable in Ω and not equal to a constant. Let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let τ be represented as in (2.5)–(2.6) by a selfadjoint relation T_0 which is definitizable over Ω'. Then the following holds.*

(i) *There exists a decomposition $\mathbb{C}^{n-s} \oplus \mathbb{C}^s$ of $\mathbb{C}^n$, $s \in \{1, \ldots, n-1\}$, a strict $\mathcal{L}(\mathbb{C}^s)$-valued function τ_s which is definitizable in Ω and a symmetric $S \in \mathcal{L}(\mathbb{C}^n)$ such that*

$$\tau(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & \tau_s(\lambda) \end{pmatrix} + S, \quad S = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}, \quad \lambda \in \mathfrak{h}(\tau) \cap \Omega,$$

holds with respect to the decomposition $\mathbb{C}^{n-s} \oplus \mathbb{C}^s$, where

$$\mathbb{C}^{n-s} = \bigcap_{\lambda \in \mathfrak{h}(\tau) \cap \Omega} \ker \frac{\tau(\lambda) - \tau(\mu_0)^*}{\lambda - \overline{\mu}_0}, \quad \mu_0 \in \mathfrak{h}(\tau) \cap \Omega.$$

(ii) *For the symmetric operator $S' := S \upharpoonright \mathrm{dom}\, S'$, $\mathrm{dom}\, S' := \mathbb{C}^{n-s}$, the relation*

$$\tau(\lambda) \upharpoonright \mathrm{dom}\, S' = S'$$

holds for all $\lambda \in \mathfrak{h}(\tau) \cap \Omega$. The operator S' is maximal in the sense that $S'' \subset \tau(\lambda)$ for all $\lambda \in \mathfrak{h}(\tau) \cap \Omega$ implies $S'' \subset S'$.

(iii) *Let π be the orthogonal projection in $\mathbb{C}^n$ onto $\mathbb{C}^s$ and let ι be the embedding of $\mathbb{C}^s$ in $\mathbb{C}^n$. Then*

$$\tau_s(\lambda) = \pi\tau(\lambda)\iota = \mathrm{Re}\, \tau_s(\lambda_0) + (\gamma'\iota)^+ \big((\lambda - \mathrm{Re}\, \lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(T_0 - \lambda)^{-1}\big)\gamma'\iota \tag{3.4}$$

holds for $\lambda_0, \lambda \in \mathfrak{h}(\tau) \cap \Omega'$ and the minimality condition

$$\mathcal{H} = \mathrm{clsp}\,\big\{\big(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1}\big)\gamma'\iota\, y \,|\, \lambda \in \rho(T_0) \cap \Omega',\, y \in \mathbb{C}^s\big\} \tag{3.5}$$

is fulfilled.

(iv) *There exists a closed symmetric operator $T \subset T_0$ of defect s and a boundary triplet $\{\mathbb{C}^s, \Gamma_0', \Gamma_1'\}$ for T^+, $T_0 = \ker \Gamma_0'$, such that τ_s coincides with the corresponding Weyl function on Ω'.*

P r o o f. (i) Since τ is not strict there exist elements $x_1, c_1 \in \mathbb{C}^n$, $\|x_1\| = 1$, such that $\tau(\lambda)x_1 = c_1$ holds for every $\lambda \in \mathfrak{h}(\tau) \cap \Omega$. We choose $y_2, \ldots, y_n \in \mathbb{C}^n$ such that $\{x_1, y_2, \ldots, y_n\}$ forms an orthonormal basis in $\mathbb{C}^n$. With respect to this basis we conclude from $\tau(\overline{\lambda}) = \tau(\lambda)^*$, $\lambda \in \mathfrak{h}(\tau) \cap \Omega$, that τ has the form

$$\tau(\lambda) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \tau_{n-1}(\lambda) & \\ 0 & & \end{pmatrix} + \begin{pmatrix} (c_1, x_1) & (y_2, c_1) & \cdots & (y_n, c_1) \\ (c_1, y_2) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ (c_1, y_n) & 0 & \cdots & 0 \end{pmatrix}, \ \lambda \in \mathfrak{h}(\tau) \cap \Omega,$$

where τ_{n-1} is an $\mathcal{L}(\mathbb{C}^{n-1})$-valued function which is definitizable in Ω.

If τ_{n-1} is not strict this consideration can be repeated with suitable elements $x_2, c_2 \in \mathrm{sp}\,\{y_2, \ldots, y_n\}$ such that $\tau_{n-1}(\lambda)x_2 = c_2$ for $\lambda \in \mathfrak{h}(\tau_1) \cap \Omega$ and $\|x_2\| = 1$. Let $\{x_2, z_3, \ldots, z_n\}$ be an orthormal basis in $\mathbb{C}^{n-1} = \mathrm{sp}\,\{y_2, \ldots, y_n\}$. Then, with respect to the orthonormal basis $\{x_1, x_2, z_3, \ldots, z_n\}$, the function τ has the form

$$\tau(\lambda) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \tau_{n-2}(\lambda) & \\ 0 & 0 & & \end{pmatrix} + \begin{pmatrix} (c_1, x_1) & (x_2, c_1) & (z_3, c_1) & \cdots & (z_n, c_1) \\ (c_1, x_2) & (c_2, x_2) & (z_3, c_2) & \cdots & (z_n, c_2) \\ (c_1, z_3) & (c_2, z_3) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ (c_1, z_n) & (c_2, z_n) & 0 & \cdots & 0 \end{pmatrix},$$

$\lambda \in \mathfrak{h}(\tau) \cap \Omega$, where τ_{n-2} is an $\mathcal{L}(\mathbb{C}^{n-2})$-valued locally definitizable function. By repeating this consideration we obtain an orthonormal basis

$$\{x_1, \dots, x_{n-s}, \tilde{z}_{n-s+1}, \dots, \tilde{z}_n\}$$

in $\mathbb{C}^n$, $1 \le s \le n-1$, subspaces

$$\mathbb{C}^{n-s} := \text{sp}\,\{x_1, \dots, x_{n-s}\} = \bigcap_{\lambda \in \mathfrak{h}(\tau)\cap\Omega} \ker \frac{\tau(\lambda) - \tau(\mu_0)^*}{\lambda - \overline{\mu}_0}$$

and $\mathbb{C}^s := \text{sp}\,\{\tilde{z}_{n-s+1}, \dots, \tilde{z}_n\}$, a strict $\mathcal{L}(\mathbb{C}^s)$-valued locally definitizable function τ_s and $S_{11} \in \mathcal{L}(\mathbb{C}^{n-s})$, $S_{11} = S_{11}^*$, $S_{12} \in \mathcal{L}(\mathbb{C}^{n-s}, \mathbb{C}^s)$ such that

$$\tau(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & \tau_s(\lambda) \end{pmatrix} + S, \quad S := \begin{pmatrix} S_{11} & S_{12}^* \\ S_{12} & 0 \end{pmatrix}, \quad \lambda \in \mathfrak{h}(\tau) \cap \Omega,$$

holds with respect to the decomposition $\mathbb{C}^{n-s} \oplus \mathbb{C}^s$.

(ii) The operator $S' := \binom{S_{11}}{S_{12}}$, dom $S' = \mathbb{C}^{n-s}$, is symmetric in $\mathbb{C}^n$ and the function τ restricted to dom S' coincides with S'. If S'' is a symmetric operator in $\mathbb{C}^n$ with $S'' \subset \tau(\lambda)$ for all $\lambda \in \mathfrak{h}(\tau) \cap \Omega$, then dom S'' is a subset of $\mathbb{C}^{n-s}$ and S'' coincides with S' on dom S''. Therefore S' is maximal.

(iii) Since τ is represented as in (2.5) by a selfadjoint relation T_0 which is definitizable over Ω' it follows that T_0 is also a representing relation for τ_s and (3.4) holds. In order to verify the minimality condition (3.5) we show that $x \in \mathbb{C}^n$ belongs to $\ker \gamma'$ if and only if $\pi x = 0$. For $x \in \ker \gamma'$ we conclude from (2.5) $\tau(\lambda)x = \text{Re}\,\tau(\lambda_0)x$ for all $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$ and therefore x belongs to

$$\bigcap_{\lambda \in \mathfrak{h}(\tau)\cap\Omega'} \ker \frac{\tau(\lambda) - \tau(\mu_0)^*}{\lambda - \overline{\mu}_0} = \mathbb{C}^{n-s}, \tag{3.6}$$

i.e. $\pi x = 0$. Conversely, if $\pi x = 0$, then x belongs to the set (3.6). But then x belongs also to the set (3.6) with μ_0 replaced by $\overline{\lambda}_0$. As in the proof of (ii)⇒(i) in Lemma 3.2 we conclude from

$$0 = \left(\frac{\tau(\lambda) - \tau(\lambda_0)}{\lambda - \lambda_0} x, y \right) = \left[\gamma' x, (1 + (\overline{\lambda} - \lambda_0)(T_0 - \overline{\lambda})^{-1})\gamma' y \right]$$

and the minimality condition (2.6) that x belongs to $\ker \gamma'$. Now the condition (3.5) follows from the minimality condition (2.6).

(iv) This assertion is a consequence of Theorem 3.3 applied to the minimal representation of the function τ_s from (iii). □

Remark 3.6 Let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued Nevanlinna function. Since $\varphi(\eta) = i(1-\eta)(1+\eta)^{-1}$, $\eta \in \mathbb{D}$, maps the open unit disc $\mathbb{D}$ onto the upper halfplane $\mathbb{C}^+$ and Im $\tau(\lambda) \ge 0$, $\lambda \in \mathbb{C}^+$, the function

$$U : \mathbb{D} \to \mathcal{L}(\mathbb{C}^n), \quad \eta \mapsto \big(\tau(\varphi(\eta)) - i\big)\big(\tau(\varphi(\eta)) + i\big)^{-1}$$

is a contractive analytic function. It is well known (cf. [27, §5, Proposition 2.1]) that there exist uniquely determined compositions $\mathbb{C}^n = \mathcal{G}_1 \oplus \mathcal{G}_2$ and $\mathbb{C}^n = \mathcal{G}_1' \oplus \mathcal{G}_2'$ such that $\eta \mapsto U(\eta) \restriction \mathcal{G}_1 : \mathcal{G}_1 \to \mathcal{G}_1'$, $\eta \in \mathbb{D}$, is a unitary constant and $\eta \mapsto U(\eta) \restriction \mathcal{G}_2 : \mathcal{G}_2 \to \mathcal{G}_2'$ is a purely contractive function. We regard $V := U(\eta) \restriction \mathcal{G}_1$, dom $V = \mathcal{G}_1$, as an isometric operator in $\mathbb{C}^n$ with ran $V = \mathcal{G}_1'$. As $\eta \mapsto U(\eta) \restriction \mathcal{G}_2$ is purely contractive it follows that $V \subset U(\eta)$ is maximal. Since

$$\tau(\varphi(\eta)) = i\big(1 + U(\eta)\big)\big(1 - U(\eta)\big)^{-1}, \quad \eta \in \mathbb{D},$$

and $\widehat{S} := i(1+V)(1-V)^{-1}$ is symmetric and $\widehat{S} \subset \tau(\lambda)$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ it follows that $\widehat{S}$ coincides with the symmetric operator S' from Theorem 3.5.

We finish this section with a simple lemma that gives a criterion for a Weyl function corresponding to a boundary triplet of a symmetric relation to be strict. Furthermore we give an example of a boundary triplet for a symmetric operator of defect one in a two-dimensional Pontryagin space where the corresponding Weyl function is equal to zero and hence in particular not strict.

Lemma 3.7 *Let A be a closed symmetric relation of finite defect n in the Krein space $\mathcal{K}$ and let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^+. Assume that the selfadjoint relation $A_0 = \ker \Gamma_0$ is definitizable over Ω and that the condition $\mathcal{K} = \operatorname{clsp}\{\mathcal{N}_\lambda \mid \lambda \in \rho(A_0) \cap \Omega\}$ is fulfilled. Then the Weyl function M corresponding to $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ is strict.*

P r o o f. If the Weyl function M would not be strict, then for every point μ_0 in $\rho(A_0)\cap\Omega$ there would exist a nonzero element

$$x_{\mu_0} \in \bigcap_{\lambda\in\rho(A_0)\cap\Omega} \ker \frac{M(\lambda) - M(\mu_0)^*}{\lambda - \overline{\mu}_0}.$$

Let γ be the γ-field corresponding to $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ and fix some $\lambda_0 \in \rho(A_0) \cap \Omega$. Setting $\mu_0 = \overline{\lambda}_0$ and making use of (2.12) we would obtain

$$0 = \left(\frac{M(\lambda) - M(\lambda_0)}{\lambda - \lambda_0} x_{\overline{\lambda}_0}, y\right) = \left[\gamma(\lambda)x_{\overline{\lambda}_0}, \gamma(\overline{\lambda}_0)y\right] = \left[\gamma(\lambda_0)x_{\overline{\lambda}_0}, \gamma(\overline{\lambda})y\right]$$

for all $y \in \mathbb{C}^n$ (cf. the proof of Lemma 3.2). Now

$$\mathcal{K} = \operatorname{clsp}\left\{\mathcal{N}_{\overline{\lambda}} \mid \lambda \in \rho(A_0) \cap \Omega\right\} = \operatorname{clsp}\left\{\gamma(\overline{\lambda})y \mid \lambda \in \rho(A_0) \cap \Omega,\, y \in \mathbb{C}^n\right\}$$

would imply $\gamma(\lambda_0)x_{\overline{\lambda}_0} = 0$ and we would get $x_{\overline{\lambda}_0} = 0$. □

