Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering
Department of Physics

Spectral theory of black holes

Spektrální teorie černých děr

Master's thesis

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Academic year: 2024/2025

ČVUT
ČESKÉ VYSOKÉ UČENÍ TECHNICKÉ V PRAZE

ZADÁNÍ DIPLOMOVÉ PRÁCE

I. OSOBNÍ A STUDIJNÍ ÚDAJE

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Zadávající katedra/ústav: **Katedra fyziky**

Studijní program: **Matematická fyzika**

II. ÚDAJE K DIPLOMOVÉ PRÁCI

Název diplomové práce:

Spektrální teorie černých děr

Název diplomové práce anglicky:

Spectral theory of black holes

Pokyny pro vypracování:

(1) Identifikace spektrálně teoretické modely v relativistické teorii gravitace (stabilita horizontu černých děr, kvazinormální módy asymptoticky anti-de-sitterovských černých děr, etc).
(2) Rigorózní definice souvisejících nesamosdružených operátorů.
(3) Spektrální a pseudospektrální analýza.

Seznam doporučené literatury:

[1] D. E. Edmunds, W. D. Evans: Spectral theory and differential operators, 2018, Oxford University Press.
[2] M. Hitrik, M. Zworski: Overdamped QNM for Schwarzschild black holes, arXiv:2406.15924 (2024).
[3] J. L. Jaramillo: Black hole horizons and quantum charged particles. Class. Quantum Grav. 32 (2015) 132001.
[4] D. Krejčiřík: Complex magnetic fields: An improved Hardy-Laptev-Weidl inequality and quasi-self-adjointness. SIAM J. Math. Anal. 51 (2019) 790-807.
[5] C. M. Warnick: On Quasinormal Modes of Asymptotically Anti-de Sitter Black Holes, Comm. Math. Phys. 333 (2015) 959-1035.

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Datum zadání diplomové práce: **30.10.2024** Termín odevzdání diplomové práce: **09.05.2025**

Platnost zadání diplomové práce: **30.10.2026**

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III. PŘEVZETÍ ZADÁNÍ

Diplomant bere na vědomí, že je povinen vypracovat diplomovou práci samostatně, bez cizí pomoci, s výjimkou poskytnutých konzultací. Seznam použité literatury, jiných pramenů a jmen konzultantů je třeba uvést v diplomové práci.

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Spektrální teorie černých děr

vypracoval samostatně a uvedl veškeré použité informační zdroje v souladu s Metodickým pokynem o dodržování etických principů při přípravě vysokoškolských závěrečných prací a Rámocovými pravidly používání umělé inteligence na ČVUT pro studijní a pedagogické účely v Bc a NM studiu.

Prohlašuji, že jsem v průběhu příprav a psaní závěrečné práce použil nástroje umělé inteligence. Vygenerovaný obsah jsem ověřil. Stvrzuji, že jsem si vědom, že za obsah závěrečné práce plně zodpovídám.

V Praze dne 03.05.2025

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....
podpis

Acknowledgment:
I would like to express my sincere gratitude to my supervisor, prof. Mgr. David Krejčiřík, Ph.D., DSc., for his expert guidance, insightful feedback, and enduring patience throughout my work. I would also like to thank Dr. Borbala Gerhat for her valuable advice and helpful suggestions.

I gratefully acknowledge the support provided by the EXPRO Grant No. 20-17749X of the Czech Science Foundation (GAČR).

Název práce:
Spektrální teorie černých děr

Autor: Bc. Jan Havel

Studijní program: Matematická fyzika

Druh práce: Diplomová práce

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Abstrakt: Ve spektroskopii černých děr hrají kvazinormální módy zásadní roli. Pomocí hyperbolických souřadnic mohou být tyto módy formulovány jako vlastní vektory nesamosdruženého operátoru, který plní roli infinitezimálního generátoru semigrupy odvozené z vlnové rovnice popisující jejich vývoj. V této práci se konkrétně zaměřujeme na černou díru v asymptoticky anti-de Sitterovském prostoročase. Příslušný spektrální problém je přeformulován jako problém invertibility diferenciálního operátoru druhého řádu, jehož vlastnosti následně zkoumáme. Nakonec okomentujeme nestabilitu kvazinormálních módů a význam pseudospektra v tomto kontextu.

Klíčová slova: černá díra, diferenciální operátor druhého řádu, kvazinormální mód, pseudospektrum, semigrupa

Title:
Spectral theory of black holes

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Abstract: In black hole spectroscopy, quasinormal modes play a central role. Within the hyperboloidal approach, these modes can be characterised as eigenvectors of a non-selfadjoint operator that serves as the infinitesimal generator of a semigroup derived from the wave equation governing their evolution. This thesis focuses specifically on asymptotically anti-de Sitter black hole spacetimes. The associated spectral problem is reformulated as an invertibility problem of a second-order differential operator, whose properties are analysed. Finally, the instability of quasinormal modes and the relevance of the pseudospectrum in this context are discussed.

Key words: black hole, pseudospectrum, quasinormal mode, second-order differential operator, semigroup

Contents

Introduction

Black holes are a central topic in gravitational physics. While much attention has traditionally been given to exact vacuum solutions describing compact mass objects, understanding real astrophysical black holes requires accounting for external perturbations, which lead to dynamical, non-stationary states [16]. Interest in black hole perturbations has increased significantly in recent decades, especially following the direct detection of gravitational waves [1]. A major focus has been on the merger of compact binaries during which black holes emit gravitational radiation in response to perturbations.

The temporal evolution of gravitational waves emitted by a perturbed black hole can be divided into three distinct stages. The initial phase is a burst of radiation associated with the violent dynamics of the system. This is followed by a prolonged phase dominated by damped oscillations, known as quasinormal modes, which encode the characteristic response of the black hole. Finally, at very late times, the waveform transitions into a power-law or exponential decay tail, reflecting the asymptotic relaxation of the perturbation [6]. Among these, quasinormal modes are of particular interest. They represent the characteristic resonant response of the black hole and are intimately tied to the structure of spacetime itself.

Quasinormal modes are traditionally defined as solutions to a perturbed wave equation with specific boundary conditions: ingoing at the horizon and outgoing at infinity. However, these solutions grow exponentially at spatial infinity or at the event horizon. To overcome this difficulty, the hyperboloidal approach reformulates the problem using a geometric foliation that regularises the behaviour of quasinormal modes, allowing them to be viewed as eigenfunctions of a non-selfadjoint operator. This operator serves as the infinitesimal generator of a C_0 semigroup that governs the time evolution of perturbations [24].

Another important aspect of quasinormal modes is their sensitivity to perturbations, known as spectral instability. Since quasinormal modes are associated with non-normal operators, small perturbations can lead to large shifts in the spectrum, a phenomenon that can be rigorously analysed using the concept of the pseudospectrum. This perspective has important implications for the stability and physical relevance of the modes in dynamical settings.

In this work, we investigate quasinormal modes through the lens of spectral theory. We focus on the analysis of a second-order differential operator arising from the spectral problem associated with the generator of time evolution. This operator has the form of a Laplace-transformed wave equation. Our main goal is to carry out a rigorous analysis of this operator in the context of asymptotically anti-de Sitter black hole spacetimes, culminating in the proof of Theorem 2.1.

The structure of this thesis is as follows. In Chapter 1, we review the definition and properties of quasinormal modes, first using the traditional formulation and then within the hyperboloidal framework. We also introduce the connection to the semigroup theory. In Chapter 2, we develop a detailed analysis of the relevant second-order differential operator. Finally, in Chapter 3, we explore the pseudospectrum and discuss its significance for the spectral instability of quasinormal modes.

Chapter 1

Quasinormal modes of black holes

Quasinormal modes, which manifest as damped oscillations, play a crucial role in the analysis of perturbed black holes. In this chapter, we explore the main approaches to their computation and examine how they can be rigorously defined within the framework of spectral theory. Our discussion is primarily based on [4, 6, 15, 19, 24].

1.1 Quasinormal modes

Rather than analysing the full non-linear dynamics of gravitational waves generated by perturbed black holes via the Einstein field equations, one often simplifies the problem by leveraging spacetime symmetries and linearising the perturbations. This leads to a more tractable master wave equation with an effective potential, governing the evolution of linear perturbations on a fixed background geometry.

In the spherically symmetric case, after separating angular variables using spherical harmonics, the radial part of the perturbation reduces to a one-dimensional wave equation. We consider a simplified (1+1)-dimensional model for the perturbation field $u(\tau, r)$ with an effective potential $V(r)$:

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial r^2} + V(r)u = 0, \tag{1.1}$$

where $r \in (-\infty, \infty)$ and $\tau \geq 0$. In this coordinate system, $r \to -\infty$ corresponds to the black hole horizon, while $r \to \infty$ denotes spatial infinity.

We assume that the potential $V(r) \geq 0$ is compactly supported, that is, $V(r) = 0$ for $|r| > R$, for some $R > 0$. For initial data $u(0, r)$ and $\partial_t u(0, r)$ with compact support, the solution remains bounded: $|u(\tau, r)| < C$, where $C > 0$. To analyse the equation, we apply the Laplace transform in time for $s > 0$:

$$\hat{u}(s, r) = \int_0^\infty e^{-s\tau} u(\tau, r)\, d\tau,$$

which transforms the wave equation into the inhomogeneous ordinary differential equation

$$s^2 \hat{u} - \hat{u}'' + V\hat{u} = su(0, r) + \partial_t u(0, r). \tag{1.2}$$

Due to the boundedness of u, the Laplace-transformed function $\hat{u}$ admits an analytic continuation to the half-plane $\Re(s) > 0$.

Let $\hat{u}_+$ and $\hat{u}_-$ be two linearly independent solution of the homogenous equation

$$s^2 \hat{u} - \hat{u}'' + V\hat{u} = 0. \tag{1.3}$$

We define the Green's function as

$$G(s,r,r') = \begin{cases} \frac{\hat{u}_-(s,r')\hat{u}_+(s,r)}{W(s)} & (r' < r), \\ \frac{\hat{u}_-(s,r)\hat{u}_+(s,r')}{W(s)} & (r' > r). \end{cases} \tag{1.4}$$

where $W(s)$ is the Wronskian of the solutions $\hat{u}_-$ and $\hat{u}_+$. The solution to the inhomogeneous equation (1.2) is then

$$\hat{u}(s,r) = \int_{-\infty}^{\infty} G(s,r,r') f(s,r')\, dr',$$

where the source term is $f(s,r) = su(0,r) + \partial_t u(0,r)$. Since the potential V vanishes outside the interval $[-R,R]$, the solution of (1.3) is simply

$$\hat{u}(s,r) = e^{\pm sr}.$$

To enforce the physical boundary conditions, that is, purely ingoing waves at the horizon ($r \to -\infty$) and purely outgoing waves at infinity ($r \to \infty$), we select the asymptotic solutions:

$$\hat{u}_+(s,r) = e^{-sr}, \quad r > R, \qquad \hat{u}_-(s,r) = e^{sr}, \quad r < -R,$$

for $\Re(s) > 0$. These choices ensure uniqueness of the Green's function and consistency with the physical interpretation.

The quasinormal frequencies are defined as the complex numbers s_n for which the two solutions become linearly dependent, that is,

$$\hat{u}_+(s_n,r) = \lambda(s_n)\hat{u}_-(s_n,r),$$

for some $\lambda(s_n) \in \mathbb{C}$. The corresponding solutions $\hat{u}_+(s_n,r)$ are known as quasinormal modes. In this case, the Wronskian vanishes and the Green's function is singular. Since the Green's function is well-defined and analytic in the region $\Re(s) > 0$, any singularities must lie in the half-plane $\Re(s) < 0$. The quasinormal frequencies are defined as the poles of the Green function analytically extended to $\Re(s) < 0$.

The corresponding solution of the original wave equation can be formally written as

$$u(\tau,r) = e^{s\tau}\hat{u}_{\text{sol}}(s,r),$$

where $\hat{u}_{\text{sol}}(s,r)$ is a solution of (1.3). To satisfy the boundary conditions, we have that

$$u(\tau,r) \sim e^{s(\tau \mp r)}, \quad r \to \pm\infty.$$

We can see that $-\Re(s)$ represents the decay rate and $\Im(s)$ dictates the oscillating frequency. However, since $\Re(s) < 0$ for quasinormal modes, these solutions exhibit exponential growth in the spatial coordinate r, conflicting with their physical interpretation as bounded perturbations. This motivates alternative formulations, such as the hyperboloidal approach, to regularize their asymptotic behaviour.