Example 3.8 We equip $\mathbb{C}^2$ with the indefinite inner product $[\cdot,\cdot] := (J\cdot,\cdot)$, where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $(\cdot,\cdot)$ is the usual scalar product. Then

$$B_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^2)$$

is selfadjoint in the Pontryagin space $(\mathbb{C}^2, [\cdot,\cdot])$ and for every $\lambda \in \mathbb{C}\backslash\{0\}$ we have

$$(B_0 - \lambda)^{-1} = \begin{pmatrix} -\lambda^{-1} & -\lambda^{-2} \\ 0 & -\lambda^{-1} \end{pmatrix} \in \mathcal{L}(\mathbb{C}^2).$$

Let $\lambda_0 \in \mathbb{C}\backslash\{0\}$, $\gamma_{\lambda_0} := (1,0)^\top \in \mathbb{C}^2$ and define for $\lambda \in \mathbb{C}\backslash\{0\}$

$$\gamma(\lambda) : \mathbb{C} \to \mathbb{C}^2, \qquad c \mapsto \left(1 + (\lambda - \lambda_0)(B_0 - \lambda)^{-1}\right)\gamma_{\lambda_0} c = c\left(\frac{\lambda_0}{\lambda}, 0\right)^\top.$$

From

$$\gamma(\mu)^+ : \mathbb{C}^2 \to \mathbb{C}, \quad (x_1, x_2)^\top \mapsto \frac{\overline{\lambda}_0}{\overline{\mu}} x_2, \quad \mu \in \mathbb{C}\backslash\{0\},$$

we obtain $\gamma(\mu)^+\gamma(\lambda) = 0$ for all $\lambda, \mu \in \mathbb{C}\backslash\{0\}$ and

$$\gamma_{\lambda_0}^+\left((\lambda - \operatorname{Re}\lambda_0) + (\lambda - \lambda_0)(\lambda - \overline{\lambda}_0)(B_0 - \lambda)^{-1}\right)\gamma_{\lambda_0} = 0.$$

Consider the closed symmetric operator

$$B := B_0 \upharpoonright \operatorname{dom} B, \qquad \operatorname{dom} B = \left\{(x_1, x_2)^\top \in \mathbb{C}^2 \mid x_2 = 0\right\}.$$

Then we have $\mathcal{N}_{\lambda,B^+} = \operatorname{ran}\gamma(\lambda)$ for all $\lambda \in \mathbb{C}\backslash\{0\}$ and hence B has defect one and $\mathcal{N}_{\lambda,B^+}[\perp]\mathcal{N}_{\mu,B^+}$ holds for all $\lambda,\mu \in \mathbb{C}\backslash\{0\}$. For a fixed $\lambda \in \mathbb{C}\backslash\{0\}$ and $f_\lambda := \left(\frac{\lambda_0}{\lambda}, 0\right)^\top \in \mathcal{N}_{\lambda,B^+}$ we write the elements $\hat f, \hat g \in B^+$ in the form

$$\hat f = \begin{pmatrix} f_0 \\ B_0 f_0 \end{pmatrix} + \alpha \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix}, \quad \hat g = \begin{pmatrix} g_0 \\ B_0 g_0 \end{pmatrix} + \beta \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix}, \quad f_0, g_0 \in \mathbb{C}^2, \quad \alpha,\beta \in \mathbb{C},$$

see (2.9). We claim that $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 \hat f := \alpha \quad \text{and} \quad \Gamma_1 \hat f := \gamma(\lambda)^+ (B_0 - \overline{\lambda}) f_0,$$

is a boundary triplet for B^+. In fact, if $\hat f, \hat g \in B^+$ are as above, then the selfadjointness of the operator B_0, $\mathcal{N}_{\lambda,B^+}[\perp]\mathcal{N}_{\lambda,B^+}$ and $\gamma(\lambda)1 = f_\lambda$ imply

$$\begin{aligned}
&\big[B_0 f_0 + \alpha\lambda f_\lambda, g_0 + \beta f_\lambda\big] - \big[f_0 + \alpha f_\lambda, B_0 g_0 + \beta\lambda f_\lambda\big] \\
&\quad = \big[(B_0 - \overline{\lambda}) f_0, \beta f_\lambda\big] - \big[\alpha f_\lambda, (B_0 - \overline{\lambda}) g_0\big] \\
&\quad = \big(\gamma(\lambda)^+ (B_0 - \overline{\lambda}) f_0, \beta\big) - \big(\alpha, \gamma(\lambda)^+ (B_0 - \overline{\lambda}) g_0\big) = (\Gamma_1 \hat f, \Gamma_0 \hat g) - (\Gamma_0 \hat f, \Gamma_1 \hat g)
\end{aligned}$$

and the surjectivity of $\Gamma = \binom{\Gamma_0}{\Gamma_1}$ follows from $\overline{\lambda} \in \rho(B_0)$. Furthermore it is not difficult to check that the definition of the mappings Γ_0 and Γ_1 does not depend on the choice of the point $\lambda \in \mathbb{C}\backslash\{0\}$. Hence it follows from $\Gamma_1 \hat f_\mu = 0$, $\hat f_\mu \in \hat{\mathcal{N}}_{\mu,B^+}$, $\mu \in \mathbb{C}\backslash\{0\}$, that the Weyl function τ corresponding to the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is identically equal to zero.

4 Boundary value problems with matrix-valued locally definitizable functions in the boundary condition

In this section we consider a class of abstract boundary value problems with boundary conditions depending on the eigenvalue parameter. Let A be a closed symmetric relation of finite defect n in the Krein space $\mathcal{K}$, let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^+ and let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued locally definitizable function. We investigate problems of the following form: For a given vector $k \in \mathcal{K}$ and $\lambda \in \mathfrak{h}(\tau)$ find $\hat f_1 = \binom{f_1}{f_1'} \in A^+$ such that

$$f_1' - \lambda f_1 = k \qquad \text{and} \qquad \tau(\lambda)\Gamma_0 \hat f_1 + \Gamma_1 \hat f_1 = 0 \tag{4.1}$$

holds. Under suitable assumptions a solution of this problem can be obtained with the help of the compressed resolvent of a selfadjoint extension $\widetilde A$ of A in a larger Krein space $\mathcal{K} \times \mathcal{H}$. Such a selfadjoint relation $\widetilde A$ is said to be a *linearization* of the boundary value problem (4.1). As the spectral properties of $\widetilde A$ are closely connected with the solvability of (4.1) we investigate the spectrum of $\widetilde A$. If, e.g., A has a locally definitizable selfadjoint extension A_0 in $\mathcal{K}$ such that the sign properties of τ and the spectral properties of A_0 are "similar" (see Definition 4.1) we show in Theorem 4.3 that $\widetilde A$ is also locally definitizable. Theorem 4.3, Theorem 4.6 – 4.7 and Theorem 5.1 – 5.2 in the next section extend results obtained with the help of the coupling method in [15] for a symmetric operator A in a Hilbert space and a Nevanlinna function τ in the boundary condition. We note that problems of the type (4.1) with scalar locally definitizable or scalar local generalized Nevanlinna functions were already treated in [3, 7], see also [10] for some more concrete problems involving indefinite Sturm-Liouville operators.

For the case that the $\mathcal{L}(\mathbb{C}^n)$-valued function τ in (4.1) is equal to a selfadjoint constant the boundary value problem can be solved with the help of the resolvent of a canonical selfadjoint extension of A. Let $\Omega = \Omega^*$ be a domain as in the beginning of Section 2.2 and assume that the selfadjoint relation $A_0 = \ker\Gamma_0$ is definitizable over Ω. Denote by γ and M be the γ-field and Weyl function corresponding to the boundary triplet $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ and assume that the function $\lambda \mapsto \det(M(\lambda) + \tau)$ is not identically equal to zero in Ω. By Theorem 2.8 the resolvent of the selfadjoint extension $A_{-\tau} := \ker(\tau\Gamma_0 + \Gamma_1)$ of A in $\mathcal{K}$ corresponding to the selfadjoint operator $-\tau$ via (2.10), (2.11) has the form

$$(A_{-\tau} - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\big(M(\lambda) + \tau\big)^{-1}\gamma(\overline{\lambda})^+$$

for all λ belonging to $\mathfrak{h}(M) \cap \mathfrak{h}((M + \tau)^{-1})$. Since A_0 is definitizable over Ω and has finite defect Theorem 2.2 implies that $A_{-\tau}$ is also definitizable over Ω. Setting $f_1 := (A_{-\tau} - \lambda)^{-1}k$ we see that

$$\hat{f}_1 := \begin{pmatrix} f_1 \\ \lambda f_1 + k \end{pmatrix} \in A_{-\tau}$$

is a solution of (4.1) with $\lambda \mapsto \tau(\lambda) = \tau$. For $\lambda \in \mathfrak{h}(M) \cap \mathfrak{h}((M + \tau)^{-1})$ this solution is unique, since if $\hat{g}_1 = \left(\begin{smallmatrix} g_1 \\ \lambda g_1 + k \end{smallmatrix}\right) \in A^+$ is also a solution of (4.1), then we have $\hat{f}_1 - \hat{g}_1 \in \hat{\mathcal{N}}_{\lambda,A^+}$ and

$$0 = \tau\Gamma_0(\hat{f}_1 - \hat{g}_1) + \Gamma_1(\hat{f}_1 - \hat{g}_1) = \big(\tau + M(\lambda)\big)\Gamma_0(\hat{f}_1 - \hat{g}_1)$$

implies $\hat{f}_1 - \hat{g}_1 \in A_0 \cap \hat{\mathcal{N}}_{\lambda,A^+}$, hence, by (2.9), $\hat{f}_1 = \hat{g}_1$.

In the sequel we shall often assume that the sign types of selfadjoint relations which are locally definitizable over Ω, and locally definitizable functions in Ω coincide outside of a discrete set in $\Omega \cap \overline{\mathbb{R}}$. A notion for this concept will be introduced in the following definition, cf. [3].

Definition 4.1 Let A_1 and A_2 be selfadjoint relations which are definitizable over Ω and let τ_1 and τ_2 be $\mathcal{L}(\mathbb{C}^n)$-valued locally definitizable functions in Ω. We shall say that *the sign types of* A_1 *and* A_2 (A_1 *and* τ_1, τ_1 *and* τ_2) *are d-compatible in* Ω if for every $\mu \in \Omega \cap \overline{\mathbb{R}}$ there exists an open connected neighborhood $I_\mu \subset \Omega \cap \overline{\mathbb{R}}$ of μ such that each component of $I_\mu \backslash \{\mu\}$ is either of positive type with respect to A_1 and A_2 (A_1 and τ_1, τ_1 and τ_2) or of negative type with respect to A_1 and A_2 (resp. A_1 and τ_1, τ_1 and τ_2).

If τ is an $\mathcal{L}(\mathbb{C}^n)$-valued locally definitizable function in Ω, Ω' is a domain as Ω, $\overline{\Omega'} \subset \Omega$, and T_0 is a minimal representing relation for τ which is definitizable over Ω', then the sign types of τ and T_0 are d-compatible in Ω' (see Theorem 2.6). A more instructive example is the d-compatibility of the sign types of local generalized Nevanlinna functions in Ω and arbitrary selfadjoint relations which are of type π_+ over Ω. An example of d-compatibility of the sign types of selfadjoint relations A_1 and A_2 follows from Theorem 2.2, that is, if A_2 is a finite dimensional perturbation in resolvent sense of the locally definitizable relation A_1, then A_2 is also locally definitizable and the sign types of A_1 and A_2 are d-compatible.

For the case of symmetric relations of defect one and scalar locally definitizable functions the following lemma reduces to [3, Proposition 3.5].

Lemma 4.2 *Let A be a closed symmetric relation of finite defect n in the Krein space $\mathcal{K}$ and let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^+ with corresponding Weyl function M. Assume that $A_0 = \ker\Gamma_0$ is definitizable over Ω and let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued definitizable function in Ω such that the sign types of τ and A_0 are d-compatible in Ω. Let Ω' be a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, and let T_0 be a minimal representing relation for τ in some Krein space $\mathcal{H}$ (see Theorem 2.6). Then the following holds.*

(i) *The selfadjoint relation $A_0 \times T_0$ in $\mathcal{K} \times \mathcal{H}$ is definitizable over Ω' and the sign types of $A_0 \times T_0$, A_0, T_0 and the functions M and τ are pairwise d-compatible in Ω'.*

(ii) *The function $M + \tau$ is definitizable in Ω and the sign types of $M + \tau$ are d-compatible with the sign types of M and τ.*

P r o o f. (i) Since A_0 and T_0 are definitizable over Ω and Ω', respectively, their resolvents satisfy the growth condition in Definition 2.1 (ii), and hence this conditionwith Ω replaced by Ω' holds also for the resolvent of $A_0 \times T_0$. By Theorem 2.6 the sign types of τ and T_0 are d-compatible in Ω'. Therefore, for every $\mu \in \Omega' \cap \overline{\mathbb{R}}$ there exists an open connected neighborhood I_μ of μ in $\overline{\mathbb{R}}$ such that each component of $I_\mu \backslash \{\mu\}$ is of the same sign type with respect to A_0 and T_0. Hence $A_0 \times T_0$ is definitizable over Ω'. It follows from relation (2.14) and Theorem 2.5 that the Weyl function M is locally definitizable over Ω and that the sign types of A_0 and M are d-compatible in Ω. Thus the sign types of $A_0 \times T_0$, A_0, T_0 and M and τ are pairwise d-compatible in Ω'.