1.2 Hyperboloidal approach

The *hyperboloidal* (or *regular slicing*) *approach* was developed to address the singular behaviour of quasinormal modes in asymptotic regions of spacetime. This method employs a

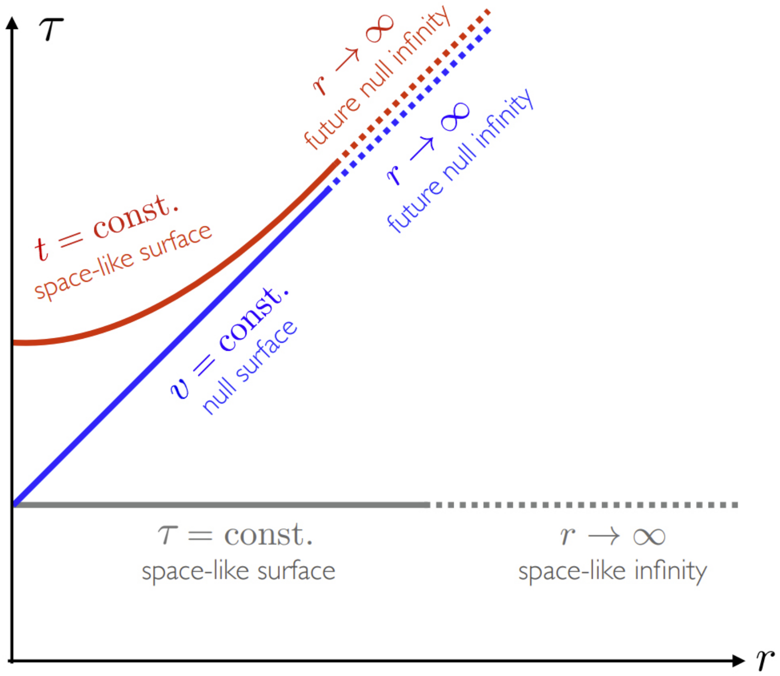

Figure 1.1: Schematic representation of a hyperboloidal spacetime slice. The coordinates τ and r denote standard time and radial coordinates respectively. The quantity $v = t - r$ defines a null "retarded time". t is the new hyperboloidal time coordinate. The grey line illustrates a standard spacelike hypersurface, while the blue line shows a null hypersurface. The red curve represents a hyperboloidal hypersurface that remains spacelike while asymptotically reaching null infinity. The figure is adapted from [13].

special foliation of spacetime, that remains regular at both the black hole event horizon and future null infinity. On one end, the slicing is adapted to horizon-regular coordinates; on the other, it reaches null infinity without encountering coordinate singularities.

The coordinate transformation underlying this approach is constructed using a *height function* and *spatial compactification*. The height function, denoted as $h(r)$, is determined by the geometry of the specific black hole spacetime. It defines a new time coordinate via the transformation

$$t = \tau + h(r),$$

where τ is the original time coordinate and t is the new hyperboloidal time. The conceptual structure of this slicing is illustrated in Figure 1.1.

The solution to the wave equation then takes the form

$$u(\tau, r) = e^{s\tau}\hat{u}_{\text{sol}}(s, r) = e^{st}e^{-sh(r)}\hat{u}_{\text{sol}}(s, r),$$

where $e^{-sh(r)}\hat{u}_{\text{sol}}(s, r)$ is now regular at both the event horizon and future null infinity. Spatial compactification is often used alongside hyperboloidal slicing to map infinite domains to finite intervals.

In the case of a Schwarzschild black hole in asymptotically flat spacetime, the metric in standard coordinates is

$$\mathrm{d}s^2 = f(\rho)\mathrm{d}t^2 - \frac{\mathrm{d}\rho^2}{f(\rho)} - \rho^2\mathrm{d}\Omega^2,$$

with

$$f(\rho) = \left(1 - \frac{2M}{\rho}\right),$$

where $\rho \in (0, \infty)$ is the standard radial coordinate. We denote by r the tortoise coordinate, defined by

$$f(\rho) = \frac{\mathrm{d}\rho}{\mathrm{d}r}.$$

In [13], a specific coordinate transformation is introduced:

$$\tau = 4Mt - 2M\left(\ln x + \ln(1-x) - \frac{1}{x}\right),$$
$$r = 2M\left(\frac{1}{x} + + \ln(1-x) - \ln x\right),$$

where $x \in [0, 1]$ is a compactified coordinate such that the event horizon lies at $x = 1$, and future null infinity at $x = 0$.

For asymptotically anti-de Sitter black hole spacetimes, as considered in [24], the wave equation (1.1) is studied over $r \in [0, \infty)$, where $r \to \infty$ corresponds to the black hole horizon and $r = 0$ to the conformal infinity. For scalar perturbations, a *Dirichlet* boundary condition $u(\tau, 0) = 0$ is imposed. The hyperboloidal transformation used in this case is:

$$x = 1 - \tanh r, \quad t = \tau - r + \tanh r,$$

with the black hole horizon located at $x = 0$ and conformal infinity at $x = 1$.

In [24], they rigorously demonstrated how quasinormal modes can be interpreted as eigenfunctions of a linear operator in the hyperboloidal framework, providing a solid foundation for spectral analysis.

1.3 Spectral theory

We follow Section 6 in [24], focusing on a concrete example of quasinormal modes in a simple (1+1)-dimensional asymptotically anti-de Sitter black hole spacetime. The spacetime is modeled as the manifold as the manifold $\mathcal{M} = [0, 1] \times \mathbb{R}_0^+$, where $\mathbb{R}_0^+ := [0, \infty)$, with coordinates (x, t), $0 \le x \le 1$, $0 \le t < \infty$. The metric is given by

$$\mathrm{d}s^2 = \frac{1}{(1-x)^2}\left[-(1-(1-x)^2)\mathrm{d}t^2 + 2(1-x)^2\mathrm{d}t\mathrm{d}x + (1+(1-x)^2)\mathrm{d}x^2\right],$$

and the mass-less wave equation for a linear scalar field $u(x, t)$ propagating on this background reads

$$(1+(1-x)^2)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(x(2-x)\frac{\partial u}{\partial x}\right) - (1-x)^2\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial}{\partial x}\left((1-x)^2\frac{\partial u}{\partial t}\right) = 0. \quad (1.5)$$

The usual notion of linear evolution equations is to associate $u(x, t)$ with a mapping $\mathbf{u}$ from t to a suitable functional space [10]. In this case, we define $\mathbf{u}$ as a mapping from $\mathbb{R}_0^+$ to $H^1(0, 1)$ by

$$[\mathbf{u}(t)](x) := u(x, t), \quad x \in [0, 1], \ t \ge 0.$$

Then, for every $g \in H^1(0,1)$ such that $g(1) = 0$, and every $h \in L^2(0,1)$, there exists a unique solution $\mathbf{u} \in L^2(\mathbb{R}_0^+, H^1(0,1))$ of the equation (1.5) satisfying the boundary condition

$$\begin{aligned} &u = g, \quad \partial_t u = h, \quad \text{on} \ \ [0,1] \times \{t = 0\}, \\ &u = 0, \quad \text{on} \ \ \{x = 1\} \times \mathbb{R}_0^+. \end{aligned} \tag{1.6}$$

This well-posedness result holds in more general settings of asymptotically anti-de Sitter black hole spacetimes as shown in [25].

Using the *Killing energy* and *redshift estimates* (see Section 3 of [24]), we obtain boundedness of the time growth of solutions. For a solution $\mathbf{u} \in L^2(\mathbb{R}_0^+, H^1(0,1))$ of (1.5) such that $\mathbf{u}' \in L^2(\mathbb{R}_0^+, L^2(0,1))$ and satisfying the boundary conditions (1.6), there exist constants $C, M > 0$ such that

$$\sup_{t\in\mathbb{R}_0^+} \left(\|\mathbf{u}(t)\|^2_{H^1(0,1)} + \|\mathbf{u}'(t)\|^2_{L^2(0,1)} \right) \le Ce^{Mt} \left(\|g\|^2_{H^1(0,1)} + \|h\|^2_{L^2(0,1)} \right), \quad \forall t \ge 0,$$

where we denoted

$$\mathbf{u}'(t) \equiv \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(t).$$

In fact, $\mathbf{u} \in C(\mathbb{R}_0^+, H^1(0,1))$ and $\mathbf{u}' \in C(\mathbb{R}_0^+, L^2(0,1))$, so the supremum is well-defined. Moreover, if the initial data is smoother, $g \in H^{k+1}(0,1)$ and $h \in H^k(0,1)$, we obtain the higher-order estimate by the *advanced redshift estimate*:

$$\sup_{t\in\mathbb{R}_0^+} \left(\|\mathbf{u}(t)\|^2_{H^{k+1}(0,1)} + \|\mathbf{u}'(t)\|^2_{H^k(0,1)} \right) \le Ce^{Mt} \left(\|g\|^2_{H^{k+1}(0,1)} + \|h\|^2_{H^k(0,1)} \right), \quad \forall t \ge 0. \tag{1.7}$$

This estimate allows us to define a C_0 semigroup. Let

$$\mathbf{H}^{k+1}(0,1) := H^{k+1}(0,1) \times H^k(0,1)$$

for $k \in \mathbb{N}_0$. We understand $H^k(0,1) \equiv L^2(0,1)$ for $k = 0$. A family $\mathcal{S} = \{\mathcal{S}(t) : t \ge 0\}$ of linear operators from $\mathbf{H}^k(0,1)$ to $\mathbf{H}^k(0,1)$ is called a C_0 **semigroup** if it satisfies the following properties:

1. $\|\mathcal{S}(t)\| < \infty, \quad \forall t \ge 0,$
2. $\mathcal{S}(t+t')f = \mathcal{S}(t)\mathcal{S}(t')f, \quad \forall f \in \mathbf{H}^k(0,1),\ \forall t, t' \ge 0,$
3. $\mathcal{S}(0)f = f, \quad \forall f \in \mathbf{H}^k(0,1),$
4. $t \longmapsto \mathcal{S}(t)f$ is continuous for $t \ge 0$ for each $f \in \mathbf{H}^k(0,1)$.

Returning to our case, we define a linear operator $\mathcal{S}(t)$ as

$$\mathcal{S}(t) : \mathbf{H}^k(0,1) \longrightarrow \mathbf{H}^k(0,1) : (g,h) \longmapsto (\mathbf{u}, \mathbf{u}')\big|_t, \quad t \ge 0.$$

The estimate (1.7) justifies such a definition. The family $\mathcal{S} = \{\mathcal{S}(t) : t \ge 0\}$ of linear operators then satisfies the conditions of a C_0 semigroup [24]. The C_0 semigroup defines a **infinitesimal generator** $\mathcal{A}$:

$$\begin{aligned} \mathrm{Dom}^k(\mathcal{A}) &:= \left\{ f \in \mathbf{H}^k(0,1) \;\middle|\; \lim_{t\to 0^+} \frac{\mathcal{S}(t)f - f}{t} \text{ exists in } \mathbf{H}^k(0,1) \right\} \\ \mathcal{A}f &:= \lim_{t\to 0^+} \frac{\mathcal{S}(t)f - f}{t}, \quad \text{for } f \in \mathrm{Dom}^k(\mathcal{A}). \end{aligned}$$

This operator has the following properties [12, Thm. 1.2.13]:

1. $\mathrm{Dom}^k(\mathcal{A})$ is dense in $\mathbf{H}^k(0,1)$,
2. $\mathcal{A}$ is a closed operator,
3. $s \in \rho(\mathcal{A})$ for each $s \in \mathbb{R}$ such that $s > M$, where M is the exponent from (1.7).

The infinitesimal generator is related to the time evolution in the following way. Formally, if we take $w(t) = \mathcal{S}(t)f$, then it solves the initial value problem

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}t}(t) &= \mathcal{A}w(t), \quad \forall t \geq 0, \\ w(0) &= f. \end{aligned}$$

Since we can rewrite the equation (1.5) as

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{1+(1-x)^2)}\left(\partial_x\left(x(2-x)\partial_x \cdot\right)\right) & \frac{1}{1+(1-x)^2)}\left((1-x)^2\partial_x + \partial_x\left((1-x)^2 \cdot\right)\right) \end{pmatrix}\begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \tag{1.8}$$

we deduce that the infinitesimal generator for the equation (1.5) has the form:

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -A_2 & -A_1 \end{pmatrix},$$

where

$$\begin{aligned} A_2 &= -\frac{1}{1+(1-x)^2)}\left(\partial_x\left(x(2-x)\partial_x \cdot\right)\right), \\ A_1 &= -\frac{1}{1+(1-x)^2)}\left((1-x)^2\partial_x \cdot + \partial_x\left((1-x)^2 \cdot\right)\right). \end{aligned}$$

In [24], the $\mathbf{H}^k$**-quasinormal spectrum** is defined as points $s \in \mathbb{C}$ such that:

1. $\Re(s) > -1/2 - k$,
2. $s \in \sigma(\mathcal{A})$.

This s is called an $\mathbf{H}^k$**-quasinormal frequency**, and the corresponding eigenvector is referred to as an $\mathbf{H}^k$**-quasinormal mode**. As the regularity increases, the construction of the quasinormal spectrum ensures that it accumulates all the quasinormal frequencies associated with the problem. Note that the quasinormal modes are, in fact, smooth functions.

Finding the spectrum of the infinitesimal generator $\mathcal{A}$ can be recast into a spectral problem of a second-order differential operator

$$\hat{L}_s = A_2 + sA_1 + s^2.$$

From the decomposition of the eigenvalue problem

$$(\mathcal{A} - s) = \begin{pmatrix} 0 & 1 \\ 1 & -A_1 - s \end{pmatrix}\begin{pmatrix} \hat{L}_s & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ -s & 1 \end{pmatrix},$$

they prove in [24] that the linear operator $\hat{L}_s$ is bijective if and only if $(\mathcal{A} - s)$ is. In the theory of operator matrices, there is a similar notion of *Schur complements* and *Frobenius–Schur factorisation* [23]. For a bounded operator matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

the **Schur complement** (of the second kind) is defined for $\lambda \in \rho(A)$ by

$$S_2(\lambda) := D - \lambda - C(A-\lambda)^{-1}B,$$

and $\lambda \in \sigma(\mathcal{A}) \setminus \sigma(A)$ if and only if $0 \in \sigma(S_2(\lambda))$ [23, Sec. 1.6]. In the case of unbounded operator matrices, it is more complicated [23, Thm. 2.3.3]. For $s \neq 0$, we formally have $\hat{L}_s = -sS_2(s)$.

We can derive $\hat{L}_s$ Laplace transformation of (1.5). Denoting

$$Lu := \partial_t^2 u + A_1 \partial_t u + A_2 u = 0,$$

we formally have

$$\begin{aligned}
\int_0^\infty e^{-st}(Lu)(x,t)\,dt &= A_2 \int_0^\infty e^{-st} u(x,t)\,dt + A_1 \int_0^\infty e^{-st} \partial_t u(x,t)\,dt \\
&\quad + \int_0^\infty e^{-st} \partial_t^2 u(x,t)\,dt \\
&= A_2 \int_0^\infty e^{-st} u(x,t)\,dt + sA_1 \int_0^\infty e^{-st} u(x,t)\,dt - A_1 u(x,0) \\
&\quad + s^2 \int_0^\infty e^{-st} u(x,t)\,dt - su(x,0) - \partial_t u(x,0) \\
&= \hat{L}_s \int_0^\infty e^{-st} u(x,t)\,dt - (A_1 + s)u(x,0) - \partial_t u(x,0).
\end{aligned}$$

In the following chapter, we further analyse the operator $L_s = w(x)\hat{L}_s$, where $w(x) = (1 + (1-x)^2)$. Since $w(x)$ is positive and bounded on $[0,1]$, invertibility of $\hat{L}_s$ is equivalent to that of L_s.

Chapter 2

Laplace-transformed operator

2.1 Definition

Taking $s \in \mathbb{C}$, we consider a differential expression

$$(\mathscr{L}_s u)(x) := -\frac{\mathrm{d}}{\mathrm{d}x}\left(x(2-x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right) - s(1-x)^2\frac{\mathrm{d}u}{\mathrm{d}x}(x) - s\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x)^2 u(x)\right) + s^2(1+(1-x)^2)\,u(x), \tag{2.1}$$

acting on smooth functions u, which are defined on the closed interval $[0,1]$. Let $C^\infty_{\mathrm{bc}}[0,1]$ be the set of all smooth functions vanishing at $x = 1$. By

$$\dot{L}_s u = \mathscr{L}_s u,$$

where $u \in C^\infty_{\mathrm{bc}}[0,1]$, we define a second-order differential operator $\dot{L}_s$ in $L^2(0,1)$ with

$$\mathrm{Dom}(\dot{L}_s) = C^\infty_{\mathrm{bc}}[0,1].$$

This operator is closable. We consider a sequence $\{u_n\}_{n=1}^\infty \subset C^\infty_{\mathrm{bc}}[0,1]$ such that for $n \to \infty$,

$$u_n \xrightarrow{L^2} 0 \quad \text{and} \quad \dot{L}_s u_n \xrightarrow{L^2} v \tag{2.2}$$

in the norm of $L^2(0,1)$, where $v \in L^2(0,1)$. The operator $\dot{L}_s$ is closable if and only if (2.2) implies $v = 0$ [14, Sec. III.5]. If we take $\varphi \in C^\infty_0(0,1)$, we have

$$(v,\varphi) = \lim_{n\to\infty}(\dot{L}_s u_n, \varphi) = \lim_{n\to\infty}(u_n, \dot{L}_{-s}\varphi) = 0$$

in the distributional sense. Since $v \in L^2(0,1)$, we find a sequence $\{\varphi_n\}_{n=1}^\infty \subset C^\infty_0(0,1)$ such that

$$\varphi_n \xrightarrow{L^2} v.$$

It follows that

$$\|v\|^2_{L^2(0,1)} = \langle v, v\rangle_{L^2(0,1)} = (\bar{v}, v) = \lim_{n\to\infty}(\bar{v},\varphi_n) = 0.$$

This proves the assertion. Denoting the closure of $\dot{L}_s$ as L_s, we have a closed and densely defined operator

$$L_s : L^2(0,1) \to L^2(0,1),$$

with

$$\mathrm{Dom}(L_s) = \overline{C^\infty_{\mathrm{bc}}[0,1]}^{\|\cdot\|_{L_s}},$$

where

$$\|\cdot\|_{L_s} := \left(\|\cdot\|^2_{L^2(0,1)} + \|\mathscr{L}_s \cdot\|^2_{L^2(0,1)}\right)^{1/2}$$

is the graph norm. The set $C^\infty_{\mathrm{bc}}[0,1]$ is called a **core** of L_s [14, Sec. III.5]. Having a differential operator with a core of smooth functions is useful for further manipulations.

In this chapter, we want to prove the following theorem, which is analogous to Lemma 1.1 in [24] in the settings of $k = 1$ with no potential.

Theorem 2.1. *Suppose that $g \in L^2(0,1)$ and $\Re(s) > -1/2$. Then either*

(i) *the equation $L_s u = g$ admits a unique solution $u \in H^1(0,1) \cap H^2_{\mathrm{loc}}(0,1]$ which vanishes at $x = 1$ for any g,*
or

(ii) *the equation $L_s u = 0$ has a non-trivial solution.*

Moreover, possibility (ii) can only occur for isolated values of s.

2.2 Injectivity

In order to prove Theorem 2.1, we review the steps in the proof of Lemma 1.1 in [24] and show them in more detail. First, we investigate the injectivity of $L_s + \gamma$ for some $\gamma \in \mathbb{R}$. Assuming $u \in C^\infty_{\mathrm{bc}}[0,1]$ and $\gamma > 1$, we multiply $L_s u + \gamma u$ by $\bar{s}\bar{u}$ and integrate it over $(0,1)$:

$$\begin{aligned}\int_0^1 \bar{s}\bar{u}(L_s u + \gamma u)\, dx = &-\int_0^1 \bar{s}\bar{u}(x(2-x)u')'\, dx - \int_0^1 |s|^2(1-x)^2 u'\bar{u}\, dx \\ &- \int_0^1 |s|^2 \bar{u}((1-x)^2 u)'\, dx + \int_0^1 \bar{s}\left[|s|^2(1+(1-x)^2) + \gamma\right] |u|^2\, dx.\end{aligned}$$

After integration by parts, and taking the real part of the equation, we obtain

$$\begin{aligned}\int_0^1 \Re\left[\bar{s}\bar{u}(L_s u + \gamma u)\right] dx = &\int_0^1 \Re(s)\left\{x(2-x)|u'|^2 + \left[|s|^2(1+(1-x)^2) + \gamma\right]|u|^2\right\} dx. \\ &+ |s|^2|u(0)|^2.\end{aligned}$$

If we assume $\Re(s) > 0$, we can divide by this the equation and we then apply the Cauchy–Schwarz and Young-type inequalities with $\varepsilon > 0$ to the left-hand side of the equation:

$$\begin{aligned}\frac{1}{\Re(s)}\int_0^1 \Re\left[\bar{s}\bar{u}(L_s u + \gamma u)\right] &\le \frac{1}{\Re(s)}\frac{\sqrt{2\varepsilon}}{\sqrt{2\varepsilon}}|s|^2\|u\|_{L^2(0,1)}\|(L_s u + \gamma u)\|_{L^2(0,1)} \\ &\le \frac{|s|^2}{4\varepsilon\Re(s)^2}\|u\|^2_{L^2(0,1)} + \varepsilon\|(L_s + \gamma)u\|^2_{L^2(0,1)}.\end{aligned}$$

Therefore,

$$\frac{|s|^2}{4\varepsilon\Re(s)^2}\|u\|^2_{L^2(0,1)} + \varepsilon\|(L_s+\gamma)u\|^2_{L^2(0,1)} \ge \int_0^1 \left\{x|u'|^2 + \gamma|u|^2\right\} dx.$$

This inequality is related to the Killing energy estimate [24]. Denoting the right-hand side of the equation as $E(u)$, we have that

$$E(u) := \int_0^1 \left\{ x|u'|^2 + \gamma |u|^2 \right\} \, dx \le \varepsilon \|(L_s + \gamma)u\|^2_{L^2(0,1)} + C^{(1)}_{s,\varepsilon} \|u\|^2_{L^2(0,1)}, \tag{2.3}$$

where $C^{(1)}_{s,\varepsilon}$ is a positive constant independent of γ.

Next, we use another estimate, which is related to the redshift estimate [24], to get u in the H^1 norm on the left-hand side. In this case, we multiply $L_s u + \gamma u$ by $-\bar{u}'$ and integrate it over $[0,1]$:

$$\begin{aligned} -\int_0^1 \bar{u}'(L_s u + \gamma u)\, dx = &\int_0^1 \bar{u}' \left(x(2-x)u' \right)' \, dx + \int_0^1 s(1-x)^2 |u'|^2 \, dx \\ &+ \int_0^1 s\bar{u}' \left((1-x)u \right)' \, dx - \int_0^1 s^2 (1 + (1-x)^2) \bar{u}' u \, dx \\ &- \int_0^1 \gamma \bar{u}' u \, dx. \end{aligned}$$

We do not yet assume anything about $s \in \mathbb{C}$. Performing integration by parts and taking the real part of the equation, we get

$$\begin{aligned} -\int_0^1 \Re \left[\bar{u}'(L_s u + \gamma u) \right] \, dx = &\int_0^1 \left[(1-x) + 2\Re(s)(1-x)^2 \right] |u'|^2 \, dx \\ &- \int_0^1 \Re \left\{ \left[s^2(1 + (1-x)^2) + 2s(1-x) \right] \bar{u}' u \right\} \, dx \\ &+ \frac{1}{2}|u'(1)|^2 + \frac{\gamma}{2}|u(0)|^2. \end{aligned}$$

By applying the Cauchy–Schwarz and Young-type inequalities for $\delta > 0$ to the terms containing u', we further obtain

$$\begin{aligned} \int_0^1 \left[(1-x) + 2\Re(s)(1-x)^2 \right] |u'|^2 \, dx \le \delta \|u'\|^2_{L^2(0,1)} &+ \frac{1}{2\delta} \|(L_s + \gamma)u\|^2_{L^2(0,1)} \\ &+ \frac{(K^{(1)}_s)^2}{2\delta} \|u\|^2_{L^2(0,1)}, \end{aligned} \tag{2.4}$$

where

$$K^{(1)}_s := \max_{x \in [0,1]} \left\{ \left| s^2 (1 + (1-x)^2) + 2s(1-x) \right| \right\}.$$

Putting $(1-x)$ aside on the left-hand side of the equation (2.4), we see that $[1 + 2\Re(s)(1-x)]$ is positive for all $x \in [0,1]$ if $\Re(s) > -1/2$. Defining a positive constant

$$K^{(2)}_s := \min_{x \in [0,1]} \{ 1 + 2\Re(s)(1-x) \},$$

and using

$$\|u\|^2_{L^2(0,1)} \le \gamma \|u\|^2_{L^2(0,1)} \le E(u),$$

we get that

$$
\begin{aligned}
K_s^{(2)}\left(\|u\|_{H^1(0,1)}^2+\gamma\|u\|_{L^2(0,1)}^2\right) &\leq \int_0^1 K_s^{(2)}(1-x)|u'|^2\,dx + K_s^{(2)}\int_0^1\left\{x|u'|^2+\gamma|u|^2\right\}\,dx \\
&\quad + K_s^{(2)}\|u\|_{L^2(0,1)}^2 \\
&\leq \int_0^1 \left[(1-x)+2\Re(s)(1-x)^2\right]|u'|^2\,dx + 2K_s^{(2)}E(u) \\
&\leq \delta\|u\|_{H^1(0,1)}^2 + \frac{1}{2\delta}\|(L_s+\gamma)u\|_{L^2(0,1)}^2 \\
&\quad + \left(\frac{(K_s^{(1)})^2}{2\delta}+2K_s^{(2)}\right)E(u).
\end{aligned}
$$

If we add $\delta\gamma\|u\|_{L^2(0,1)}^2$ to the right-hand side of the equation and consider $\delta < K_s^{(2)}$, we deduce that

$$
\|u\|_{H^1(0,1)}^2+\gamma\|u\|_{L^2(0,1)}^2 \leq C_{s,\delta}^{(2)}\left(\|(L_s+\gamma)u\|_{L^2(0,1)}^2+E(u)\right) \tag{2.5}
$$

holds with a positive constant $C_{s,\delta}^{(2)}$ independent of γ.