(ii) By (i) the sign types of M and τ are d-compatible in Ω, hence every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ possesses an open connected neighborhood I_μ in $\overline{\mathbb{R}}$ such that each component of $I_\mu \backslash \{\mu\}$ is of the same sign type with respect to M and τ. It follows from Definition 2.3 that the sign types are the same with respect to the function $M + \tau$. The growth properties of M and τ near to open connected subsets Δ, $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, (see Definition 2.4 (ii)) imply that $M + \tau$ fulfils the second condition in Definition 2.4 and therefore $M + \tau$ is definitizable in Ω. □

The next theorem is the main result of this section. We construct a linearization $\widetilde{A}$ of the boundary value problem (4.1) and investigate its spectral properties. A special feature here is that we do not assume that the matrix function τ in the boundary condition of (4.1) is strict (see Theorem 4.6 for the special case that τ is strict).

Theorem 4.3 *Let A be a closed symmetric relation of finite defect n in the Krein space $\mathcal{K}$ and assume that there exists a selfadjoint extension A_0 of A which is definitizable over Ω. Let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^+, $A_0 = \ker \Gamma_0$, and denote by γ and M the corresponding γ-field and Weyl function, respectively.*
Let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued function which is definitizable in Ω and not equal to a constant and assume that the sign types of τ and A_0 are d-compatible in Ω. Choose $s \in 1, \ldots, n$, a strict $\mathcal{L}(\mathbb{C}^s)$-valued function τ_s and a symmetric $S \in \mathcal{L}(\mathbb{C}^n)$ such that

$$\tau(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & \tau_s(\lambda) \end{pmatrix} + S, \quad S = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}, \quad \lambda \in \mathfrak{h}(\tau) \cap \Omega,$$

holds with respect to the decomposition $\mathbb{C}^n = \mathbb{C}^{n-s} \oplus \mathbb{C}^s$ (cf. Theorem 3.5). Let Ω' be a domain as Ω, $\overline{\Omega'} \subset \Omega$, and choose a closed symmetric operator T in a Krein space $\mathcal{H}$ and a boundary triplet $\{\mathbb{C}^s, \Gamma_0', \Gamma_1'\}$ for T^+ such that τ_s is the corresponding Weyl function and $T_0 = \ker \Gamma_0'$ is a minimal representing relation for τ_s which is definitizable over Ω'.
Let π be the orthogonal projection in $\mathbb{C}^n$ onto $\mathbb{C}^s$ and let ι be the embedding of $\mathbb{C}^s$ in $\mathbb{C}^n$. Assume that the functions $\lambda \mapsto \det(M(\lambda) + S)$ and $\lambda \mapsto \det(\pi(M(\lambda) + S)^{-1}\iota)$ are not identically equal to zero in Ω. Let

$$M_s(\lambda) := \big(\pi(M(\lambda) + S)^{-1}\iota\big)^{-1}, \tag{4.2}$$

suppose that the function $\lambda \mapsto \det\big(M_s(\lambda) + \tau_s(\lambda)\big)$ is not identically equal to zero in Ω and define

$$\mathfrak{h}_0 := \mathfrak{h}(M_s) \cap \mathfrak{h}(\tau_s) \cap \mathfrak{h}\big((M_s + \tau_s)^{-1}\big).$$

Then the relation

$$\widetilde{A} = \Big\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,\big|\, (1 - \pi)(S\Gamma_0 + \Gamma_1)\hat{f}_1 = 0, \\ \pi(S\Gamma_0 + \Gamma_1)\hat{f}_1 - \Gamma_1'\hat{f}_2 = \pi\Gamma_0\hat{f}_1 + \Gamma_0'\hat{f}_2 = 0 \Big\} \tag{4.3}$$

is a selfadjoint extension of A in $\mathcal{K} \times \mathcal{H}$ which is definitizable over Ω', the sign types of $\widetilde{A}$ are d-compatible with the sign types of A_0 and τ in Ω' and $\widetilde{A}$ fulfils the minimality condition

$$\mathcal{K} \times \mathcal{H} = \operatorname{clsp}\big\{\mathcal{K}, (\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}} \,|\, \lambda \in \rho(\widetilde{A}) \cap \Omega'\big\}. \tag{4.4}$$

The set $\Omega' \backslash (\overline{\mathbb{R}} \cup \mathfrak{h}_0)$ is finite. For every $k \in \mathcal{K}$ and every $\lambda \in \mathfrak{h}_0 \cap \Omega'$ the unique solution of the λ-dependent boundary value problem

$$f_1' - \lambda f_1 = k, \quad \tau(\lambda)\Gamma_0\hat{f}_1 + \Gamma_1\hat{f}_1 = 0, \quad \hat{f}_1 = \begin{pmatrix} f_1 \\ f_1' \end{pmatrix} \in A^+, \tag{4.5}$$

is given by

$$f_1 = P_{\mathcal{K}}(\widetilde{A}-\lambda)^{-1}\{k,0\} = (B_0-\lambda)^{-1}k - \gamma_s(\lambda)\big(M_s(\lambda)+\tau_s(\lambda)\big)^{-1}\gamma_s(\overline{\lambda})^+ k,$$
$$f_1' = \lambda f_1 + k, \tag{4.6}$$

where

$$B_0 = \Big\{\hat f_1 \in A^+ \,\big|\, \pi\Gamma_0\hat f_1 = (1-\pi)(S\Gamma_0+\Gamma_1)\hat f_1 = 0\Big\} \tag{4.7}$$

is a selfadjoint extension of A in $\mathcal{K}$ which is definitizable over Ω, γ_s is the analytic continuation of the function $\lambda\mapsto\gamma(\lambda)(M(\lambda)+S)^{-1}\iota M_s(\lambda)$ onto $\mathfrak{h}(M_s)$ and $P_{\mathcal{K}}$ is the orthogonal projection onto the first component of $\mathcal{K}\times\mathcal{H}$.

Proof. **1.** In this first step of the proof we define a symmetric intermediate extension B of A, a boundary triplet for B^+ such that B_0 in (4.7) is the fixed canonical extension, M_s in (4.2) is the corresponding Weyl function and we show that the set $\Omega'\backslash(\overline{\mathbb{R}}\cup\mathfrak{h}_0)$ is finite.

Note first that $\{\mathbb{C}^n, S\Gamma_0+\Gamma_1, -\Gamma_0\}$ is a boundary triplet for A^+. By Theorem 2.8 a point $\lambda\in\rho(A_0)$ belongs to the resolvent set of the selfadjoint relation

$$B_1 := \ker\big(S\Gamma_0+\Gamma_1\big) \tag{4.8}$$

if and only if $(M(\lambda)+S)^{-1}\in\mathcal{L}(\mathbb{C}^n)$. By our assumptions A_0 is definitizable over Ω and the function $\lambda\mapsto\det(M(\lambda)+S)$ is not identically equal to zero in Ω. Therefore $\rho(B_1)\cap\Omega\neq\emptyset$ and by Theorem 2.2 the selfadjoint relation B_1 is definitizable over Ω. The γ-field and Weyl function corresponding to the boundary triplet $\{\mathbb{C}^n, S\Gamma_0+\Gamma_1, -\Gamma_0\}$ are defined for all $\lambda\in\rho(B_1)$. For $\lambda\in\rho(A_0)\cap\rho(B_1) = \mathfrak{h}(M)\cap\mathfrak{h}((M+S)^{-1})$ it is not difficult to verify that they are given by

$$\lambda\mapsto\gamma(\lambda)(M(\lambda)+S)^{-1} \quad\text{and}\quad \lambda\mapsto -(M(\lambda)+S)^{-1},$$

respectively, (cf. [15, §3.3]). It follows from [2, Theorem 2.5] or relation (2.14) and Theorem 2.5 that the $\mathcal{L}(\mathbb{C}^n)$-valued function $\lambda\mapsto-(M(\lambda)+S)^{-1}$ is definitizable over Ω.

We define a symmetric relation B, $A\subset B\subset B_1$, in $\mathcal{K}$ by

$$B := \Big\{\hat f_1\in A^+ \,\big|\, \pi\Gamma_0\hat f_1 = (S\Gamma_0+\Gamma_1)\hat f_1 = 0\Big\}.$$

Then the adjoint relation $B^+\subset A^+$ is given by

$$\begin{aligned} B^+ &= \left\{\hat f_1 = \begin{pmatrix} f_1\\ f_1'\end{pmatrix}\in A^+ \,\middle|\, [g_1', f_1] = [f_1', g_1] \text{ for all } \hat g_1 = \begin{pmatrix} g_1\\ g_1'\end{pmatrix}\in B\right\}\\ &= \Big\{\hat f_1\in A^+ \,\big|\, \big(\Gamma_0\hat g_1, \Gamma_1\hat f_1\big) = \big(\Gamma_1\hat g_1, \Gamma_0\hat f_1\big) \text{ for all } \hat g_1\in B\Big\}\\ &= \Big\{\hat f_1\in A^+ \,\big|\, (1-\pi)(S\Gamma_0+\Gamma_1)\hat f_1 = 0\Big\}. \end{aligned} \tag{4.9}$$

An application of [15, Corollary 4.2] to A^+ and the boundary triplet $\{\mathbb{C}^n, S\Gamma_0+\Gamma_1, -\Gamma_0\}$ shows that $\{\mathbb{C}^s, (S\Gamma_0+\Gamma_1)|_{B^+}, -\pi\Gamma_0|_{B^+}\}$ is a boundary triplet for B^+ with corresponding γ-field

$$\lambda\mapsto\gamma(\lambda)(M(\lambda)+S)^{-1}\iota, \quad \lambda\in\rho(A_0)\cap\rho(B_1),$$

and Weyl function

$$\lambda\mapsto-\pi(M(\lambda)+S)^{-1}\iota, \quad \lambda\in\rho(A_0)\cap\rho(B_1).$$

Hence $\{\mathbb{C}^s, \pi\Gamma_0|_{B^+}, (S\Gamma_0+\Gamma_1)|_{B^+}\}$ is also a boundary triplet for B^+. By our assumptions the function $\lambda\mapsto\pi(M(\lambda)+S)^{-1}\iota$ is invertible for some $\mu\in\Omega$. Then, by Theorem 2.8, the resolvent set of the selfadjoint relation

$$B_0 = \ker\big(\pi\Gamma_0|_{B^+}\big) = \Big\{\hat f_1\in A^+ \,\big|\, \pi\Gamma_0\hat f_1 = (1-\pi)(S\Gamma_0+\Gamma_1)\hat f_1 = 0\Big\}$$

in Ω is nonempty and we conclude from Theorem 2.2 that B_0 is definitizable over Ω.

The γ-field γ_s and Weyl function M_s corresponding to $\{\mathbb{C}^s, \pi\Gamma_0|_{B^+}, (S\Gamma_0 + \Gamma_1)|_{B^+}\}$ are defined for all $\lambda \in \rho(B_0)$. For $\lambda \in \rho(A_0) \cap \rho(B_1) \cap \rho(B_0)$ we have

$$\gamma_s(\lambda) = \gamma(\lambda)\big(M(\lambda) + S\big)^{-1}\iota\big(\pi(M(\lambda) + S)^{-1}\iota\big)^{-1} \in \mathcal{L}(\mathbb{C}^s, \mathcal{K}),$$

and

$$M_s(\lambda) = \big(\pi(M(\lambda) + S)^{-1}\iota\big)^{-1} \in \mathcal{L}(\mathbb{C}^s).$$

Since A_0, B_1 and B_0 are definitizable over Ω the set of nonreal points in Ω not belonging to

$$\rho(A_0) \cap \rho(B_1) \cap \rho(B_0) = \mathfrak{h}(M) \cap \mathfrak{h}\big((M + S)^{-1}\big) \cap \mathfrak{h}(M_s)$$

is discrete and does not accumulate to $\Omega \cap \overline{\mathbb{R}}$. Moreover the local definitizability of B_0 over Ω implies that the function M_s is definitizable over Ω and that the sign types of M_s are d-compatible with the sign types of B_0 and A_0 in Ω, see e.g. Lemma 4.2.

It follows from $\pi\tau(\lambda)\iota = \tau_s(\lambda)$, $\lambda \in \mathfrak{h}(\tau)$, that τ_s is a locally definitizable function in Ω and that the sign types of τ and τ_s are d-compatible. By our assumptions the sign types of A_0 and τ are d-compatible in Ω. Hence the sign types of M_s and τ_s are also d-compatible in Ω and it follows as in the proof of Lemma 4.2 (ii) that the function

$$\lambda \mapsto M_s(\lambda) + \tau_s(\lambda)$$

is definitizable in Ω. By our assumptions $\lambda \mapsto \det(M_s(\lambda) + \tau_s(\lambda))$ is not identically equal to zero in Ω. Therefore we can apply [2, Theorem 2.5] and it follows that

$$\lambda \mapsto -\big(M_s(\lambda) + \tau_s(\lambda)\big)^{-1}$$

is also a locally definitizable function in Ω. Hence the nonreal poles of this function in Ω do not accumulate to $\Omega \cap \overline{\mathbb{R}}$. By our assumptions Ω' is a domain with the same properties as Ω such that $\overline{\Omega'} \subset \Omega$. This implies that there are only finitely many nonreal points which do not belong to

$$\mathfrak{h}_0 \cap \Omega' = \mathfrak{h}(M_s) \cap \mathfrak{h}(\tau_s) \cap \mathfrak{h}\big((M_s + \tau_s)^{-1}\big) \cap \Omega'$$

and therefore the set $\Omega' \backslash (\overline{\mathbb{R}} \cup \mathfrak{h}_0)$ is finite.

2. In this step we define a boundary triplet for the relation $B^+ \times T^+$ in $\mathcal{K} \times \mathcal{H}$, where T^+ is as in the assumptions of the theorem, and we construct a selfadjoint relation $\widetilde{A} \subset B^+ \times T^+$ in $\mathcal{K} \times \mathcal{H}$ such that the compressed resolvent of $\widetilde{A}$ onto $\mathcal{K}$ yields the unique solution of the boundary value problem (4.5).