We now combine both estimates. In order to do that, we shift the Killing energy estimate (2.3) to the half-plane $\Re(s) > -1/2$:

$$
E(u) \leq \varepsilon\|(L_{s+\frac{1}{2}}+\gamma)u\|_{L^2(0,1)}^2 + C_{s+\frac{1}{2},\varepsilon}^{(1)}\|u\|_{L^2(0,1)}^2. \tag{2.6}
$$

If we rewrite the operator $L_{s+\frac{1}{2}}$ as

$$
L_{s+\frac{1}{2}}u = L_s u - (1-x)^2u' + (1-x)u + \left(s+\frac{1}{4}\right)(1+(1-x)^2)u,
$$

we see that

$$
\|(L_{s+\frac{1}{2}}+\gamma)u\|_{L^2(0,1)} \leq \|(L_s+\gamma)u\|_{L^2(0,1)} + \|u'\|_{L^2(0,1)} + K_s^{(3)}\|u\|_{L^2(0,1)},
$$

where

$$
K_s^{(3)} := \max_{x\in I}\left\{\left|(1-x)+\left(s+\frac{1}{4}\right)(1+(1-x)^2)\right|\right\}.
$$

Using a Young-type inequality, we get

$$
\begin{aligned}
\|(L_{s+\frac{1}{2}}+\gamma)u\|_{L^2(0,1)}^2 &\leq 3\|u'\|_{L^2(0,1)}^2 + 3\|(L_s+\gamma)u\|_{L^2(0,1)}^2 + 3(K_s^{(3)})^2\|u\|_{L^2(0,1)}^2 \\
&\leq 3\|u\|_{H^1(0,1)}^2 + 3\|(L_s+\gamma)u\|_{L^2(0,1)}^2 + 3(K_s^{(3)})^2\|u\|_{L^2(0,1)}^2.
\end{aligned}
$$

Plugging this back to the equation (2.6), we deduce that there exists a positive constant $C_{s,\varepsilon}^{(3)}$ independent of γ such that

$$
E(u) \leq 3\varepsilon\|u\|_{H^1(0,1)}^2 + C_{s,\varepsilon}^{(3)}\left(\|(L_s+\gamma)u\|_{L^2(0,1)}^2+\|u\|_{L^2(0,1)}^2\right) \tag{2.7}
$$

holds.

If we put (2.5) and (2.7) together, consider $\varepsilon < 1/(3C_{s,\delta}^{(2)})$, and add $3\varepsilon C_{s,\delta}^{(2)}\gamma\|u\|_{L^2(0,1)}^2$ to the right-hand side of the merged inequality, we obtain

$$
\|u\|_{H^1(0,1)}^2+\gamma\|u\|_{L^2(0,1)}^2 \leq C_{s,\varepsilon,\delta}^{(4)}\left(\|(L_s+\gamma)u\|_{L^2(0,1)}^2+\|u\|_{L^2(0,1)}^2\right), \tag{2.8}
$$

where $C^{(4)}_{s,\varepsilon,\delta}$ is a positive constant independent of γ. Finally, we take $\gamma \geq C^{(4)}_{s,\varepsilon,\delta}$ and get

$$\|u\|^2_{H^1(0,1)} \leq C^{(4)}_{s,\varepsilon,\delta}\|(L_s+\gamma)u\|^2_{L^2(0,1)}. \tag{2.9}$$

The equation (2.9) holds for all $u \in C^\infty_{\mathrm{bc}}[0,1]$ if $\Re(s) > -1/2$. Taking $\tilde{u} \in \mathrm{Dom}(L_s)$, there exists a Cauchy sequence $\{u_n\}_{n=1}^\infty \subset C^\infty_{\mathrm{bc}}[0,1]$ such that it converges to $\tilde{u}$ in the graph norm, that is,

$$u_n \xrightarrow{L_s} \tilde{u}.$$

It follows from the equation (2.9) that the sequence $\{u_n\}_{n=1}^\infty$ is also Cauchy in the H^1 norm. Since $H^1(0,1)$ is complete, there exists $\hat{u} \in H^1(0,1)$ such that

$$u_n \xrightarrow{H^1} \hat{u}.$$

Considering the $\mathrm{Dom}(L_s)$ and $H^1(0,1)$ as subsets of $L^2(0,1)$, we get $\tilde{u} = \hat{u}$. Therefore,

$$\mathrm{Dom}(L_s) \subseteq H^1(0,1),$$

and (2.9) holds for all $u \in \mathrm{Dom}(L_s)$. This implies that for $\Re(s) > -1/2$, the operator $L_s + \gamma$ is injective for sufficiently large γ.

2.3 Surjectivity

The next step in the proof of Theorem 2.1 is to show that for $\Re(s) > -1/2$, the operator $L_s + \gamma$ is surjective for sufficiently large γ. It follows from the equation (2.9) that the range of $L_s + \gamma$ is closed. Considering a Cauchy sequence $\{v_n\}_{n=1}^\infty \subset \mathrm{Ran}(L_s+\gamma)$ in the L^2 norm, there exists $v \in L^2(0,1)$ such that

$$v_n \xrightarrow{L^2} v.$$

We denote the inverse image of $\{v_n\}_{n=1}^\infty$ as $\{u_n\}_{n=1}^\infty \subset \mathrm{Dom}(L_s)$, that is, $(L_s+\gamma)u_n = v_n$. As a consequence of the equation (2.9), the sequence $\{u_n\}_{n=1}^\infty$ is also Cauchy in the L^2 norm. Therefore, there exists $u \in L^2(0,1)$ such that

$$u_n \xrightarrow{L^2} u.$$

Since the operator $L_s + \gamma$ is closed, we get $(L_s+\gamma)u = v$ and $v \in \mathrm{Ran}(L_s+\gamma)$. Thus the range of $L_s + \gamma$ is closed and we can prove its surjectivity by injectivity of the adjoint operator since

$$\mathrm{Ker}(L_s^* + \gamma)^\perp = \mathrm{Ran}(L_s+\gamma).$$

Using [24, Lemma 4.5], we define a second-order differential operator $\dot{L}^\dagger_s$, $s \in \mathbb{C}$, in $L^2(0,1)$ with the domain

$$C^\infty_{\mathrm{bc}*}[0,1] := \{u \in C^\infty[0,1] \mid u(0) = u(1) = 0\}$$

by

$$\dot{L}^\dagger_s u = \mathscr{L}^\dagger_s u, \quad u \in C^\infty_{\mathrm{bc}*}[0,1],$$

where

$$(\mathscr{L}^\dagger_s u)(x) := -\frac{\mathrm{d}}{\mathrm{d}x}\left(x(2-x)\frac{\mathrm{d}u}{\mathrm{d}x}(x)\right) + \bar{s}(1-x)^2\frac{\mathrm{d}u}{\mathrm{d}x}(x) + \bar{s}\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x)^2u(x)\right) + \bar{s}^2(1+(1-x)^2)u(x).$$

If we use an argument similar to that in the case of L_s, we see that $\dot{L}_s^\dagger$ is closable and we denote the closure as $L_s^\dagger$.

In [24], they use the operator $L_s^\dagger$ as the adjoint. The reason for defining the core of $L_s^\dagger$ as $C^\infty_{\mathrm{bc}*}[0,1]$ can be seen from the following equality. If we take $v \in C^\infty[0,1]$ and $u \in C^\infty_{\mathrm{bc}}[0,1]$, we get by integration by parts

$$\langle v, \mathscr{L}_s u\rangle_{L^2(0,1)} - \langle \mathscr{L}_s^\dagger v, u\rangle_{L^2(0,1)} = -\bar{v}(1)u'(1) - 2s\bar{v}(0)u(0).$$

Taking the function v from $C^\infty_{\mathrm{bc}*}[0,1]$, the boundary terms vanish. Considering $\tilde{v} \in \mathrm{Dom}(L_s^\dagger)$ and $\tilde{u} \in \mathrm{Dom}(L_s)$, there exist Cauchy sequences $\{v_n\}_{n=1}^\infty \subset C^\infty_{\mathrm{bc}*}[0,1]$ and $\{u_m\}_{m=1}^\infty \subset C^\infty_{\mathrm{bc}}[0,1]$ such that

$$v_n \xrightarrow{L_s^\dagger} \tilde{v}, \quad u_m \xrightarrow{L_s} \tilde{u}.$$

Having the equality

$$\langle v_n, L_s u_m\rangle_{L^2(0,1)} = \langle L_s^\dagger v_n, u_m\rangle_{L^2(0,1)}$$

for every $n, m \in \mathbb{N}$, we can take the limits $n, m \to \infty$ since all the sequences are bounded, and obtain

$$\langle \tilde{v}, L_s\tilde{u}\rangle_{L^2(0,1)} = \langle L_s^\dagger \tilde{v}, \tilde{u}\rangle_{L^2(0,1)}.$$

Therefore,

$$L_s^\dagger \subset L_s^*$$

holds for every $s \in \mathbb{C}$.

However, we find concrete examples of s such that

$$L_s^* \not\subset L_s^\dagger.$$

In order to show that, we use solutions of $\mathscr{L}_s u = 0$ and the fact that $\mathrm{Dom}(L_s) \subseteq H^1(0,1)$ for $\Re(s) > -1/2$.