Let T be a symmetric relation with defect s in the Krein space $\mathcal{H}$ and let $\{\mathbb{C}^s, \Gamma_0', \Gamma_1'\}$ be a boundary triplet for T^+ such that τ_s is the corresponding Weyl function (see Theorem 3.3 and Theorem 3.5) and $T_0 = \ker \Gamma_0'$ is a minimal representing relation for τ_s which is definitizable over the domain Ω', $\overline{\Omega'} \subset \Omega$. The γ-field corresponding to the boundary triplet $\{\mathbb{C}^s, \Gamma_0', \Gamma_1'\}$ will be denoted by γ'. It is not difficult to see that $\{\mathbb{C}^s \times \mathbb{C}^s, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ are the mappings from $B^+ \times T^+$ into $\mathbb{C}^s \times \mathbb{C}^s$ defined by

$$\widetilde{\Gamma}_0\{\hat{f}_1, \hat{f}_2\} := \begin{pmatrix} \pi\Gamma_0\hat{f}_1 \\ \Gamma_0'\hat{f}_2 \end{pmatrix}, \qquad \hat{f}_1 \in B^+,\ \hat{f}_2 \in T^+, \tag{4.10}$$

and

$$\widetilde{\Gamma}_1\{\hat{f}_1, \hat{f}_2\} := \begin{pmatrix} (S\Gamma_0 + \Gamma_1)\hat{f}_1 \\ \Gamma_1'\hat{f}_2 \end{pmatrix}, \qquad \hat{f}_1 \in B^+,\ \hat{f}_2 \in T^+, \tag{4.11}$$

is a boundary triplet for $B^+ \times T^+$. The γ-field $\widetilde{\gamma}$ and the Weyl function $\widetilde{M}$ corresponding to the boundary triplet $\{\mathbb{C}^s \times \mathbb{C}^s, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ are defined for all $\lambda \in \rho(B_0) \cap \rho(T_0) \cap \Omega' = \mathfrak{h}(M_s) \cap \mathfrak{h}(\tau_s) \cap \Omega'$ and are given by

$$\lambda \mapsto \widetilde{\gamma}(\lambda) = \begin{pmatrix} \gamma_s(\lambda) & 0 \\ 0 & \gamma'(\lambda) \end{pmatrix} \in \mathcal{L}(\mathbb{C}^s \times \mathbb{C}^s, \mathcal{K} \times \mathcal{H}), \quad \lambda \in \rho(M_s) \cap \mathfrak{h}(\tau_s) \cap \Omega', \tag{4.12}$$

and

$$\lambda \mapsto \widetilde{M}(\lambda) = \begin{pmatrix} M_s(\lambda) & 0 \\ 0 & \tau_s(\lambda) \end{pmatrix} \in \mathcal{L}(\mathbb{C}^s \times \mathbb{C}^s), \quad \lambda \in \rho(M_s) \cap \mathfrak{h}(\tau_s) \cap \Omega',$$

respectively.

It is straightforward to check that the relation

$$\Theta = \left\{ \begin{pmatrix} \{u, -u\} \\ \{v, v\} \end{pmatrix} \,\middle|\, u, v \in \mathbb{C}^s \right\} \tag{4.13}$$

in $\mathbb{C}^s \times \mathbb{C}^s$ is selfadjoint. The selfadjoint relation $\widetilde{A} = \widetilde{\Gamma}^{-1}\Theta \subset B^+ \times T^+$, $\widetilde{\Gamma} = (\widetilde{\Gamma}_0, \widetilde{\Gamma}_1)^\top$, in $\mathcal{K} \times \mathcal{H}$ corresponding to Θ via (2.10) is given by

$$\begin{aligned} \widetilde{A} &= \Big\{ \{\hat{f}_1, \hat{f}_2\} \in B^+ \times T^+ \,|\, \pi\Gamma_0 \hat{f}_1 + \Gamma_0' \hat{f}_2 = (S\Gamma_0 + \Gamma_1)\hat{f}_1 - \Gamma_1' \hat{f}_2 = 0 \Big\} \\ &= \Big\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,|\, (1 - \pi)(S\Gamma_0 + \Gamma_1)\hat{f}_1 = 0, \\ &\qquad\qquad (S\Gamma_0 + \Gamma_1)\hat{f}_1 - \Gamma_1' \hat{f}_2 = \pi\Gamma_0 \hat{f}_1 + \Gamma_0' \hat{f}_2 = 0 \Big\}. \end{aligned} \tag{4.14}$$

We have

$$\big(\Theta - \widetilde{M}(\lambda)\big)^{-1} = \left\{ \begin{pmatrix} \{v - M_s(\lambda)u, v + \tau_s(\lambda)u\} \\ \{u, -u\} \end{pmatrix} \,\middle|\, u, v \in \mathbb{C}^s \right\}.$$

Setting $x = v - M_s(\lambda)u$ and $y = v + \tau_s(\lambda)u$ we obtain

$$u = -\big(M_s(\lambda) + \tau_s(\lambda)\big)^{-1} x + \big(M_s(\lambda) + \tau_s(\lambda)\big)^{-1} y, \qquad \lambda \in \mathfrak{h}_0 \cap \Omega'.$$

For $\lambda \in \mathfrak{h}_0 \cap \Omega'$ the set $\big\{\{v - M_s(\lambda)u, v + \tau_s(\lambda)u\} \,|\, u, v \in \mathbb{C}^s\big\}$ coincides with $\mathbb{C}^s \times \mathbb{C}^s$ and therefore $\big(\Theta - \widetilde{M}(\lambda)\big)^{-1}$ is the matrix-valued function

$$\begin{pmatrix} -\big(M_s(\lambda) + \tau_s(\lambda)\big)^{-1} & \big(M_s(\lambda) + \tau_s(\lambda)\big)^{-1} \\ \big(M_s(\lambda) + \tau_s(\lambda)\big)^{-1} & -\big(M_s(\lambda) + \tau_s(\lambda)\big)^{-1} \end{pmatrix} \in \mathcal{L}(\mathbb{C}^s \times \mathbb{C}^s). \tag{4.15}$$

Note that by Theorem 2.8 a point $\lambda \in \mathfrak{h}(\widetilde{M}) \cap \Omega' = \mathfrak{h}(M_s) \cap \mathfrak{h}(\tau_s) \cap \Omega'$ belongs to $\rho(\widetilde{A})$ if and only if $(\Theta - \widetilde{M}(\lambda))^{-1} \in \mathcal{L}(\mathbb{C}^s \times \mathbb{C}^s)$. Hence we obtain $(\mathfrak{h}_0 \cap \Omega') \subset \rho(\widetilde{A})$ and for every $\lambda \in \mathfrak{h}_0 \cap \Omega'$ Theorem 2.8 implies

$$(\widetilde{A} - \lambda)^{-1} = \begin{pmatrix} (B_0 - \lambda)^{-1} & 0 \\ 0 & (T_0 - \lambda)^{-1} \end{pmatrix} + \widetilde{\gamma}(\lambda)\big(\Theta - \widetilde{M}(\lambda)\big)^{-1} \widetilde{\gamma}(\overline{\lambda})^+. \tag{4.16}$$

Let us show that the compressed resolvent of $\widetilde{A}$ onto $\mathcal{K}$ has the form (4.6) and is a solution of (4.5). From (4.12), (4.15) and (4.16) we obtain

$$P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}|_{\mathcal{K}} = (B_0 - \lambda)^{-1} - \gamma_s(\lambda)\big(M_s(\lambda) + \tau_s(\lambda)\big)^{-1} \gamma_s(\overline{\lambda})^+$$

for $\lambda \in \mathfrak{h}_0 \cap \Omega'$. For a given $k \in \mathcal{K}$ and $\lambda \in \mathfrak{h}_0 \cap \Omega'$ we define

$$f_1 := P_{\mathcal{K}}(\widetilde{A} - \lambda)^{-1}\{k, 0\} \quad \text{and} \quad f_2 := P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}\{k, 0\}.$$

Then

$$\begin{pmatrix} \{f_1, f_2\} \\ \{\lambda f_1 + k, \lambda f_2\} \end{pmatrix} \in \widetilde{A}$$

and from $\widetilde{A} \subset B^+ \times T^+$ we obtain

$$\hat{f}_1 := \begin{pmatrix} f_1 \\ \lambda f_1 + k \end{pmatrix} \in B^+ \quad \text{and} \quad \hat{f}_2 := \begin{pmatrix} f_2 \\ \lambda f_2 \end{pmatrix} \in \hat{\mathcal{N}}_{\lambda, T^+}.$$

It remains to check that the boundary condition $\tau(\lambda)\Gamma_0 \hat{f}_1 + \Gamma_1 \hat{f}_1 = 0$ is fulfilled. From the decomposition of τ in the asssumptions of the theorem we see that

$$\tau(\lambda)\Gamma_0 \hat{f}_1 + \Gamma_1 \hat{f}_1 = \begin{pmatrix} (1-\pi)(S\Gamma_0 + \Gamma_1)\hat{f}_1 \\ \tau_s(\lambda)\pi\Gamma_0 \hat{f}_1 + \pi(S\Gamma_0 + \Gamma_1)\hat{f}_1 \end{pmatrix} = 0 \tag{4.17}$$

has to be verified. As $\hat{f}_1 \in B^+ \subset A^+$ we have $(1-\pi)(S\Gamma_0 + \Gamma_1)\hat{f}_1 = 0$. The form of $\widetilde{A}$ in (4.14) and the fact that τ_s is the Weyl function corresponding to T and the boundary triplet $\{\mathbb{C}^s, \Gamma_0', \Gamma_1'\}$ implies

$$\pi(S\Gamma_0 + \Gamma_1)\hat{f}_1 = \Gamma_1' \hat{f}_2 = \tau_s(\lambda)\Gamma_0' \hat{f}_2 = -\tau_s(\lambda)\pi\Gamma_0 \hat{f}_1$$

for $\lambda \in \mathfrak{h}_0 \cap \Omega'$. We have shown that for $k \in \mathcal{K}$ and $\lambda \in \mathfrak{h}_0 \cap \Omega'$ the vector

$$\hat{f}_1 = \begin{pmatrix} f_1 \\ \lambda f_1 + k \end{pmatrix} \in A^+ \tag{4.18}$$

is a solution of the boundary value problem (4.5).

Let us verify that this solution $\hat{f}_1 \in A^+$ is unique. Assume that the vector $\hat{g}_1 = \left(\begin{smallmatrix} g_1 \\ \lambda g_1 + k \end{smallmatrix}\right) \in A^+$ is also a solution of (4.5), $\lambda \in \mathfrak{h}_0 \cap \Omega'$. Then $\hat{f}_1 - \hat{g}_1$ belongs to $\hat{\mathcal{N}}_{\lambda, A^+}$ and the relations

$$\begin{aligned} (1-\pi)(S\Gamma_0 + \Gamma_1)(\hat{f}_1 - \hat{g}_1) &= 0, \\ \tau_s(\lambda)\pi\Gamma_0(\hat{f}_1 - \hat{g}_1) + \pi(S\Gamma_0 + \Gamma_1)(\hat{f}_1 - \hat{g}_1) &= 0 \end{aligned} \tag{4.19}$$

are fulfilled. The first relation in (4.19) implies $\hat{f}_1 - \hat{g}_1 \in \hat{\mathcal{N}}_{\lambda, B^+}$. Since M_s is the Weyl function corresponding to $\{\mathbb{C}^s, \pi\Gamma_0|_{B^+}, (S\Gamma_0 + \Gamma_1)|_{B^+}\}$ the second relation in (4.19) can be written as

$$\big(M_s(\lambda) + \tau_s(\lambda)\big)\pi\Gamma_0(\hat{f}_1 - \hat{g}_1) = 0.$$

As $\ker(M_s(\lambda) + \tau_s(\lambda)) = \{0\}$, $\lambda \in \mathfrak{h}_0 \cap \Omega'$, we obtain $\hat{f}_1 - \hat{g}_1 \in \ker(\pi\Gamma_0|_{B^+}) = B_0$. But for $\lambda \in \mathfrak{h}_0 \cap \Omega'$ we have $B^+ = B_0 \,\widehat{+}\, \hat{\mathcal{N}}_{\lambda, B^+}$ and as $\hat{f}_1 - \hat{g}_1 \in \hat{\mathcal{N}}_{\lambda, B^+}$ we conclude $\hat{f}_1 = \hat{g}_1$, that is, the solution (4.18) of (4.5) is unique.

3. It remains to show that $\widetilde{A}$ is definitizable over Ω', that the sign types of $\widetilde{A}$ are d-compatible with the sign types of A_0 and τ in Ω' and that the minimality condition

$$\mathcal{K} \times \mathcal{H} = \operatorname{clsp}\big\{\mathcal{K}, (\widetilde{A} - \lambda)^{-1}\{k, 0\} \,|\, \lambda \in \rho(\widetilde{A}) \cap \Omega',\, k \in \mathcal{K}\big\} \tag{4.20}$$

holds.

Let us first verify, that $\widetilde{A}$ fulfils (4.20). As T_0 is a minimal representing relation for τ_s we have

$$\mathcal{H} = \operatorname{clsp}\big\{\big(1 + (\lambda - \lambda_0)(T_0 - \lambda)^{-1}\big)\gamma' x \,|\, \lambda \in \rho(T_0) \cap \Omega',\, x \in \mathbb{C}^s\big\} \tag{4.21}$$

and by (2.12) and (4.21)

$$\mathcal{H} = \operatorname{clsp}\{\gamma'(\lambda)x \mid \lambda \in \rho(T_0) \cap \Omega',\, x \in \mathbb{C}^s\} \tag{4.22}$$

holds. The set $\rho(T_0) \cap \Omega'$ in (4.22) can be replaced by $\rho(\widetilde{A}) \cap \Omega'$. From (4.12), (4.15) and (4.16) we obtain

$$P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}\{k, 0\} = \gamma'(\lambda)\big(M_s(\lambda) + \tau_s(\lambda)\big)^{-1}\gamma_s(\overline{\lambda})^+ k$$

for $k \in \mathcal{K}$ and $\lambda \in \mathfrak{h}_0 \cap \Omega'$. As $\gamma_s(\overline{\lambda})$ is injective $\operatorname{ran} \gamma_s(\overline{\lambda})^+ = \mathbb{C}^s$ follows. Making use of (4.22) we obtain

$$\mathcal{H} = \operatorname{clsp}\{P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1}\{k, 0\} \mid \lambda \in \rho(\widetilde{A}) \cap \Omega',\, k \in \mathcal{K}\},$$

and therefore $\widetilde{A}$ fulfils the minimality condition (4.20).

As the sign types of A_0 are d-compatible with the sign types of τ in Ω, Lemma 4.2 (i) implies that $A_0 \times T_0$ is definitizable over Ω' and the sign types of $A_0 \times T_0$ are d-compatible with the sign types of A_0 and τ in Ω'. For $\lambda \in \mathfrak{h}_0 \cap \Omega'$ we have

$$\dim\big(\operatorname{ran}\big((\widetilde{A} - \lambda)^{-1} - ((A_0 \times T_0) - \lambda)^{-1}\big)\big) < \infty$$

and therefore Theorem 2.2 implies that $\widetilde{A}$ is definitizable over Ω' and that the sign types of $\widetilde{A}$, A_0 and τ are d-compatible in Ω'. □

Remark 4.4 For a symmetric operator A in a Hilbert space and an $\mathcal{L}(\mathbb{C}^n)$-valued Nevanlinna function τ a result very similar to Theorem 4.3 was proved in [15]. For the case that τ is not strict it is sufficient to consider the relation-valued Nevanlinna function

$$\lambda \mapsto \tau(\lambda) = \left\{\begin{pmatrix} 0 \\ x \end{pmatrix} \mid x \in \mathbb{C}^{n-s}\right\} \widehat{+}\, \tau_s(\lambda), \quad \lambda \in \mathbb{C}\backslash\mathbb{R},$$

where τ_s is a strict $\mathcal{L}(\mathbb{C}^s)$-valued Nevanlinna function. Similarly to the proof of Theorem 4.3 the λ-dependent boundary value problem

$$f_1' - \lambda f_1 = k, \quad \tau(\lambda)\Gamma_0 \hat{f}_1 + \Gamma_1 \hat{f}_1 = 0, \quad \hat{f}_1 = \begin{pmatrix} f_1 \\ f_1' \end{pmatrix} \in A^*,$$

can be rewritten in the form

$$f_1' - \lambda f_1 = k, \quad \tau_s(\lambda)\Gamma_0 \hat{f}_1 + \pi\Gamma_1 \hat{f}_1 = 0, \quad \hat{f}_1 = \begin{pmatrix} f_1 \\ f_1' \end{pmatrix} \in C^*,$$

where $C^* = \{\hat{f}_1 \in A^* \mid (1 - \pi)\Gamma_0 \hat{f}_1 = 0\}$ and π is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^s$. Note, that the selfadjoint relation $A_0 = \ker \Gamma_0$ is a restriction of C^* whereas in the proof of Theorem 4.3 (if τ is not strict) the relation A_0 is not a restriction of B^+.

If $k \in \mathcal{K}$ in (4.5) is zero, then (4.6) yields the trivial solution $f_1 = f_1' = 0$ (which is unique if λ belongs to $\mathfrak{h}_0 \cap \Omega'$). In the next theorem we show that the nontrivial solutions of this "homogeneous" boundary value problem are closely connected with the eigenvalues and eigenvectors of $\widetilde{A}$.

Theorem 4.5 *Let A, $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ and τ, T, $\{\mathbb{C}^s, \Gamma_0', \Gamma_1'\}$ be as in Theorem 4.3. Then the following assertions hold.*

(i) *If $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$ is an eigenvalue of the selfadjoint relation $\widetilde{A}$ in (4.3) and $\{f_1, f_2\} \in \ker(\widetilde{A} - \lambda)$ is a corresponding eigenvector, then the vector $\hat{f}_1 = \binom{f_1}{\lambda f_1} \in A^+$ is a nontrivial solution of the homogeneous boundary value problem*

$$f_1' - \lambda f_1 = 0, \quad \tau(\lambda)\Gamma_0 \hat{f}_1 + \Gamma_1 \hat{f}_1 = 0, \quad \begin{pmatrix} f_1 \\ f_1' \end{pmatrix} \in A^+. \tag{4.23}$$

(ii) *If* $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$ *and* $\hat{f}_1 = \binom{f_1}{f_1'} \in A^+$ *is a nontrivial solution of the homogeneous boundary value problem* (4.23)*, then* λ *is an eigenvalue of the selfadjoint relation* $\widetilde{A}$ *in* (4.3).

(iii) *The symmetric relation* $B = \{\hat{f}_1 \in A^+ \,|\, \pi\Gamma_0\hat{f}_1 = (S\Gamma_0 + \Gamma_1)\hat{f}_1 = 0\}$ *is an* $(n-s)$*-dimensional extension of* A*, has defect* s *and the symmetric relation* $B \times T$ *in* $\mathcal{K} \times \mathcal{H}$ *has defect* $2s$*. If* M_s *is the function defined in* (4.2) *and* $\lambda \in \mathfrak{h}(M_s) \cap \mathfrak{h}(\tau) \cap \Omega'$ *is an eigenvalue of* $\widetilde{A}$*, then the dimension of the corresponding eigenspace is not larger than* s.

Proof. (i) Assume that $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$ is an eigenvalue of

$$\widetilde{A} = \Big\{\{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,\big|\, (1-\pi)(S\Gamma_0 + \Gamma_1)\hat{f}_1 = 0, \\ \pi(S\Gamma_0 + \Gamma_1)\hat{f}_1 - \Gamma_1'\hat{f}_2 = \pi\Gamma_0\hat{f}_1 + \Gamma_0'\hat{f}_2 = 0\Big\} \tag{4.24}$$

and let $\hat{f}_1 = \binom{f_1}{\lambda f_1} \in A^+$, $\hat{f}_2 = \binom{f_2}{\lambda f_2} \in T^+$ such that $\{\hat{f}_1, \hat{f}_2\} \in \widetilde{A}$ is a nontrivial element of $\widetilde{A}$. Then $\hat{f}_1 \neq 0$, as otherwise (4.24) would imply

$$0 = \pi(S\Gamma_0 + \Gamma_1)\hat{f}_1 = \Gamma_1'\hat{f}_2 \qquad \text{and} \qquad 0 = -\pi\Gamma_0\hat{f}_1 = \Gamma_0'\hat{f}_2,$$

but then we would have $\hat{f}_2 \in \ker\Gamma_0' \cap \ker\Gamma_1' = T$. This is impossible since T_0 is a minimal representing relation for τ and therefore T is an operator without eigenvalues.

As $\{\hat{f}_1, \hat{f}_2\} \in \widetilde{A}$, $\hat{f}_2 \in \hat{\mathcal{N}}_{\lambda,T^+}$ and τ_s is the Weyl function corresponding to the boundary triplet $\{\mathbb{C}^s, \Gamma_0', \Gamma_1'\}$ the relations

$$(1-\pi)(S\Gamma_0 + \Gamma_1)\hat{f}_1 = 0 \tag{4.25}$$

and

$$\tau_s(\lambda)\pi\Gamma_0\hat{f}_1 = -\tau_s(\lambda)\Gamma_0'\hat{f}_2 = -\Gamma_1'\hat{f}_2 = -\pi(S\Gamma_0 + \Gamma_1)\hat{f}_1 \tag{4.26}$$

hold. Hence $\tau(\lambda)\Gamma_0\hat{f}_1 + \Gamma_1\hat{f}_1 = 0$ (cf. (4.17)) and $\hat{f}_1 = \binom{f_1}{\lambda f_1} \in A^+$ is a nontrivial solution of the homogeneous boundary value problem (4.23).

(ii) Let $\hat{f}_1 = \binom{f_1}{\lambda f_1} \in A^+$ be a nontrivial solution of (4.23). Then the relation $\tau(\lambda)\Gamma_0\hat{f}_1 + \Gamma_1\hat{f}_1 = 0$ is fulfilled and by (4.17) the relations (4.25) and (4.26) hold. As λ belongs to $\mathfrak{h}(\tau)\cap\Omega' = \rho(T_0)\cap\Omega'$ we have $T^+ = T_0 \,\widehat{+}\, \hat{\mathcal{N}}_{\lambda,T^+}$, where $T_0 = \ker\Gamma_0'$. Hence there exists a vector $\hat{f}_2 \in \hat{\mathcal{N}}_{\lambda,T^+}$ such that $\Gamma_0'\hat{f}_2 = -\pi\Gamma_0\hat{f}_1$. From (4.26), $\hat{f}_2 \in \hat{\mathcal{N}}_{\lambda,T^+}$ and the fact that τ_s is the Weyl function corresponding to the boundary triplet $\{\mathbb{C}^s, \Gamma_0', \Gamma_1'\}$ we conclude

$$\pi(S\Gamma_0 + \Gamma_1)\hat{f}_1 = -\tau_s(\lambda)\pi\Gamma_0\hat{f}_1 = \tau_s(\lambda)\Gamma_0'\hat{f}_2 = \Gamma_1'\hat{f}_2.$$

This relation, (4.25) and $\Gamma_0'\hat{f}_2 = -\pi\Gamma_0\hat{f}_1$ imply $\{\hat{f}_1, \hat{f}_2\} \in \widetilde{A}$. From $\hat{f}_1 = \binom{f_1}{\lambda f_1}$ and $\hat{f}_2 = \binom{f_2}{\lambda f_2} \in \hat{\mathcal{N}}_{\lambda,T^+}$ it follows that λ is an eigenvalue of $\widetilde{A}$.

(iii) In part 1 of the proof of Theorem 4.3 we have already shown that $\{\mathbb{C}^s, \pi\Gamma_0|_{B^+}, (S\Gamma_0 + \Gamma_1)|_{B^+}\}$ is a boundary triplet for B^+. Therefore B has defect s and is an $(n-s)$-dimensional extension of A. Since T is an operator of defect s it follows that $B \times T$ has defect $2s$.

We have to show that for an eigenvalue $\lambda \in \mathfrak{h}(M_s) \cap \mathfrak{h}(\tau) \cap \Omega'$ of $\widetilde{A}$ the dimension of the eigenspace is less or equal to s. Assume that

$$\{\hat{f}_1^{(i)}, \hat{f}_2^{(i)}\} \in \widetilde{A}, \quad \hat{f}_1^{(i)} = \begin{pmatrix} f_1^{(i)} \\ \lambda f_1^{(i)} \end{pmatrix}, \;\; \hat{f}_2^{(i)} = \begin{pmatrix} f_2^{(i)} \\ \lambda f_2^{(i)} \end{pmatrix}, \;\; i = 1, \dots, s+1,$$

are linearly independent eigenvectors corresponding to $\lambda \in \sigma_p(\widetilde{A})$. As in part (i) of the proof one verifies $\hat{f}_1^{(i)} \neq 0$ for $i = 1, \dots, s+1$. From $\widetilde{A} \subset B^+ \times T^+$ we get $\hat{f}_1^{(i)} \in \hat{\mathcal{N}}_{\lambda,B^+}$. Note that the set $\mathfrak{h}(M_s)$ coincides with the resolvent set of $B_0 = \ker(\pi\Gamma_0|_{B^+})$ (see (4.7)). Hence we have $B^+ = B_0 \,\widehat{+}\, \hat{\mathcal{N}}_{\lambda,B^+}$ for $\lambda \in \mathfrak{h}(M_s)$. As each vector $\{\hat{f}_1^{(i)}, \hat{f}_2^{(i)}\} \in \widetilde{A}$ has the property

$$\pi\Gamma_0 \hat{f}_1^{(i)} = -\Gamma_0' \hat{f}_2^{(i)}, \quad i = 1, \dots, s+1,$$

we obtain $\Gamma_0' \hat{f}_2^{(i)} \neq 0$ for $i = 1, \dots, s+1$, as otherwise $\pi\Gamma_0 \hat{f}_1^{(i)} = 0$ would imply $\hat{f}_1^{(i)} \in B_0 \cap \hat{\mathcal{N}}_{\lambda,B^+}$, i.e. $\hat{f}_1^{(i)} = 0$. By $T^+ = T_0 \,\widehat{+}\, \hat{\mathcal{N}}_{\lambda,T^+}$, $\lambda \in \mathfrak{h}(\tau) \cap \Omega'$, we even have $\hat{f}_2^{(i)} \neq 0$ for $i = 1, \dots, s+1$.