Proposition 2.2. *Let s be a complex number such that $-1/2 < \Re(s) < 0$ or $s = 0$. Then L_s is not an extension of $(L_s^\dagger)^*$.*

Proof. We know that $u \in \mathrm{Ker}((L_s^\dagger)^*) \subseteq \mathrm{Dom}((L_s^\dagger)^*)$ if and only if

$$\langle L_s^\dagger v, u\rangle_{L^2(0,1)} = 0$$

for all $v \in \mathrm{Dom}(L_s^\dagger)$. From the definition of $L_s^\dagger$, it is sufficient to take functions from its core, that is, $C^\infty_{\mathrm{bc}*}[0,1]$. We first consider $s = 0$. If we take $u(x) = \ln[x/(2-x)] \in L^2(0,1)$, then for all $v \in C^\infty_{\mathrm{bc}*}[0,1]$

$$\begin{aligned}\langle L_0^\dagger v, u\rangle_{L^2(0,1)} &= \lim_{\varepsilon\to 0}\int_\varepsilon^1 -(x(2-x)\bar{v}')'u\,dx = \lim_{\varepsilon\to 0}\left\{-\left[x(2-x)\bar{v}'u\right]_\varepsilon^1 + \int_\varepsilon^1 x(2-x)\bar{v}'u'\,dx\right\}\\ &= \lim_{\varepsilon\to 0}\left\{\varepsilon(2-\varepsilon)\bar{v}'(\varepsilon)\ln\left(\frac{\varepsilon}{2-\varepsilon}\right) + \int_\varepsilon^1 \bar{v}'\,dx\right\} = -\lim_{\varepsilon\to 0}\bar{v}(\varepsilon) = 0.\end{aligned}$$

This implies $u \in \mathrm{Ker}((L_0^\dagger)^*)$, but since $u \notin H^1(0,1)$, we get that

$$\mathrm{Dom}((L_0^\dagger)^*) \not\subset \mathrm{Dom}(L_0).$$

Considering now $s \in \mathbb{C}$ with $\Re(s) < 0$, we take $u(x) = e^{sx}[1 - (2-x)^s x^{-s}] \in L^2(0,1)$. We have that

$$\lim_{x \to 0} u(x) = 1,$$

and thus for all $v \in C^\infty_{\mathrm{bc}*}[0,1]$ we obtain

$$\begin{aligned}\langle L_s^\dagger v, u\rangle_{L^2(0,1)} &= \langle v, \mathscr{L}_s u\rangle_{L^2(0,1)} + \lim_{\varepsilon\to 0}\left\{\left[-x(2-x)\bar{v}'u\right]_\varepsilon^1 + \left[x(2-x)\bar{v}\,u'\right]_\varepsilon^1 + \left[2s(1-x)^2\bar{v}\,u\right]_\varepsilon^1\right\}\\
&= \lim_{\varepsilon\to 0}\left\{(1-s)\varepsilon(2-\varepsilon)\bar{v}'(\varepsilon)e^{s\varepsilon}\left[1-\left(\frac{2-\varepsilon}{\varepsilon}\right)^s\right]\right\}\\
&\quad - \lim_{\varepsilon\to 0}\left\{\varepsilon(2-\varepsilon)\bar{v}(\varepsilon)e^{s\varepsilon}\frac{2}{\varepsilon^2}\left(\frac{2-\varepsilon}{\varepsilon}\right)^{s-1}\right\}\\
&\quad - \lim_{\varepsilon\to 0}\left\{2s(1-\varepsilon)^2\bar{v}(\varepsilon)e^{s\varepsilon}\left[1-\left(\frac{2-\varepsilon}{\varepsilon}\right)^s\right]\right\}\\
&= 0.\end{aligned}$$

This gives us $u \in \mathrm{Dom}((L_s^\dagger)^*)$, but $u \notin H^1(0,1)$ for $s \geq -1/2$. Thus

$$\mathrm{Dom}((L_s^\dagger)^*) \not\subset \mathrm{Dom}(L_s)$$

if $-1/2 < \Re(s) < 0$. □

Injectivity of $L_s^\dagger + \gamma$

The advantage of using the operator $L_s^\dagger$ as the adjoint would be that we could repeat the estimates of $L_s + \gamma$ in Section 2.2 to prove the injectivity of $L_s^\dagger + \gamma$. We show that if $\Re(s) \neq 1/2$, then $L_s^\dagger + \gamma$ is injective for sufficiently large γ and the domain of $L_s^\dagger$ lies in $H^1(0,1)$. We first consider $\gamma > 1$ and $u \in C^\infty_{\mathrm{bc}*}[0,1]$, and derive the estimate corresponding to the Killing energy estimate. In this case, we multiply $L_s^\dagger u + \gamma u$ by $(-s\bar{u})$, integrate it over $[0,1]$, and take the real part. Assuming $\Re(s) \neq 0$, we obtain

$$\frac{1}{\Re(s)}\int_0^1 \Re\left[\bar{s}\bar{u}(L_s u + \gamma u)\right]dx = \int_0^1 \left\{x(2-x)|u'|^2 + \left[|s|^2(1+(1-x)^2)+\gamma\right]|u|^2\right\}dx.$$

Since u vanishes at 0, there is no boundary term and we get that for $\varepsilon > 0$ and $\Re(s) \neq 0$,

$$E(u) = \int_0^1 \left\{x|u'|^2 + \gamma|u|^2\right\}dx \leq \varepsilon\|(L_s^\dagger + \gamma)u\|^2_{L^2(0,1)} + C^{*(1)}_{s,\varepsilon}\|u\|^2_{L^2(0,1)} \tag{2.10}$$

holds with a positive constant $C^{*(1)}_{s,\varepsilon}$ independent of γ. This means that this estimate works for both complex half-planes, except for the line $\Re(s) = 0$ in between.

In the second estimate which corresponds to the redshift estimate, we multiply $L_s^\dagger u + \gamma u$ by $(-\bar{u}')$, integrate it over $[0,1]$, and take the real part:

$$\begin{aligned}-\int_0^1 \Re\left[\bar{u}'(L_s^\dagger u + \gamma u)\right]dx &= -\int_0^1 \Re\left\{\left[\bar{s}^2(1+(1-x)^2) - 2\bar{s}(1-x)\right]\bar{u}'u\right\}dx\\
&\quad + \frac{1}{2}|u'(1)|^2 + \int_0^1 \left[(1-x) - 2\Re(s)(1-x)^2\right]|u'|^2\,dx.\end{aligned} \tag{2.11}$$

This estimate should ensure that the prefactor of $|u'|^2$ does not degenerate at 0 and we get u in the H^1 norm on the entire interval $[0,1]$. In Section 2.2, we use the positivity of the prefactor after putting $(1-x)$ aside. In this case, we have

$$1 - 2\Re(s)(1-x) > 0$$

for all $x \in [0,1]$ if $\Re(s) < 1/2$. Following the steps in the proof of injectivity of $L_s + \gamma$, and shifting the first estimate by $s \to s - 1/2$, we deduce that for sufficiently large γ, there exists a positive constant C such that for all $u \in C^\infty_{\mathrm{bc}*}[0,1]$

$$\|u\|^2_{H^1(0,1)} \leq C\|(L_s^\dagger + \gamma)u\|^2_{L^2(0,1)}, \tag{2.12}$$

assuming $\Re(s) < 1/2$. Since $C^\infty_{\mathrm{bc}*}[0,1]$ is the core of $L_s^\dagger$, the inequality (2.12) holds for all $u \in \mathrm{Dom}(L_s^\dagger)$ and implies

$$\mathrm{Dom}(L_s^\dagger) \subseteq H^1(0,1). \tag{2.13}$$

The vanishing boundary condition at 0 helps us to derive this result also for $\Re(s) > 1/2$. We again consider $\gamma > 1$ and $u \in C^\infty_{\mathrm{bc}*}[0,1]$. We now would like to change the sign of the equation (2.11). However, the expression $-1 + 2\Re(s)(1-x)$ with $\Re(s) > 1/2$ is positive only for x in a neighbourhood of 0 which depends on s. Other problem is that the boundary term $u'(1)$ would have the wrong sign to get the estimate. Nevertheless, both issues can be resolved by a cutoff function. Note that the redshift estimate is used for a neighbourhood of 0 and the Killing energy estimate controls the rest.

For every $s \in \mathbb{C}$ with $\Re(s) > 1/2$, there exists ε_s such that $\Re(s) > \varepsilon_s > 1/2$. We consider a smooth cutoff function χ_s such that $0 \leq \chi_s(x) \leq 1$ for all $x \in [0,1]$, $\chi_s \equiv 1$ in $[0, \delta_s/2]$, and $\mathrm{supp}(\chi_s) \subset [0, \delta_s]$, where $\delta_s := 1 - 1/(2\varepsilon_s)$. Then, the H^1 norm of u can be decomposed in the following way:

$$\begin{aligned}\|u\|_{H^1(0,1)} &= \|\chi_s u + (1-\chi_s)u\|_{H^1(0,1)} \leq \|\chi_s u\|_{H^1(0,1)} + \|(1-\chi_s)u\|_{H^1(\delta_s/2,1)} \\ &\leq 2\|\chi_s u\|_{H^1(0,1)} + \|u\|_{H^1(\delta_s/2,1)}.\end{aligned}$$

We start with the last term on the right-hand side of this equation. Since $\delta_s/2$ is positive, we have that

$$\begin{aligned}\|u\|^2_{H^1(\delta_s/2,1)} + \gamma\|u\|^2_{L^2(0,1)} &\leq \frac{2}{\delta_s}\int_{\delta_s/2}^1 \frac{\delta_s}{2}|u'|^2\,dx + \|u\|^2_{L^2(0,1)} + \gamma\|u\|^2_{L^2(0,1)} \\ &\leq \frac{2}{\delta_s}\int_{\delta_s/2}^1 x|u'|^2\,dx + \|u\|^2_{L^2(0,1)} + \gamma\frac{2}{\delta_s}\|u\|^2_{L^2(0,1)} \\ &\leq \frac{2}{\delta_s}E(u) + \|u\|^2_{L^2(0,1)}.\end{aligned}$$

Using the estimate (2.10), there exists a positive constant $\hat{C}_s^{(1)}$ such that

$$\|u\|^2_{H^1(\delta_s/2,1)} + \gamma\|u\|^2_{L^2(0,1)} \leq \frac{2}{\delta_s}\|(L_s^\dagger + \gamma)u\|^2_{L^2(0,1)} + \left(C^{*(1)}_{s,1}\frac{2}{\delta_s} + 1\right)\|u\|^2_{L^2(0,1)}$$

holds for $\Re(s) \neq 0$. If we take γ such that

$$\gamma \geq C^{*(1)}_{s,1}\frac{2}{\delta_s} + 1,$$

we obtain

$$\|u\|_{H^1(\delta_s/2,1)} \leq \sqrt{\frac{2}{\delta_s}}\|(L_s^\dagger + \gamma)u\|_{L^2(0,1)}. \tag{2.14}$$

Now we return to $\|\chi_s u\|_{H^1(0,1)}$. We plug $(\chi_s u)$ into the equation (2.11) and get

$$\begin{aligned}
-\int_0^1 \Re\left[(\chi_s\bar{u})'[L_s^\dagger(\chi_s u) + \gamma(\chi_s u)]\right] dx = &-\int_0^1 \Re\left\{[\bar{s}^2(1+(1-x)^2)](\chi_s\bar{u}')(\chi_s u)\right\} dx \\
&+ \int_0^1 \Re\left\{2\bar{s}(1-x)(\chi_s\bar{u}')(\chi_s u)\right\} dx \\
&+ \int_0^1 \left[(1-x) - 2\Re(s)(1-x)^2\right] |(\chi_s u)'|^2 \, dx.
\end{aligned}$$

Since $-1 + 2\Re(s)(1-x)$ is positive for all x in $[0, \delta_s]$, we deduce that

$$\|\chi_s u\|_{H^1(0,1)}^2 + \gamma\|\chi_s u\|_{L^2(0,1)}^2 \leq C_s^{*(2)} \left(\|(L_s^\dagger + \gamma)(\chi_s u)\|_{L^2(0,1)}^2 + E(\chi_s u)\right),$$

where $C_s^{*(2)}$ is a positive constant independent of γ. Therefore, we get

$$\begin{aligned}
\|\chi_s u\|_{H^1(0,1)}^2 + \gamma\|u\|_{L^2(0,1)}^2 &\leq \|\chi_s u\|_{H^1(0,1)}^2 + \gamma\|\chi_s u\|_{L^2(0,1)}^2 + \gamma\|u\|_{L^2(0,1)}^2 \\
&\leq C_s^{*(2)} \left(\|(L_s^\dagger + \gamma)(\chi_s u)\|_{L^2(0,1)}^2 + E(\chi_s u)\right) + E(u).
\end{aligned}$$

It follows from the properties of the function χ_s that there exist positive constants $K_{\chi_s}^{*(1)}$, $K_{\chi_s}^{*(2)}$, $K_{\chi_s}^{*(3)}$ such that

$$E(\chi_s u) \leq 2E(u) + K_{\chi_s}^{*(1)}\|u\|_{L^2(0,1)}^2,$$

and

$$\begin{aligned}
\|(L_s^\dagger + \gamma)(\chi_s u)\|_{L^2(0,1)}^2 &= \|(L_s^\dagger + \gamma)u\|_{L^2(0,\delta_s/2)}^2 + \|(L_s^\dagger + \gamma)(\chi_s u)\|_{L^2(\delta_s/2,1)}^2 \\
&\leq 2\|(L_s^\dagger + \gamma)u\|_{L^2(0,1)}^2 + K_{\chi_s}^{*(2)}\|u\|_{H^1(\delta_s/2,1)}^2 + K_{\chi_s}^{*(3)}\|u\|_{L^2(0,1)}^2 \\
&\leq 2\|(L_s^\dagger + \gamma)u\|_{L^2(0,1)}^2 + \frac{2K_{\chi_s}^{*(2)}}{\delta_s}E(u) + (K_{\chi_s}^{*(3)} + 1)\|u\|_{L^2(0,1)}^2.
\end{aligned}$$

Thus the inequality

$$\|\chi_s u\|_{H^1(0,1)}^2 + \gamma\|u\|_{L^2(0,1)}^2 \leq C_s^{*(2)} \left(\|(L_s^\dagger + \gamma)u\|_{L^2(0,1)}^2 + \|u\|_{L^2(0,1)}^2\right)$$

holds for a positive constant $C_s^{*(2)}$. Taking $\gamma \geq C_s^{*(2)}$, we get that

$$\|\chi_s u\|_{H^1(0,1)} \leq \sqrt{C_s^{*(2)}}\|(L_s^\dagger + \gamma)u\|_{L^2(0,1)}. \tag{2.15}$$

Putting (2.14) and (2.15) together, we obtain

$$\|u\|_{H^1(0,1)} \leq \left(2\sqrt{C_s^{*(2)}} + \sqrt{\frac{2}{\delta_s}}\right)\|(L_s^\dagger + \gamma)u\|_{L^2(0,1)}. \tag{2.16}$$

This gives us the injectivity of $L_s^\dagger + \gamma$ for sufficiently large γ and

$$\mathrm{Dom}(L_s^\dagger) \subseteq H^1(0,1)$$

if $\Re(s) > 1/2$.