Since the symmetric operator T has defect s and the vectors $\hat{f}_2^{(i)}$, $i = 1, \dots, s+1$, belong to $\hat{\mathcal{N}}_{\lambda,T^+}$, there exists $k \in \{1, \dots s+1\}$ and numbers $\alpha_j \in \mathbb{C}$ such that

$$\hat{f}_2^{(k)} = \sum_{\substack{j=1\\ j\neq k}}^{s+1} \alpha_j \hat{f}_2^{(j)} \tag{4.27}$$

holds. From $\pi\Gamma_0 \hat{f}_1^{(i)} = -\Gamma_0' \hat{f}_2^{(i)}$, $i = 1, \dots, s+1$, we conclude

$$\pi\Gamma_0 \hat{f}_1^{(k)} = -\Gamma_0' \hat{f}_2^{(k)} = -\sum_{\substack{j=1\\ j\neq k}}^{s+1} \alpha_j \Gamma_0' \hat{f}_2^{(j)} = \pi\Gamma_0 \sum_{\substack{j=1\\ j\neq k}}^{s+1} \alpha_j \hat{f}_1^{(j)}$$

and from $\hat{f}_1^{(k)} \in \hat{\mathcal{N}}_{\lambda,B^+}$ and $\sum \alpha_j \hat{f}_1^{(j)} \in \hat{\mathcal{N}}_{\lambda,B^+}$ we obtain that

$$\hat{f}_1^{(k)} - \sum_{\substack{j=1\\ j\neq k}}^{s+1} \alpha_j \hat{f}_1^{(j)} \in \hat{\mathcal{N}}_{\lambda,B^+}$$

belongs to $\ker(\pi\Gamma_0|_{B^+})$. Again making use of $B^+ = B_0 \,\widehat{+}\, \hat{\mathcal{N}}_{\lambda,B^+}$, $\lambda \in \mathfrak{h}(M_s)$, we find

$$\hat{f}_1^{(k)} = \sum_{\substack{j=1\\ j\neq k}}^{s+1} \alpha_j \hat{f}_1^{(j)} \tag{4.28}$$

and from (4.27) and (4.28) we conclude

$$\{\hat{f}_1^{(k)}, \hat{f}_2^{(k)}\} = \sum_{\substack{j=1\\ j\neq k}}^{s+1} \alpha_j \{\hat{f}_1^{(j)}, \hat{f}_2^{(j)}\},$$

a contradiction to the assumption that the vectors $\{\hat{f}_1^{(i)}, \hat{f}_2^{(i)}\}$, $i = 1, \dots, s+1$, are linearly independent. □

If the function τ in the boundary condition of (4.5) is strict, then in the assumptions of Theorem 4.3 we have $s = n$, $S = 0$, $\pi = I_{\mathbb{C}^n}$ and $\iota = I_{\mathbb{C}^n}$. In this case it can be shown that the assumptions on the invertibility of the functions $\lambda \mapsto M(\lambda) + S = M(\lambda)$ and $\lambda \mapsto \pi(M(\lambda) + S)^{-1}\iota = M(\lambda)^{-1}$ can be dropped and Theorem 4.3 reduces to the following theorem.

Theorem 4.6 *Let A be a closed symmetric relation of finite defect n in the Krein space $\mathcal{K}$ and assume that there exists a selfadjoint extension A_0 of A which is definitizable over Ω. Let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^+, $A_0 = \ker \Gamma_0$, and denote by γ and M the corresponding γ-field and Weyl function, respectively.*
Let τ be a strict $\mathcal{L}(\mathbb{C}^n)$-valued function which is definitizable in Ω, let Ω' be a domain as Ω, $\overline{\Omega'} \subset \Omega$, and choose a closed symmetric operator T in a Krein space $\mathcal{H}$ and a boundary triplet $\{\mathbb{C}^n, \Gamma_0', \Gamma_1'\}$ for T^+ such that τ is the corresponding Weyl function and $T_0 = \ker \Gamma_0'$ is a minimal representing relation

for τ which is definitizable over Ω'. Assume that the sign types of τ and A_0 are d-compatible in Ω, that the function $\lambda \mapsto \det(M(\lambda)+\tau(\lambda))$ is not identically equal to zero in Ω and set

$$\mathfrak{h}_0 = \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}\big((M+\tau)^{-1}\big). \tag{4.29}$$

Then the relation

$$\widetilde{A} = \Big\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,\Big|\, \Gamma_1 \hat{f}_1 - \Gamma_1' \hat{f}_2 = \Gamma_0 \hat{f}_1 + \Gamma_0' \hat{f}_2 = 0 \Big\} \tag{4.30}$$

is a selfadjoint extension of A in $\mathcal{K}\times\mathcal{H}$ which is definitizable over Ω', the sign types of $\widetilde{A}$ are d-compatible with the sign types of A_0 and τ in Ω' and $\widetilde{A}$ fulfils the minimality condition (4.4)*. The set $\Omega'\backslash(\overline{\mathbb{R}} \cup \mathfrak{h}_0)$ is finite. For every $k \in \mathcal{K}$ and every $\lambda \in \mathfrak{h}_0 \cap \Omega'$ the unique solution of the λ-dependent boundary value problem*

$$f_1' - \lambda f_1 = k, \quad \tau(\lambda)\Gamma_0 \hat{f}_1 + \Gamma_1 \hat{f}_1 = 0, \quad \hat{f}_1 = \begin{pmatrix} f_1 \\ f_1' \end{pmatrix} \in A^+, \tag{4.31}$$

is given by

$$\begin{aligned} f_1 &= P_{\mathcal{K}}(\widetilde{A}-\lambda)^{-1}\{k,0\} = (A_0-\lambda)^{-1}k - \gamma(\lambda)\big(M(\lambda)+\tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ k, \\ f_1' &= \lambda f_1 + k, \end{aligned} \tag{4.32}$$

where $P_{\mathcal{K}}$ is the orthogonal projection onto the first component of $\mathcal{K}\times\mathcal{H}$.

The next theorem is a variant of Theorem 4.3 and Theorem 4.6 for the case that A has defect one. Under the additional assumption $\mathcal{K} = \operatorname{clsp}\{\mathcal{N}_{\lambda,A^+} \mid \lambda \in \Omega\}$ Theorem 4.7 was proved in [3].

Theorem 4.7 *Let A be a closed symmetric relation of defect one in the Krein space $\mathcal{K}$ and assume that there exists a selfadjoint extension A_0 of A which is definitizable over Ω. Let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^+, $A_0 = \ker \Gamma_0$, and denote by γ and M the corresponding γ-field and Weyl function, respectively.*
Let τ be a (scalar) locally definitizable function in Ω which is not equal to a constant, let Ω' be a domain as Ω, $\overline{\Omega'} \subset \Omega$, and choose a closed symmetric operator T in a Krein space $\mathcal{H}$ and a boundary triplet $\{\mathbb{C}, \Gamma_0', \Gamma_1'\}$ for T^+ such that τ is the corresponding Weyl function and $T_0 = \ker \Gamma_0'$ is a minimal representing relation for τ which is definitizable over Ω'. Assume that the sign types of τ and A_0 are d-compatible in Ω, that the function $M+\tau$ is not identically equal to zero in Ω and let $\mathfrak{h}_0$ be as in (4.29)*.*

Then $\widetilde{A}$ in (4.30) *is a selfadjoint extension of A in $\mathcal{K}\times\mathcal{H}$ which definitizable over Ω', the sign types of $\widetilde{A}$ are d-compatible with the sign types of A_0 and τ in Ω' and $\widetilde{A}$ fulfils the minimality condition* (4.4)*. The set $\Omega'\backslash(\overline{\mathbb{R}} \cup \mathfrak{h}_0)$ is finite. For every $k \in \mathcal{K}$ and every $\lambda \in \mathfrak{h}_0 \cap \Omega'$ the unique solution of the λ-dependent boundary value problem* (4.31) *is given by* (4.32)*.*

We finish this section with some remarks concerning Theorem 4.3, Theorem 4.6 and Theorem 4.7.

Remark 4.8 Let A and $\widetilde{A}$ be as in Theorem 4.3. Assume that $\widetilde{B}$ is a selfadjoint extension of A in some Krein space $\mathcal{K}\times\widetilde{\mathcal{H}}$ which is definitizable over Ω' such that the compression of the resolvent of $\widetilde{B}$ onto the Krein space $\mathcal{K}$ yields a solution of the boundary value problem (4.5) and that $\widetilde{B}$ fulfils the minimality condition (4.4) with $\mathcal{K}\times\mathcal{H}$ and $\rho(\widetilde{A})\cap\Omega'$ replaced by $\mathcal{K}\times\widetilde{\mathcal{H}}$ and $\rho(\widetilde{B})\cap\Omega'$, respectively. Denote the local spectral functions of $\widetilde{A}$ and $\widetilde{B}$ on $\Omega'\cap\overline{\mathbb{R}}$ by $E_{\widetilde{A}}$ and $E_{\widetilde{B}}$, respectively (see [36] for the definition and properties of the local spectral function).

There exists a densely defined closed isometric operator V from $\mathcal{K}\times\mathcal{H}$ into $\mathcal{K}\times\widetilde{\mathcal{H}}$ such that for each closed connected set $\Delta \subset \Omega'\cap\overline{\mathbb{R}}$, where $E_{\widetilde{A}}(\Delta)$ (and hence also $E_{\widetilde{B}}(\Delta)$) is defined, V is reduced by

$$E_{\widetilde{A}}(\Delta)(\mathcal{K}\times\mathcal{H}) \times E_{\widetilde{B}}(\Delta)(\mathcal{K}\times\widetilde{\mathcal{H}}).$$

The closed isometric operator $V_1 := V \cap \big(E_{\widetilde{A}}(\Delta)(\mathcal{K}\times\mathcal{H}) \times E_{\widetilde{B}}(\Delta)(\mathcal{K}\times\widetilde{\mathcal{H}})\big)$ intertwines the resolvents of

$$\widetilde{A}_1 := \widetilde{A} \cap \big(E_{\widetilde{A}}(\Delta)(\mathcal{K}\times\mathcal{H})\big)^2 \quad \text{and} \quad \widetilde{B}_1 := \widetilde{B} \cap \big(E_{\widetilde{B}}(\Delta)(\mathcal{K}\times\widetilde{\mathcal{H}})\big)^2,$$

i.e. for $\lambda \in \rho(\widetilde{A}_1) \cap \rho(\widetilde{B}_1) \cap \Omega'$ we have $V_1(\widetilde{A}_1 - \lambda)^{-1}x = (\widetilde{B}_1 - \lambda)^{-1}V_1 x$ for all $x \in \operatorname{dom} V_1$. In particular, the ranks of positivity and negativity of the inner products $[\cdot,\cdot]_{\mathcal{K}\times\mathcal{H}}$ and $[\cdot,\cdot]_{\mathcal{K}\times\widetilde{\mathcal{H}}}$ on the subspaces $E_{\widetilde{A}}(\Delta)(\mathcal{K}\times\mathcal{H})$ and $E_{\widetilde{B}}(\Delta)(\mathcal{K}\times\widetilde{\mathcal{H}})$ coincide.

If, in addition to the assumptions above, $(E_{\widetilde{A}}(\Delta)(\mathcal{K}\times\mathcal{H}), [\cdot,\cdot]_{\mathcal{K}\times\mathcal{H}})$ is a Pontryagin space, then also $E_{\widetilde{B}}(\Delta)(\mathcal{K}\times\widetilde{\mathcal{H}})$ equipped with the inner product from $\mathcal{K}\times\widetilde{\mathcal{H}}$ is a Pontryagin space and by [31, Theorem 6.2] the operator V_1 is an isometric isomorphism of $E_{\widetilde{A}}(\Delta)(\mathcal{K}\times\mathcal{H})$ onto $E_{\widetilde{B}}(\Delta)(\mathcal{K}\times\widetilde{\mathcal{H}})$, i.e. $\widetilde{A}_1$ and $\widetilde{B}_1$ are isometrically equivalent.

Remark 4.9 Let the assumptions be as in Theorem 4.3 and assume, in addition, that A is a densely defined operator. Then the linearization $\widetilde{A}$ of the boundary value problem (4.5) is a selfadjoint operator in $\mathcal{K}\times\mathcal{H}$.

In fact, let B^+ be as in (4.9), let mul $(B^+\times T^+)$ be the multivalued part of $B^+\times T^+$ and let

$$\hat{\mathcal{N}}_\infty := \left\{ \begin{pmatrix} 0 \\ h \end{pmatrix} \,\Big|\, h \in \operatorname{mul}\,(B^+\times T^+) \right\}.$$

Let $\widetilde{\Gamma}_0$, $\widetilde{\Gamma}_1$ and Θ be as in (4.10)-(4.11) and (4.13), respectively. As $A\times T$ is an (in general not densely defined) operator, by [14, Proposition 2.1] it is sufficient to show that $(\widetilde{\Gamma}_0,\widetilde{\Gamma}_1)^\top\hat{\mathcal{N}}_\infty \cap \Theta = \{0\}$ holds. Since B^+ is a restriction of A^+ and A^+ is by our assumption an operator we find that mul $(B^+\times T^+)$ coincides with $\{\{0,f\}\,|\,f\in\operatorname{mul}T^+\}$ and therefore

$$\hat{\mathcal{N}}_\infty = \left\{ \begin{pmatrix} \{0,0\} \\ \{0,f\} \end{pmatrix} \,\Big|\, f \in \operatorname{mul}\,T^+ \right\}.$$

Hence we obtain

$$\begin{pmatrix} \widetilde{\Gamma}_0 \\ \widetilde{\Gamma}_1 \end{pmatrix} \hat{\mathcal{N}}_\infty \cap \Theta = \left\{ \begin{pmatrix} \{0, \Gamma_0'(\begin{smallmatrix} 0 \\ f \end{smallmatrix})\} \\ \{0, \Gamma_1'(\begin{smallmatrix} 0 \\ f \end{smallmatrix})\} \end{pmatrix} \,\Big|\, f\in \operatorname{mul}\,T^+ \right\} \cap \left\{ \begin{pmatrix} \{u,-u\} \\ \{v,v\} \end{pmatrix} \,\Big|\, u,v\in\mathbb{C}^s \right\}$$

and it follows that $\widetilde{A}$ is an operator.