2.4 Adjoint operator

In this section, we fully characterise the adjoint operator L_s^* for $\Re(s) > -1/2$ by the theory of second-order linear differential operators [8, Sec. III.10]. Considering the equation

$$\mathscr{L}_s u = -\left(x(2-x)u'\right)' - s(1-x)^2 u' - s\left((1-x)^2 u\right)' + s^2(1+(1-x)^2)\, u \tag{2.17}$$

on the interval $(0,1)$, $\mathscr{L}_s u$ is defined almost everywhere if u and $x(2-x)u'$ belong to $\mathrm{AC}_{\mathrm{loc}}(0,1)$. Moreover, the coefficients satisfy such conditions that the following statement holds. For any $g \in L^1_{\mathrm{loc}}(0,1)$, $c_1, c_2 \in \mathbb{C}$ and $x_0 \in (0,1)$, the equation $\mathscr{L}_s u = g$ has a unique solution which satisfies $u(x_0) = c_1$ and $(x(2-x)u')(x_0) = c_2$ [8, Thm. III.10.1].

The formal adjoint of $\mathscr{L}_s$ is

$$\mathscr{L}_s^\dagger v = -\left(x(2-x)v'\right)' + \bar{s}(1-x)^2 v' + \bar{s}\left((1-x)^2 v\right)' + \bar{s}^2(1+(1-x)^2)\, v.$$

Considering $u, v, x(2-x)u', x(2-x)v' \in \mathrm{AC}[a,b]$, $0 < a < b < 1$, we obtain the *Green's formula*

$$\int_a^b [\bar{v}(\mathscr{L}_s u) - (\overline{\mathscr{L}_s^\dagger v})u] = [u,v](b) - [u,v](a),$$

where

$$[u,v](x) := x(2-x)u(x)\overline{v'(x)} - x(2-x)u'(x)\overline{v(x)} - 2s(1-x)^2 u(x)\overline{v(x)}.$$

We eventually want to integrate over the entire interval $(0,1)$. In order to do this, we should examine the behaviour of the coefficients of $\mathscr{L}_s$ at the boundary points. The equation (2.17) is **regular** at 1, that is,

$$1 \in \mathbb{R}; \frac{1}{x(2-x)}, -\frac{2s(1-x)^2}{x(2-x)}, -2s(1-x) + s^2(1+(1-x)^2) \in L^1_{\mathrm{loc}}(0,1],$$

and **singular** at 0 since

$$\frac{1}{x(2-x)} \notin L^1_{\mathrm{loc}}[0,1).$$

Thus we shall consider the functions which are absolutely continuous up to 1, that is, $\mathrm{AC}_{\mathrm{loc}}(0,1]$.

Let $L_{s,\max}$ be a second-order differential operator defined on the domain

$$\mathrm{Dom}(L_{s,\max}) := \left\{u \in L^2(0,1) \mid u(x),\, x(2-x)u'(x) \in \mathrm{AC}_{\mathrm{loc}}(0,1];\, \mathscr{L}_s u \in L^2(0,1)\right\},$$

which acts as $L_{s,\max} u = \mathscr{L}_s u$, $u \in \mathrm{Dom}(L_{s,\max})$. This domain is maximal where we can define an operator in $L^2(0,1)$ from the differential expression $\mathscr{L}_s$. In the same way, we define the maximal operator from the formal adjoint $\mathscr{L}_s^\dagger$ with the domain

$$\mathrm{Dom}(L_{s,\max}^\dagger) := \left\{u \in L^2(0,1) \mid v(x),\, x(2-x)v'(x) \in \mathrm{AC}_{\mathrm{loc}}(0,1];\, \mathscr{L}_s^\dagger v \in L^2(0,1)\right\}.$$

Considering $u \in \mathrm{Dom}(L_{s,\max})$ and $v \in \mathrm{Dom}(L_{s,\max}^\dagger)$, we have the Green's formula

$$\int_0^1 [\bar{v}(L_{s,\max} u) - (\overline{L_{s,\max}^\dagger v})u] = [u,v](1) - \lim_{x\to 0^+}[u,v](x),$$

and the limit

$$\lim_{x\to 0^+}[u,v](x)$$

exists.

We denote the restriction of $L^{(\dagger)}_{s,\max}$ to the functions

$$\left\{u \in \mathrm{Dom}(L^{(\dagger)}_{s,\max}) \mid u(1) = u'(1) = 0,\ u \equiv 0 \text{ outside a compact subset of } (0,1]\right\}$$

as $L'^{(\dagger)}_{s,\min}$. Such operators are closable and we have

$$L^*_{s,\min} = L^\dagger_{s,\max}, \quad (L^\dagger_{s,\min})^* = L_{s,\max},$$

where $L^{(\dagger)}_{s,\min} := \overline{L'^{(\dagger)}_{s,\min}}$ is the minimal (closed) operator. Since the coefficients of $\mathscr{L}_s$ satisfy the assumptions of [8, Thm. III.10.18], the set of smooth functions that vanish outside a compact subinterval of $(0,1)$, that is, $C_0^\infty(0,1)$, is the core of the minimal operators. Moreover, using [8, Thm. III.10.19], we deduce that absolutely continuous functions are related to the functions from Sobolev spaces and get

$$\begin{aligned}\mathrm{Dom}(L_{s,\max}) &= \left\{u \in L^2(0,1) \mid u \in H^2_{\mathrm{loc}}(0,1],\, \mathscr{L}_s u \in L^2(0,1)\right\},\\ \mathrm{Dom}(L^\dagger_{s,\max}) &= \left\{v \in L^2(0,1) \mid v \in H^2_{\mathrm{loc}}(0,1],\, \mathscr{L}^\dagger_s v \in L^2(0,1)\right\}.\end{aligned}$$

We now return to our operator L_s. It follows from its definition that

$$L_{s,\min} \subset L_s \subset L_{s,\max}.$$

Thus the adjoint operator to L_s has to be a restriction of $L^\dagger_{s,\max}$, that is,

$$\mathrm{Dom}(L^*_s) \subseteq \left\{v \in L^2(0,1) \mid v \in H^2_{\mathrm{loc}}(0,1],\, \mathscr{L}^\dagger_s v \in L^2(0,1)\right\}.$$

and $L^*_s v = \mathscr{L}^\dagger_s v$, $v \in \mathrm{Dom}(L^*_s)$. In the next theorem, we determine the boundary conditions satisfied by the functions in its domain.

Proposition 2.3. *Let us define*

$$\mathrm{D}^*_s := \Big\{v \in L^2(0,1) \mid v \in H^2_{\mathrm{loc}}(0,1],\, \mathscr{L}^\dagger_s v \in L^2(0,1), v(1) = 0,$$
$$\lim_{x\to 0^+} \left[x(2-x)v'(x) - 2\bar{s}(1-x)^2 v(x)\right] = 0\Big\}.$$

Then $\mathrm{Dom}(L^*_s) = \mathrm{D}^*_s$ *for* $\Re(s) > -1/2$.

Proof. First, we fix $v \in \mathrm{Dom}(L^*_s)$. Then there exists $\eta \in L^2(0,1)$ such that

$$\langle v, L_s u\rangle_{L^2(0,1)} = \langle \eta, u\rangle_{L^2(0,1)} \tag{2.18}$$

for all $u \in C^\infty_{\mathrm{bc}}[0,1]$. Using the Green's formula, we get that $\eta = \mathscr{L}^\dagger_s v$ and the boundary terms vanish, that is,

$$-u'(1)\overline{v(1)} + \lim_{x\to 0^+}\left(x(2-x)u'(x)\overline{v(x)} - x(2-x)u(x)\overline{v'(x)} + 2s(1-x)^2 u(x)\overline{v(x)}\right) = 0. \tag{2.19}$$

Choosing $u \in C^\infty_{\text{bc}}[0,1]$ such that $u \equiv 0$ around $x = 0$ and $u \equiv x - 1$ around $x = 1$, it follows that $v(1) = 0$. On the other hand, with $u \in C^\infty_{\text{bc}}[0,1]$ such that $u \equiv 1$ around $x = 0$ and $u \equiv 0$ around $x = 1$, we conclude

$$\lim_{x\to 0^+} \left[x(2-x)v'(x) - 2\bar{s}(1-x)^2 v(x)\right] = 0. \tag{2.20}$$

This leads to the inclusion $\mathrm{Dom}(L_s^*) \subseteq \mathrm{D}_s^*$.

In order to derive the other inclusion, one may repeat the computations in the Green's formula with $v \in \mathrm{D}_s^*$ and $u \in C^\infty_{\text{bc}}[0,1]$. The only thing to show is that the condition (2.20) implies

$$\lim_{x\to 0^+} x(2-x)v(x) = 0.$$

In that case the boundary terms (2.19) vanish for all $u \in C^\infty_{\text{bc}}[0,1]$. Since this is a core of L_s, we get $\mathrm{D}_s^* \subseteq \mathrm{Dom}(L_s^*)$. We set

$$h(x) := e^{2\bar{s}x}\left(\frac{2-x}{x}\right)^{\bar{s}} v(x),$$

where we use the convention $x^0 = 1$ for all $x \in [0,1]$. Then

$$e^{-2\bar{s}x} x^{\bar{s}+1}(2-x)^{-\bar{s}+1} h' = x(2-x)v'(x) - 2\bar{s}(1-x)^2 v(x),$$

and thus it follows from $v \in H^2_{\text{loc}}(0,1]$ and the limit (2.20) that there exists a continuous function $a : (0,1] \to \mathbb{C}$ such that

$$h'(x) = \frac{a(x)}{e^{-2\bar{s}x}x^{\bar{s}+1}(2-x)^{-\bar{s}+1}}, \qquad \lim_{x\to 0^+} a(x) = 0.$$

Since $h(1) = e^{2\bar{s}}v(1) = 0$, we further obtain

$$\begin{aligned} |h(x)| &= \left|\int_x^1 \frac{a(t)}{e^{-2\bar{s}t}(2-t)^{-\bar{s}+1}t^{\bar{s}+1}}\,dt\right| \le \left\|\frac{a(x)}{e^{-2\bar{s}x}(2-x)^{-\bar{s}+1}}\right\|_{L^\infty(0,1)} \int_x^1 t^{-\Re(s)-1}\,dt \\ &\le \left\|\frac{a(x)}{e^{-2\bar{s}x}(2-x)^{-\bar{s}+1}}\right\|_{L^\infty(0,1)} \left|\frac{1}{\Re(s)}\right| \left(1 + x^{-\Re(s)}\right). \end{aligned}$$

Therefore, for $\Re(s) > -1/2$,

$$\lim_{x\to 0^+} |x(2-x)v(x)| = 2^{-\Re(s)+1} \lim_{x\to 0^+} x^{\Re(s)+1}|h(x)| \le C_s \lim_{x\to 0^+} x^{\Re(s)+1}(1 + x^{-\Re(s)}) = 0,$$

where C_s is a positive constant. □

Corollary 2.4. *For $s \neq 0$ with $-1/2 < \Re(s) < 1/2$, $\mathrm{Dom}(L_s^*) \neq \mathrm{Dom}(L_s^\dagger)$.*

Proof. We consider a cutoff function $\chi \in C^\infty[0,1]$ such that $\chi \equiv 1$ around $x = 0$ and $\chi \equiv 0$ around $x = 1$. Denoting

$$\tilde{v}(x) := \chi(x)\, e^{2\bar{s}x}\left(\frac{2-x}{x}\right)^{\bar{s}},$$

for $\Re(s) > -1/2$, we find that $\tilde{v}$ belongs to $\mathrm{Dom}(L_s^*)$ by Proposition 2.3. However, for $\Re(s) \le 1/2$, $\tilde{v}$ does not belong to $H^1(0,1)$. Since $\mathrm{Dom}(L_s^\dagger) \subseteq H^1(0,1)$ if $\Re(s) \neq 1/2$, we get the desired result. □

Together with Proposition 2.2, we have that $L_s^\dagger \neq L_s^*$ for $-1/2 < \Re(s) < 1/2$. We look more closely at two special cases: $s = 0$ and $s \in \mathbb{C}$ such that $\Re(s) > 1/2$.