5 Boundary value problems with matrix-valued (local) generalized Nevanlinna functions in the boundary condition

In this section we consider boundary value problems of the form (4.1) where an $\mathcal{L}(\mathbb{C}^n)$-valued local generalized Nevanlinna function τ appears in the boundary condition. Theorem 5.1 below is a variant of Theorem 4.3 and Theorem 4.6. For simplicity we consider only the case where τ is strict in this theorem. By Theorem 3.3 τ is the Weyl function corresponding to a closed symmetric operator T in some Krein space $\mathcal{H}$ and a boundary triplet $\{\mathbb{C}^n, \Gamma_0', \Gamma_1'\}$, where $\ker\Gamma_0'$ is a selfadjoint relation which is locally of type π_+. The proof of Theorem 5.1 is very similar to the proof of Theorem 4.6. Instead of Theorem 2.2 on finite dimensional perturbations of locally definitizable selfadjoint relations here one has to use [6, Theorem 2.4] on compact (and finite dimensional) perturbations of selfadjoint relations which are locally of type π_+.

Let again Ω be a domain in $\overline{\mathbb{C}}$ symmetric with respect to the real axis such that $\Omega\cap\overline{\mathbb{R}}\neq\emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected.

Theorem 5.1 *Let A be a closed symmetric relation of finite defect n in the Krein space $\mathcal{K}$ and assume that there exists a selfadjoint extension A_0 of A which is of type π_+ over Ω. Let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^+, $A_0 = \ker\Gamma_0$, and denote by γ and M the corresponding γ-field and Weyl function, respectively.*
Let τ be a strict $\mathcal{L}(\mathbb{C}^n)$-valued local generalized Nevanlinna function in Ω, let Ω' be a domain as Ω, $\overline{\Omega'}\subset\Omega$, and choose a closed symmetric operator T in a Krein space $\mathcal{H}$ and a boundary triplet $\{\mathbb{C}^n,\Gamma_0',\Gamma_1'\}$ for T^+ such that τ is the corresponding Weyl function and $T_0 = \ker\Gamma_0'$ is a minimal representing relation

for τ which is of type π_+ over Ω'. Assume that the function $\lambda \mapsto \det(M(\lambda)+\tau(\lambda))$ is not identically equal to zero in Ω and set

$$\mathfrak{h}_0 = \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}\big((M+\tau)^{-1}\big).$$

Then the relation

$$\widetilde{A} = \Big\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,\big|\, \Gamma_1 \hat{f}_1 - \Gamma_1' \hat{f}_2 = \Gamma_0 \hat{f}_1 + \Gamma_0' \hat{f}_2 = 0 \Big\}$$

is a selfadjoint extension of A in $\mathcal{K}\times\mathcal{H}$ which is of type π_+ over Ω' and $\widetilde{A}$ fulfils the minimality condition (4.4)*. The set $\Omega' \backslash (\overline{\mathbb{R}} \cup \mathfrak{h}_0)$ is finite. For every $k \in \mathcal{K}$ and every $\lambda \in \mathfrak{h}_0 \cap \Omega'$ the unique solution of the λ-dependent boundary value problem*

$$f_1' - \lambda f_1 = k, \quad \tau(\lambda)\Gamma_0 \hat{f}_1 + \Gamma_1 \hat{f}_1 = 0, \quad \hat{f}_1 = \begin{pmatrix} f_1 \\ f_1' \end{pmatrix} \in A^+,$$

is given by

$$\begin{aligned} f_1 &= P_{\mathcal{K}}(\widetilde{A}-\lambda)^{-1}\{k,0\} = (A_0-\lambda)^{-1}k - \gamma(\lambda)\big(M(\lambda)+\tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ k, \\ f_1' &= \lambda f_1 + k, \end{aligned}$$

where $P_{\mathcal{K}}$ is the orthogonal projection onto the first component of $\mathcal{K}\times\mathcal{H}$.

We remark that by the second example below Definition 4.1 here the assumption that the sign types of τ and A_0 are d-compatible in Ω from Theorem 4.3 and Theorem 4.6 is fulfilled. For the case that A is of defect one Theorem 5.1 was proved in [7]. We do not formulate a variant of Theorem 4.7 for this special case.

In the next theorem we consider the special case that $\mathcal{K}$ is a Pontryagin space (of finite rank of negativity) and τ is a (in general non-strict) matrix-valued generalized Nevanlinna function. In contrast to the previous theorems we assume that A is a densely defined operator. Then all canonical selfadjoint extensions of A are operators in the Pontryagin space $\mathcal{K}$ and, in particular, their resolvent sets are nonempty.

Theorem 5.2 *Let A be a densely defined closed symmetric operator of defect n in the Pontryagin space $\mathcal{K}$. Let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^+, $A_0 = \ker\Gamma_0$, and denote by γ and M the corresponding γ-field and Weyl function, respectively.*
Let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued generalized Nevanlinna function which is not equal to a constant. Choose $s \in 1,\dots,n$, a strict $\mathcal{L}(\mathbb{C}^s)$-valued generalized Nevanlinna function τ_s and a selfadjoint $S \in \mathcal{L}(\mathbb{C}^n)$ such that

$$\tau(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & \tau_s(\lambda) \end{pmatrix} + S, \quad S = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix},$$

holds with respect to the decomposition $\mathbb{C}^n = \mathbb{C}^{n-s} \oplus \mathbb{C}^s$ (cf. Theorem 3.5*). Let T be a closed symmetric operator of defect s in a Pontryagin space $\mathcal{H}$ and let $\{\mathbb{C}^s, \Gamma_0', \Gamma_1'\}$ be a boundary triplet for T^+ such that τ_s is the corresponding Weyl function and $T_0 = \ker\Gamma_0'$ is a minimal representing relation for τ_s (cf. Corollary* 3.4*).*
Let π be the orthogonal projection in $\mathbb{C}^n$ onto $\mathbb{C}^s$ and let ι be the embedding of $\mathbb{C}^s$ in $\mathbb{C}^n$. Then the functions $\lambda \mapsto \det(M(\lambda)+S)$ and $\lambda \mapsto \det(\pi(M(\lambda)+S)^{-1}\iota)$ are not identically equal to zero. Let $M_s(\lambda) := (\pi(M(\lambda)+S)^{-1}\iota)^{-1}$, assume that the function $\lambda \mapsto \det\big(M_s(\lambda)+\tau_s(\lambda)\big)$ is not identically equal to zero and set

$$\mathfrak{h}_0 = \mathfrak{h}(M_s) \cap \mathfrak{h}(\tau_s) \cap \mathfrak{h}\big((M_s+\tau_s)^{-1}\big).$$

Then the operator

$$\begin{aligned} \widetilde{A} = \Big\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \,\big|\, & (1-\pi)(S\Gamma_0 + \Gamma_1)\hat{f}_1 = 0, \\ & \pi(S\Gamma_0+\Gamma_1)\hat{f}_1 - \Gamma_1'\hat{f}_2 = \pi\Gamma_0\hat{f}_1 + \Gamma_0'\hat{f}_2 = 0 \Big\} \end{aligned}$$

is a selfadjoint extension of A *in the Pontryagin space* $\mathcal{K}\times\mathcal{H}$ *which fulfils the minimality condition* (4.4)*. The set* $\mathbb{C}\backslash(\mathbb{R}\cup\mathfrak{h}_0)$ *is finite. For every* $k\in\mathcal{K}$ *and every* $\lambda\in\mathfrak{h}_0$ *the unique solution of the* λ*-dependent boundary value problem*

$$f_1'-\lambda f_1=k,\quad \tau(\lambda)\Gamma_0\hat{f}_1+\Gamma_1\hat{f}_1=0,\quad \hat{f}_1=\begin{pmatrix}f_1\\ f_1'\end{pmatrix}\in A^+,$$

is given by

$$f_1=P_{\mathcal{K}}(\widetilde{A}-\lambda)^{-1}\{k,0\}=(B_0-\lambda)^{-1}k-\gamma_s(\lambda)\big(M_s(\lambda)+\tau_s(\lambda)\big)^{-1}\gamma_s^+(\overline{\lambda})k,$$
$$f_1'=\lambda f_1+k,$$

where

$$B_0=\Big\{\hat{f}_1\in A^+\,\big|\,\pi\Gamma_0\hat{f}_1=(1-\pi)(S\Gamma_0+\Gamma_1)\hat{f}_1=0\Big\}$$

is a selfadjoint extension of A *in* $\mathcal{K}$*,* γ_s *is the analytic continuation of* $\lambda\mapsto\gamma(\lambda)(M(\lambda)+S)^{-1}\iota M_s(\lambda)$ *onto* $\mathfrak{h}(M_s)$ *and* $P_{\mathcal{K}}$ *is the orthogonal projection onto the first component of* $\mathcal{K}\times\mathcal{H}$*.*

Proof. Here the selfadjoint extensions A_0, B_0 and B_1 of A (cf. (4.7) and (4.8)) are selfadjoint operators in the Pontryagin space $\mathcal{K}$. Hence with the exception of finitely many points $\mathbb{C}\backslash\mathbb{R}$ belongs to the resolvent set of A_0, B_0 and B_1. From $\rho(A_0)\cap\rho(B_1)=\mathfrak{h}(M)\cap\mathfrak{h}((M+S)^{-1})$ and

$$\rho(A_0)\cap\rho(B_1)\cap\rho(B_0)=\mathfrak{h}(M)\cap\mathfrak{h}\big((M+S)^{-1}\big)\cap\mathfrak{h}(M_s)$$

(see part 1 of the proof of Theorem 4.3) we find that the assumptions on the invertibility of the functions $\lambda\mapsto M(\lambda)+S$ and $\lambda\mapsto\pi(M(\lambda)+S)^{-1}\iota$ from Theorem 4.3 are automatically fulfilled. It follows as in Remark 4.9 that $\widetilde{A}$ is an operator. The remaining assertions follow from Theorem 4.3. □

6 Indefinite Sturm-Liouville operators with eigenvalue depending interface conditions

In this section we show that the general results from the previous sections can be applied to classes of boundary value problems for singular indefinite Sturm-Liouville differential expressions on $\mathbb{R}$ of the form

$$\operatorname{sgn}(\cdot)\left(-\frac{d^2}{dx^2}+q\right),\tag{6.1}$$

where $q\in L^1_{\text{loc}}(\mathbb{R})$ is assumed to be a real valued function. For this equip $L^2(\mathbb{R})$ with the indefinite inner product

$$[f,g]:=\int_{\mathbb{R}}f(x)\overline{g(x)}\operatorname{sgn}(x)dx,\quad f,g\in L^2(\mathbb{R}),$$

and denote the corresponding Krein space $(L^2(\mathbb{R}),[\cdot,\cdot])$ by $L^2_{\text{sgn}}(\mathbb{R})$. As a fundamental symmetry in $L^2_{\text{sgn}}(\mathbb{R})$ we choose $(Jf)(x):=\operatorname{sgn}(x)f(x)$, $f\in L^2_{\text{sgn}}(\mathbb{R})$; then $[J\cdot,\cdot]$ coincides with the usual Hilbert scalar product on $L^2(\mathbb{R})$.

Let us assume that the differential expression $\ell:=-\frac{d^2}{dx^2}+q$ is in the limit point case at both singular endpoints ∞ and $-\infty$, cf., e.g., [47]. Then it is well-known that the operator $D_0y:=\ell(y)$ defined on the usual maximal domain

$$\mathcal{D}_{\max}(\mathbb{R})=\big\{y\in L^2(\mathbb{R}):y,y'\in AC(\mathbb{R}),\,\ell(y)\in L^2(\mathbb{R})\big\},\tag{6.2}$$

is selfadjoint in the Hilbert space $L^2(\mathbb{R})$, and hence

$$\begin{aligned}(A_0 y)(x) :=& (JD_0 y)(x) = \operatorname{sgn}(x)\big(-y''(x) + q(x)y(x)\big),\\ \operatorname{dom} A_0 =& \operatorname{dom} JD_0 = \mathcal{D}_{\max}(\mathbb{R}),\end{aligned} \tag{6.3}$$

is selfadjoint in the Krein space $L^2_{\rm sgn}(\mathbb{R})$. Under the assumption that the limits $\lim_{x\to\pm\infty} q(x)$ exist, the regions of definitizability of A_0 are characterized in the next theorem. In the present form Theorem 6.1 can be found in [5], see also [9, 12, 39, 40].

Theorem 6.1 *Suppose that the limits*

$$q_\infty := \lim_{x\to\infty} q(x) \quad \text{and} \quad q_{-\infty} := \lim_{x\to-\infty} q(x)$$

exist and that the functions $x \mapsto q(x) - q_\infty$ *and* $x \mapsto q(x) - q_{-\infty}$ *belong to* $L^1((b,\infty))$ *and* $L^1((-\infty,a))$ *for some* $a, b \in \mathbb{R}$*, respectively. Then the following holds.*

(i) *If* $q_\infty \le -q_{-\infty}$*, then* A_0 *is definitizable over* $\overline{\mathbb{C}}\backslash[q_\infty, -q_{-\infty}]$*.*

(ii) *If* $-q_{-\infty} < q_\infty$*, then* A_0 *is definitizable and* $\sigma(A_0) \cap (-q_{-\infty}, q_\infty)$ *consists of eigenvalues of* A_0 *with* q_∞ *and* $-q_{-\infty}$ *as only possible accumulation points.*

Furthermore, A_0 *is of type* π_+ *over* $\overline{\mathbb{C}}\backslash[-\infty, -q_{-\infty}]$ *and of type* π_- *over* $\overline{\mathbb{C}}\backslash[q_\infty, \infty]$*.*

Let $c \in \mathbb{R}$ be fixed and let τ be an $\mathcal{L}(\mathbb{C}^2)$-valued function which is definitizable over some domain Ω. We will consider boundary value problems of the following form: For a given function $k \in L^2(\mathbb{R})$ and $\lambda \in \mathfrak{h}(\tau)$ find a function $f_1 \in \mathcal{D}_{\max}((c,\infty)) \times \mathcal{D}_{\max}((-\infty,c))$ such that

$$\operatorname{sgn}(x)\big(-f_1''(x) + q(x)f_1(x)\big) - \lambda f_1(x) = k(x), \qquad x \in \mathbb{R}, \tag{6.4}$$

and

$$\tau(\lambda)\begin{pmatrix} f_1(c+) - f_1(c-) \\ f_1'(c+) - f_1'(c-)\end{pmatrix} = \begin{pmatrix} -f_1'(c+) \\ f_1(c-)\end{pmatrix} \tag{6.5}$$

holds. Here the subsets $\mathcal{D}_{\max}((c,\infty))$ and $\mathcal{D}_{\max}((-\infty,c))$ of $L^2((c,\infty))$ and $L^2((-\infty,c))$, respectively, are defined analogously to (6.2). With the help of the next lemma the λ-dependent boundary value problem (6.4)-(6.5) can be rewritten in the general form (4.1). The proof of Lemma 6.2 is straightforward and we leave it to the reader.