Adjoint of L_0

For $s = 0$, the operator L_0 has a simple form

$$\mathscr{L}_0 u = -(x(2-x)u')'.$$

From the definition of the operator, we can see that it is symmetric. Indeed, for all $u \in C_{\mathrm{bc}}^\infty[0,1]$,

$$\langle u, L_0 u\rangle_{L^2(0,1)} = \langle L_0 u, u\rangle_{L^2(0,1)}.$$

By Proposition 2.3, the domain of L_0^* is the set

$$\mathrm{D}_0^* = \left\{ v \in L^2(0,1) \mid v \in H^2_{\mathrm{loc}}(0,1],\, \mathscr{L}_0 v \in L^2(0,1), v(1) = 0,\, \lim_{x\to 0^+} \left[x(2-x)v'(x)\right] = 0 \right\}.$$

As a consequence of Lemma A.2, we get $\mathrm{Dom}(L_0^*) \subseteq H^1(0,1)$, and use this to prove the following proposition.

Proposition 2.5. *The operator* L_0 *is selfadjoint and* $\mathrm{Dom}(L_0) = \mathrm{D}_0^*$.

Proof. We know that L_0 is a closed symmetric operator and $\mathrm{Dom}(L_0^*) = \mathrm{D}_0^*$. In order to prove that L_0 is selfadjoint, it is sufficient to show that L_0^* is also symmetric, that is, $L_0^* \subset L_0$. Using the Green's formula, we get that for $u \in \mathrm{Dom}(L_0^*)$,

$$\langle u, L_0^* u\rangle_{L^2(0,1)} - \langle L_0^* u, u\rangle_{L^2(0,1)} = \lim_{x\to 0^+} \left[x(2-x)u'(x)\overline{u(x)} - x(2-x)u(x)\overline{u'(x)} \right].$$

Since $\mathrm{Dom}(L_0^*) \subseteq H^1(0,1)$, $u \in \mathrm{AC}[0,1]$, and therefore the limit

$$\lim_{x\to 0^+} u(x)$$

is finite. This implies that the boundary term vanishes and the proposition follows. □

We see from this proposition that the domain of L_0, which is defined as a closure of the core $C_{\mathrm{bc}}^\infty[0,1]$, can be characterised as follows:

$$\mathrm{Dom}(L_0) = \left\{ u \in H^1(0,1) \mid u \in H^2_{\mathrm{loc}}(0,1],\, \mathscr{L}_0 u \in L^2(0,1), u(1) = 0 \right\}. \tag{2.21}$$

Considering $u \in \mathrm{Dom}(L_0)$, we get

$$\langle u, L_0 u\rangle_{L^2(0,1)} = \int_0^1 x(2-x)|u'(x)|^2\, dx \geq 0,$$

that is, L_0 is bounded from below. One may show that this operator corresponds to the *Friedrichs extension* [8, Sec. IV.2], [14, Sec. VI.2]. In the theory of linear operators, there is the notion of sesquilinear forms. By *The first representation theorem*, there exists a relationship between semibounded symmetric forms and semibounded selfadjoint operators.

Let l be a symmetric form defined on the product space $\mathscr{H} \times \mathscr{H}$, where $\mathscr{H}$ is a Hilbert space, with the domain $\mathrm{D}(l)$. Then l is said to be **lower semibounded** if there exists $c \in \mathbb{R}$ such that for all $u \in \mathrm{D}(l)$,

$$l[u,u] \geq c\|u\|^2.$$

Let $\{u_n\}_{n=1}^{\infty} \subset \mathrm{D}(l)$ be a sequence such that for $n, m \to \infty$,

$$u_n \to \tilde{u}$$

in the norm of $\mathscr{H}$ for some $\tilde{u} \in \mathscr{H}$, and

$$l[u_n - u_m] \to 0.$$

The lower semibounded, symmetric form l is said to be **closed** if the above assumptions imply $\tilde{u} \in \mathrm{D}(l)$. A lower semibounded, symmetric form is called **closable** if it has a closed extension.

In our case, we define the symmetric form l_0 by

$$l_0[u,v] = \langle u, \dot{L}_0 v\rangle_{L^2(0,1)}, \quad u, v \in \mathrm{D}(l_0),$$

where $\mathrm{D}(l_0) := \mathrm{Dom}(\dot{L}_0) = C_{\mathrm{bc}}^{\infty}[0,1]$. By [14, Thm. VI.1.27], the form l_0 is lower semibounded and closable, since $\dot{L}_0$ has the same properties. We denote the closure of l_0 by l_F. The domain of l_F is the closure of $\mathrm{D}(l_0)$ in the graph norm

$$\|\cdot\|_F := \left(\|\cdot\|^2_{L^2(0,1)} + \|(x(2-x))^{1/2}(\cdot)'\|^2_{L^2(0,1)}\right)^{1/2},$$

that is,

$$\mathrm{D}(l_F) = \overline{C_{\mathrm{bc}}^{\infty}[0,1]}^{\|\cdot\|_F}.$$

By the first representation theorem, there exists a unique selfadjoint operator L_F bounded from below such that $\mathrm{Dom}(L_F) \subset \mathrm{D}(l_F)$ and

$$l_F[u,v] = \langle u, L_F v\rangle_{L^2(0,1)}$$

for every $u \in \mathrm{D}(l_F)$ and $v \in \mathrm{Dom}(L_F)$. The operator L_F is thus defined on

$$\mathrm{Dom}(L_F) = \left\{u \in \mathrm{D}(l_F) \mid L_F u \in L^2(0,1)\right\}.$$

We show that $L_F = L_0$. Since they are both self-adjoint, it is sufficient to prove that

$$\mathrm{Dom}(L_F) \subset \mathrm{Dom}(L_0).$$

We first define

$$\mathrm{D}_F := \left\{u \in L^2(0,1) \mid u \in H^1_{\mathrm{loc}}(0,1];\ u(1) = 0;\ \sqrt{x(2-x)}u', (x(2-x)u')' \in L^2(0,1)\right\}$$

and show that $\mathrm{Dom}(L_F) \subset \mathrm{D}_F$. Considering $u \in \mathrm{Dom}(L_F)$, we know $\sqrt{x(2-x)}u' \in L^2(0,1)$, and thus $u \in H^1_{\mathrm{loc}}(0,1]$. It follows from the definition of $\mathrm{D}(l_F)$ that for every $u \in \mathrm{Dom}(L_F)$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset C_{\mathrm{bc}}^{\infty}[0,1]$ such that for every closed interval $K \subset (0,1]$

$$\lim_{n\to\infty} \|u - u_n\|_{H^1(K)} = 0.$$

Since we can estimate the values of the functions by the H^1 norm, that is, for every $x \in K$,

$$|u(x) - u_n(x)| \leq C\|u - u_n\|_{H^1(K)}, \tag{2.22}$$

where C is a positive constant independent of x and u, we deduce that $u \in \mathrm{Dom}(L_F)$ inherits the vanishing boundary condition at $x = 1$. This result is related to the notion of *traces* [10, Sec. 5.5]. Next, there exists $\eta \in L^2(0,1)$ such that for all $\varphi \in \mathrm{D}(l_F)$,

$$\int_0^1 x(2-x)\bar{\varphi}' u' \, dx = \int_0^1 \bar{\varphi}\eta \, dx.$$

Since $C_0^\infty(0,1) \subset \mathrm{D}(l_F)$, we know that there exists $(x(2-x)u')' = -\eta \in L^2(0,1)$ and therefore $u \in \mathrm{D}_F$.

To prove that $\mathrm{D}_F \subset \mathrm{Dom}(L_0)$, it suffices to show that for all $u \in \mathrm{D}_F$,

$$\lim_{x \to 0^+} x(2-x)u'(x) = 0.$$

If we take $u \in \mathrm{D}_F$, we get that

$$x(2-x)u'(x) = \int_0^x g(t)\, dt + c, \quad c \in \mathbb{C},$$

where $g(x) := (x(2-x)u'(x))'$. Therefore, the limit exists. If we suppose $c \neq 0$, then there exists $\delta \in (0, |c|)$ and $\varepsilon > 0$ such that for all $x \in (0, \varepsilon)$,

$$|x(2-x)u'(x)| \geq |c| - \delta > 0.$$

Hence

$$\int_0^\varepsilon x(2-x)|u'(x)|^2 \, dx \geq \int_0^\varepsilon \frac{1}{x(2-x)} \left(|c| - \delta\right) dx = \infty,$$

but this is a contradiction to $\sqrt{x(2-x)}u' \in L^2(0,1)$. Altogether, we get that $L_F = L_0$.

The operator L_F is called the Friedrichs extension of $\dot{L}_0$. The Friedrichs extension is useful for finding a selfadjoint extension of a symmetric operator. Instead of proving selfadjointness of an operator, it is easier to prove that a form is closed and symmetric. Also, the Friedrichs extension has the smallest form domain among all other selfadjoint extensions of a given operator (and their associated forms). This implies its physical relevance in quantum mechanics [18, Rem. 1.6].

Adjoint of L_s for $\Re(s) > 1/2$

Proposition 2.6. *For $\Re(s) > 1/2$,*

$$\mathrm{Dom}(L_s^*) = \left\{ v \in H^1(0,1) \mid v \in H^2_{\mathrm{loc}}(0,1],\, \mathscr{L}_0 v \in L^2(0,1),\, v(0) = v(1) = 0 \right\}. \tag{2.23}$$

Proof. We first prove that $\mathrm{Dom}(L_s^*) \subseteq H^1(0,1)$. Considering $v \in \mathrm{Dom}(L_s^*) = \mathrm{D}_s^*$, we have $\mathscr{L}_s^\dagger v \in L^2(0,1)$. Hence

$$\begin{aligned} g(x) &:= (\mathscr{L}_s^\dagger v)(x) + 2\bar{s}(1-x)v(x) - \bar{s}^2(1 + (1-x)^2)v(x) \\ &= \left(-x(2-x)v'(x) + 2\bar{s}(1-x)^2 v(x)\right)' \in L^2(0,1). \end{aligned}$$

If we define

$$h(x) := e^{2\bar{s}x}\left(\frac{2-x}{x}\right)^{\bar{s}} v(x),$$

we get that

$$e^{-2\bar{s}x}(2-x)^{-\bar{s}+1}x^{\bar{s}+1}h'(x) = x(2-x)v'(x) - 2\bar{s}(1-x)^2 v(x) = -\int_0^x g(t)\,dt \tag{2.24}$$

since

$$\lim_{x\to 0^+}[x(2-x)v'(x) - 2\bar{s}(1-x)^2 v(x)] = 0.$$

It follows from $h(1) = e^{2\bar{s}}v(1) = 0$ that

$$h(x) = \int_x^1 e^{2\bar{s}y}(2-y)^{\bar{s}-1}y^{-\bar{s}-1}\int_0^y g(t)\,dt\,dy,$$

that is,

$$v(x) = e^{-2\bar{s}x}\left(\frac{2-x}{x}\right)^{-\bar{s}}\int_x^1 e^{2\bar{s}y}(2-y)^{\bar{s}-1}y^{-\bar{s}-1}\int_0^y g(t)\,dt\,dy.$$

If we plug this into the equation (2.24), we obtain

$$\begin{aligned} v'(x) = &-\frac{1}{x(2-x)}\int_0^x g(t)\,dt \\ &+ 2\bar{s}e^{-2\bar{s}x}(1-x)^2(2-x)^{-\bar{s}-1}x^{\bar{s}-1}\int_x^1 e^{2\bar{s}y}(2-y)^{\bar{s}-1}y^{-\bar{s}-1}\int_0^y g(t)\,dt\,dy. \end{aligned}$$