Lemma 6.2 *The operator*

$$\begin{aligned}(Af_1)(x) =& \operatorname{sgn}(x)\big(-f_1''(x) + q(x)f_1(x)\big),\\ \operatorname{dom} A =& \big\{f_1 \in \mathcal{D}_{\max}(\mathbb{R}) : f_1(c) = f_1'(c) = 0\big\},\end{aligned}$$

is a densely defined closed symmetric operator of defect two in the Krein space $L^2_{\rm sgn}(\mathbb{R})$ *and the adjoint operator* A^+ *is given by*

$$(A^+ f_1)(x) = \operatorname{sgn}(x)\big(-f_1''(x) + q(x)f_1(x)\big), \quad \operatorname{dom} A^+ = \mathcal{D}_{\max}((c,\infty)) \times \mathcal{D}_{\max}((-\infty,c)).$$

The triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$*, where*

$$\Gamma_0 \hat f_1 := \begin{pmatrix} f_1(c+) - f_1(c-) \\ f_1'(c+) - f_1'(c-)\end{pmatrix}, \qquad \Gamma_1 \hat f_1 := \begin{pmatrix} f_1'(c+) \\ -f_1(c-)\end{pmatrix}, \qquad \hat f_1 = \begin{pmatrix} f_1 \\ A^+ f_1\end{pmatrix},$$

is a boundary triplet for A^+*, and the selfadjoint operator* $\ker\Gamma_0$ *coincides with the indefinite Sturm-Liouville operator* A_0 *in* (6.3)*. Furthermore, the boundary value problem* (6.4)-(6.5) *is equivalent to*

$$(A^+ - \lambda)f_1 = k, \quad \tau(\lambda)\Gamma_0 \hat f_1 + \Gamma_1 \hat f_1 = 0, \qquad \hat f_1 = \begin{pmatrix} f_1 \\ A^+ f_1\end{pmatrix}.$$

In the sequel it will be assumed that $\ell = -\frac{d^2}{dx^2} + q$ satisfies the conditions in Theorem 6.1 and that $q_\infty \le -q_{-\infty}$ holds, i.e., A_0 is definitizable over $\overline{\mathbb{C}}\backslash[q_\infty, -q_{-\infty}]$. Moreover, if τ is definitizable over some domain Ω, $\Omega \subset \overline{\mathbb{C}}\backslash[q_\infty, -q_{-\infty}]$ and the sign types of τ and A_0 are d-compatible in Ω, then Theorem 4.3 or Theorem 4.6, respectively, can be applied. More precisely, if, e.g., τ is strict, Ω' is a domain with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, T is a closed symmetric operator in a Krein space $\mathcal{H}$ and $\{\mathbb{C}^2, \Gamma_0', \Gamma_1'\}$ is a boundary triplet for T^+ such that τ is the corresponding Weyl function, $T_0 = \ker \Gamma_0'$ is a minimal representing relation for τ which is definitizable over Ω', and the function $\lambda \mapsto \det(M(\lambda) + \tau(\lambda))$ is not identically equal to zero in Ω, where M is the Weyl function of $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$, then

$$\widetilde{A} = \left\{ \{\hat{f}_1, \hat{f}_2\} \in A^+ \times T^+ \middle| \begin{pmatrix} f_1'(c+) \\ -f_1(c-) \end{pmatrix} - \Gamma_1' \hat{f}_2 = \begin{pmatrix} f_1(c+) - f_1(c-) \\ f_1'(c+) - f_1'(c-) \end{pmatrix} + \Gamma_0' \hat{f}_2 = 0 \right\} \tag{6.6}$$

is a selfadjoint extension of A in $L^2_{\text{sgn}}(\mathbb{R}) \times \mathcal{H}$ which is definitizable over Ω', the sign types of $\widetilde{A}$ are d-compatible with the sign types of A_0 and τ in Ω', and for every $\lambda \in \mathfrak{h}(M) \cap \mathfrak{h}(\tau) \cap \mathfrak{h}((M+\tau)^{-1}) \cap \Omega'$ the unique solution of the boundary value problem (6.4)-(6.5) is given by

$$f_1 = P_{L^2_{\text{sgn}}}(\widetilde{A} - \lambda)^{-1}\{k, 0\} = (A_0 - \lambda)^{-1}k - \gamma(\lambda)\big(M(\lambda) + \tau(\lambda)\big)^{-1}\gamma(\overline{\lambda})^+ k, \tag{6.7}$$

cf. Theorem 4.6. Here γ denotes the γ-field corresponding to the boundary triplet $\{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ from Lemma 6.2. Note also that by Remark 4.9 here $\widetilde{A}$ is an operator,

$$\begin{aligned} \widetilde{A}\{f_1, f_2\} &= \big\{\operatorname{sgn}\big(-f_1'' + q f_1\big), f_2'\big\}, \\ \operatorname{dom} \widetilde{A} &= \left\{ \{f_1, f_2\} \in \mathcal{D}_{\max}(\mathbb{R}) \times \operatorname{dom} T^+ \middle| \begin{pmatrix} f_1'(c+) \\ -f_1(c-) \end{pmatrix} - \Gamma_1' \hat{f}_2 = 0, \right. \\ &\qquad \left. \begin{pmatrix} f_1(c+) - f_1(c-) \\ f_1'(c+) - f_1'(c-) \end{pmatrix} + \Gamma_0' \hat{f}_2 = 0 \right\}, \end{aligned} \tag{6.8}$$

where $\hat{f}_2 = \binom{f_2}{f_2'} \in T^+$.

In the following three examples we briefly consider some special types of λ-dependent boundary conditions of the form (6.5). Namely, first of all it is assumed that τ is the difference of two generalized Nevanlinna functions, so that the sign types of A_0 and τ become locally d-compatible, secondly a simple situation of a non-strict function τ is discussed, and thirdly τ is assumed to be a generalized Nevanlinna function.

Example 6.3 Assume that $\tau = G_1 - G_2$ is the difference of two $\mathcal{L}(\mathbb{C}^2)$-valued generalized Nevanlinna functions G_1 and G_2 such that

$$\Delta_- := \mathfrak{h}(G_1) \cap (-\infty, q_\infty) \quad \text{and} \quad \Delta_+ := \mathfrak{h}(G_2) \cap (-q_{-\infty}, \infty)$$

are nonempty intervals. Suppose that τ is strict and that $\lambda \mapsto \det(M(\lambda) + \tau(\lambda))$ is not identically equal to zero. It is not difficult to see that τ is definitizable over

$$\Omega := (\mathbb{C}\backslash\mathbb{R}) \cup (\Delta_- \cup \Delta_+)$$

and that $(\mathbb{C}\backslash\mathbb{R}) \cup \Delta_+$ is of type π_+ with respect to τ and $(\mathbb{C}\backslash\mathbb{R}) \cup \Delta_-$ is of type π_- with respect to τ. Let Ω' be a domain with the usual properties, $\overline{\Omega'} \subset \Omega$, and choose $\mathcal{H}$, $T \subset T^+$ and $\{\mathbb{C}^2, \Gamma_0', \Gamma_1'\}$ as above. Then the linearization $\widetilde{A}$ in (6.8) of the boundary value problem (6.4)-(6.5) is locally definitizable over Ω'. Moreover, $\widetilde{A}$ is of type π_+ over $\Omega'\backslash\Delta_-$ and of type π_- over $\Omega'\backslash\Delta_+$.

Example 6.4 Let $\alpha \in \mathbb{R}$ and assume that τ has the form

$$\tau(\lambda) = \begin{pmatrix} \alpha & 0 \\ 0 & \tau_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \tau_{22}(\lambda) \end{pmatrix} + S, \qquad S = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.9}$$

where τ_{22} is a scalar function which is definitizable over $\overline{\mathbb{C}}\backslash[q_\infty,-q_{-\infty}]$ and not equal to a constant. It is assumed that the sign types of τ_{22} and A_0 are d-compatible in $\overline{\mathbb{C}}\backslash[q_\infty,-q_{-\infty}]$. Let Ω' be a domain with the usual properties, $\overline{\Omega'}\subset\overline{\mathbb{C}}\backslash[q_\infty,-q_{-\infty}]$, let T be a closed symmetric operator of defect one in a Krein space $\mathcal{H}$ and let $\{\mathbb{C},\Gamma_0',\Gamma_1'\}$ be a boundary triplet for T^+ such that τ coincides with the corresponding Weyl function on Ω' and $T_0=\ker\Gamma_0'$ is a locally definitizable minimal representing relation for τ_{22}. Furthermore, suppose that the functions $\lambda\mapsto\det(M(\lambda)+S)$ and $\lambda\mapsto((M(\lambda)+S)^{-1})_{22}$ are not identically equal to zero in $\overline{\mathbb{C}}\backslash[q_\infty,-q_{-\infty}]$, let $M_s(\lambda)=(((M(\lambda)+S)^{-1})_{22})^{-1}$ and suppose that $M_s+\tau_{22}$ is also not identically equal to zero in $\overline{\mathbb{C}}\backslash[q_\infty,-q_{-\infty}]$. Then the linearization $\widetilde{A}$ in (6.8) is given by

$$\widetilde{A}\{f_1,f_2\}=\bigl\{\operatorname{sgn}\bigl(-f_1''+qf_1\bigr),f_2'\bigr\},$$
$$\operatorname{dom}\widetilde{A}=\left\{\{f_1,f_2\}\in\mathcal{D}_{\max}(\mathbb{R})\times\operatorname{dom}T^+\;\middle|\;\begin{matrix}\alpha(f_1(c_+)-f_1(c_-))=-f_1'(c_+),\\ \alpha(f_1'(c_+)-f_1'(c_-))-f_1(c_-)=\Gamma_1'\hat{f}_2,\\ f_1'(c_+)-f_1'(c_-)=-\Gamma_0'\hat{f}_2\end{matrix}\right\},$$

where $\hat{f}_2=\binom{f_2}{f_2'}\in T^+$, and $\widetilde{A}$ is definitizable over Ω', its sign types are d-compatible with the sign types of A_0, τ_{22} and τ, and for every $\lambda\in\mathfrak{h}(M_s)\cap\mathfrak{h}(\tau_{22})\cap\mathfrak{h}((M_s+\tau_{22})^{-1})\cap\Omega'$ the unique solution f_1 of (6.4)-(6.5) is given by

$$f_1=P_{L^2_{\mathrm{sgn}}}(\widetilde{A}-\lambda)^{-1}\{k,0\}=(B_0-\lambda)^{-1}k-\gamma_s(\lambda)\bigl(M_s(\lambda)+\tau_{22}(\lambda)\bigr)^{-1}\gamma_s(\overline{\lambda})^+k,$$

where

$$(B_0f_1)(x)=\operatorname{sgn}(x)\bigl(-f_1''(x)+q(x)f_1(x)\bigr),$$
$$\operatorname{dom}B_0=\left\{f_1\in\mathcal{D}_{\max}((c,\infty))\times\mathcal{D}_{\max}((-\infty,c))\;\middle|\;\begin{matrix}f_1'(c_+)-f_1'(c_-)=0,\\ \alpha(f_1(c_+)-f_1(c_-))=-f_1'(c_+)\end{matrix}\right\},$$

is a selfadjoint extension of A in $L^2_{\mathrm{sgn}}(\mathbb{R})$ which is definitizable over $\overline{\mathbb{C}}\backslash[q_\infty,-q_{-\infty}]$ and γ_s is the analytic continuation of the function $\lambda\mapsto\gamma(\lambda)(M(\lambda)+S)^{-1}\binom{0}{M_s(\lambda)}$ onto $\mathfrak{h}(M_s)$.

Example 6.5 Let again τ be of the form (6.9) and assume that the function τ_{22} is a scalar generalized Nevanlinna function which is not equal to a constant. Then τ_{22} is the Weyl function corresponding to a boundary triplet $\{\mathbb{C},\Gamma_0',\Gamma_1'\}$ of a closed symmetric operator T of defect one in a Pontryagin space $\mathcal{H}$. Suppose that $\lambda\mapsto M(\lambda)+S$ satisfies the conditions in Theorem 4.3 or Example 6.4, respectively. Then the linearization $\widetilde{A}$ in Example 6.4 is definitizable over $\overline{\mathbb{C}}\backslash[-\infty,-q_{-\infty}]$ and $\overline{\mathbb{C}}\backslash[-\infty,-q_{-\infty}]$ is of type π_+ with respect to $\widetilde{A}$. If, in addition, τ_{22} is holomorphic on $(-\infty,q_\infty)$ with the possible exception of finitely many isolated poles, then $\widetilde{A}$ is definitizable over $\overline{\mathbb{C}}\backslash[q_\infty,-q_{-\infty}]$, $\overline{\mathbb{C}}\backslash[-\infty,-q_{-\infty}]$ is of type π_+ with respect to $\widetilde{A}$ and $\overline{\mathbb{C}}\backslash[q_\infty,\infty]$ is of type π_- with respect to $\widetilde{A}$.

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