Using Lemma A.1 and Lemma A.3, we deduce that $v' \in L^2(0,1)$. The fact that $v \in H^1(0,1)$ implies the rest. We get

$$(\mathscr{L}_0 v)(x) = (\mathscr{L}_s^\dagger v)(x) - 2\bar{s}(1-x)^2 v'(x) + 2\bar{s}(1-x)v(x) - \bar{s}^2(1+(1-x)^2)v(x) \in L^2(0,1),$$

and it follows from Lemma A.2 that

$$\lim_{x\to 0^+} x(2-x)v'(x) = 0, \tag{2.25}$$

therefore, $v(0) = 0$. □

This proposition helps us to show that $L_s^* + \gamma$ is injective for a sufficiently large γ if $\Re(s) > 1/2$. Using a similar approach as in the Killing energy estimate, we get that for $v \in \mathrm{Dom}(L_s^*)$,

$$\begin{aligned} \Re\langle \bar{s}v, (L_s^* + \gamma)v\rangle = &\Re(s)\int_0^1 x(2-x)|v'|^2\,dx + \Re(s)\int_0^1 [\gamma + |s|^2(1+(1-x)^2)]|v|^2\,dx \\ &+ \Re\left\{\lim_{x\to 0^+}\left[s\,x(2-x)\bar{v}(x)v'(x) - |s|^2(1-x)^2|v(x)|^2\right]\right\}. \end{aligned}$$

where $\gamma > 0$. If we consider $\Re(s) > 1/2$, it follows from (2.23) that the boundary term vanishes. Thus we deduce that there exists a positive constant C_s independent of γ such that for $v \in \mathrm{Dom}(L_s^*)$,

$$E(v) \le C_s\left(\|(L_s^* + \gamma)v\|^2_{L^2(0,1)} + \|v\|^2_{L^2(0,1)}\right).$$

It follows from

$$\gamma\|v\|^2_{L^2(0,1)} \le E(v)$$

that if we take $\gamma = C_s + 1$, then

$$\|v\|^2_{L^2(0,1)} \le C_s\|(L_s^* + \gamma)v\|^2_{L^2(0,1)}. \tag{2.26}$$

2.5 Higher regularity

It follows from (2.9) and (2.26) that for $\Re(s) > 1/2$, the operator $L_s + \gamma$ is bijective for sufficiently large γ. Although injectivity holds for $\Re(s) > -1/2$, we proved surjectivity using the adjoint operator only for $\Re(s) > 1/2$. In order to show that $L_s + \gamma$ is also surjective for $-1/2 < \Re(s) \leq 1/2$, the operator L_s is considered on more regular spaces [24]. This eventually leads to the proof of Lemma 1.1 in [24] for general k.

We define

$$L_s : \mathrm{Dom}^{k+1}(L_s) \subset H^k(0,1) \to H^k(0,1), \quad k \geq 1,$$

where $\mathrm{Dom}^{k+1}(L_s)$ is the closure of $C^\infty_{\mathrm{bc}}[0,1]$ in the graph norm

$$\| \cdot \|^2_{k,L_s} := \| \cdot \|^2_{H^k(0,1)} + \|\mathscr{L}_s \cdot \|^2_{H^k(0,1)}.$$

The operator is indeed closable. We take a sequence $\{u_n\}_{n=1}^\infty \subset C^\infty_{\mathrm{bc}}[0,1]$ such that

$$u_n \xrightarrow{H^k} 0, \quad L_s u_n \xrightarrow{H^k} u,$$

where $u \in H^k(0,1)$. If we consider $\varphi \in C_0^\infty(0,1)$, we get that for every $i \in \{0, \dots, k\}$,

$$(u^{(i)}, \varphi) = (-1)^i (u, \varphi^{(i)}) = \lim_{n\to\infty} (-1)^i (L_s u_n, \varphi^{(i)}) = \lim_{n\to\infty} (-1)^i (u_n, L_{-s}(\varphi^{(i)})) = 0,$$

where we used the notation

$$u^{(i)} \equiv \frac{\mathrm{d}^i u}{\mathrm{d}x^i}.$$

Since $u^{(i)} \in L^2(0,1)$, we find a sequence $\{\varphi_{i,n}\}_{n=1}^\infty \subset C_0^\infty(0,1)$ such that

$$\varphi_{i,n} \xrightarrow{L^2} u^{(i)},$$

and therefore,

$$\|u^{(i)}\|^2_{L^2(0,1)} = \langle u^{(i)}, u^{(i)} \rangle_{L^2(0,1)} = (\bar{u}^{(i)}, u^{(i)}) = \lim_{n\to\infty} (\bar{u}^{(i)}, \varphi_{i,n}) = 0.$$

Altogether we have

$$\|u\|^2_{H^k(0,1)} = \sum_{i=0}^{k} \|u^{(i)}\|^2_{L^2(0,1)} = 0,$$

which implies $u = 0$.

Injectivity

We prove that the operator $L_s + \gamma$ considered in more regular spaces is injective in a larger half-plane of $\mathbb{C}$. In particular, for every $s \in \mathbb{C}$ with $\Re(s) > -1/2 - k$, there exist constants $\gamma > 0$ and $C > 0$ such that for all $u \in \mathrm{Dom}^{k+1}(L_s)$

$$\|u\|^2_{H^{k+1}(0,1)} \leq C \|(L_s + \gamma)u\|^2_{H^k(0,1)}. \tag{2.27}$$

We first differentiate the equation (2.1) and get that for $u \in C^\infty_{\mathrm{bc}}[0,1]$,

$$\begin{aligned}(\mathscr{L}_s u)'(x) = &-(x(2-x)u''(x))' - [2(1-x) + 2s(1-x)^2]u''(x) \\ &+ [s + 2 - sx^2 + s^2(1 + (1-x)^2)]u'(x) - [2s + 2s^2(1-x)]u(x).\end{aligned}$$

For general $k \geq 2$, we have

$$\begin{aligned}(\mathscr{L}_s u)^{(k)}(x) = &-(x(2-x)u^{(k+1)}(x))' - [2k(1-x) + 2s(1-x)^2]u^{(k+1)}(x)\\ &+ [2k^2 + 2s(2k+1)(1-x) + s^2(1+(1-x)^2)]u^{(k)}(x)\\ &- [2sk^2 + 2s^2k(1-x)]u^{(k-1)}(x) + s^2k(k-1)u^{(k-2)}(x).\end{aligned}$$

We then repeat the estimates from the proof in Section 2.2. In this section, we will assume that all of the constants are positive and we will explicitly write only their dependence on γ. The first part of the proof is the Killing energy estimate. We take $u \in C^\infty_{\mathrm{bc}}[0,1]$ and $\gamma > 1$. For $k = 0$ and $\Re(s) > 0$, we got

$$E(u) = \int_0^1 \left\{x|u'|^2 + \gamma|u|^2\right\} dx \leq \varepsilon\|(L_s+\gamma)u\|^2_{L^2(0,1)} + C^{(0)}\|u\|^2_{L^2(0,1)}, \tag{2.28}$$

by multiplying $L_s u + \gamma u$ by $\bar{s}\bar{u}$, integrate it over $(0,1)$ and taking the real part. For $k \geq 1$, we take

$$\Re\left\{\int_0^1 \bar{s}(\overline{u^{(k)}})[(L_s u + \gamma u)^{(k)}]\, dx\right\},$$

from which we obtain

$$\begin{aligned}E(u^{(k)}) \leq\ &\varepsilon\|(L_s u + \gamma u)^{(k)}\|^2_{L^2(0,1)} + C^{(k)}\|u\|^2_{H^k(0,1)}\\ &+ \tilde{\varepsilon}\|u^{(k+1)}\|^2_{L^2(0,1)} + \frac{1}{\Re(s)}\Re\left[\bar{s}(\overline{u^{(k)}(1)})(u^{(k+1)}(1))\right]\end{aligned} \tag{2.29}$$

for $\Re(s) \neq 0$.

Note that we do not assume that the real part of s has to be positive. In the case $k = 0$, the structure of the operator L_s allows us to integrate by parts so that we did not get the term $\|u'\|^2_{L^2(0,1)}$ on the right-hand side of the equation (2.28). Instead we get the boundary term $|u(0)|^2$ and the positivity of $\Re(s)$ insures the proper sign of it. This is no longer possible in case of a general k and we obtain $\|u^{(k+1)}\|^2_{L^2(0,1)}$ on the right-hand side of the equation (2.29). Nevertheless, it is not a problem since we can make $\tilde{\varepsilon}$ arbitrarily small. There is also an additional boundary term at 1. In the case $k = 0$, this vanishes since $u(1) = 0$, but we do not assume that the higher derivatives also vanish at 1. We will deal with this term later with a different approach.

The second part of the proof is the redshift estimate. For $k = 0$, we used

$$-\Re\left[\int_0^1 (\overline{u'})(L_s u + \gamma u)\, dx\right]$$

to get

$$\int_0^1 \left[(1-x) + 2\Re(s)(1-x)^2\right]|u'|^2\, dx \leq \delta\|u'\|^2_{L^2(0,1)} + \tilde{C}^{(0)}\left(\|(L_s+\gamma)u\|^2_{L^2(0,1)} + \|u\|^2_{L^2(0,1)}\right).$$

Putting $(1-x)$ aside, $[1+2\Re(s)(1-x)]$ is always positive if $\Re(s) > -1/2$, so by adding a multiple of $E(u)$, we deduce that

$$\|u\|^2_{H^1(0,1)} + \gamma\|u\|^2_{L^2(0,1)} \leq \tilde{C}^{(0)}_1\left(\|(L_s+\gamma)u\|^2_{L^2(0,1)} + E(u)\right).$$

For $k \geq 1$, we use

$$-\Re\left\{\int_0^1 (\overline{u^{(k+1)}})[(L_s u + \gamma u)^{(k)}]\, dx\right\},$$

and obtain

$$\int_0^1 \left[(1-x)(2k+1) + 2\Re(s)(1-x)^2\right] |u^{(k+1)}|^2 \, dx$$
$$\leq \delta \|u^{(k+1)}\|^2_{L^2(0,1)} + \tilde{C}^{(k)} \left(\|(L_s u + \gamma u)^{(k)}\|^2_{L^2(0,1)} + \|u\|^2_{H^k(0,1)} \right) + \frac{1}{2}\gamma |u^{(k)}(1)|^2.$$

Putting $(1-x)$ aside again, $[2k+1+2\Re(s)(1-x)]$ is always positive if $\Re(s) > -1/2 - k$. We also have another boundary term at 1, but this estimate is used to show the H^{k+1} regularity in the neighbourhood of 0 and therefore we can use the cutoff function to erase it. If we add a multiple of $E(u^{(k)})$, we deduce that

$$\|u^{(k)}\|^2_{H^1(0,1)} + \gamma \|u^{(k)}\|^2_{L^2(0,1)} \leq \tilde{C}_1^{(k)} \left(\|(L_s u + \gamma u)^{(k)}\|^2_{L^2(0,1)} + E(u^{(k)}) + \|u\|^2_{H^k(0,1)} \right)$$
$$+ \tilde{C}_2^{(k)} \frac{1}{2} \gamma |u^{(k)}(1)|^2.$$

The third part of the proof is the shifted first estimate. Denoting $s_k := -1/2 - k$, we have

$$E(u^{(k)}) \leq \varepsilon \|(\hat{L}_{s-s_k} u + \gamma u)^{(k)}\|^2_{L^2(0,1)} + \tilde{\varepsilon} \|u^{(k+1)}\|^2_{L^2(0,1)} + C^{(k)} \|u\|^2_{H^k(0,1)}$$
$$+ \frac{1}{\Re(s) - s_k} \Re \left[(\bar{s} - s_k)(\overline{u^{(k)}(1)})(u^{(k+1)}(1)) \right]$$

for $\Re(s) > s_k$. The shifted operator $\hat{L}_{s-s_k}$ can be decomposed as follows:

$$\|(\hat{L}_{s-s_k} u + \gamma u)^{(k)}\|^2_{L^2(0,1)} \leq \tilde{C}_3^{(k)} \left(\|(L_s u + \gamma u)^{(k)}\|^2_{L^2(0,1)} + \|u^{(k)}\|^2_{H^1(0,1)} + \|u\|^2_{H^k(0,1)} \right).$$

Since the ε can be arbitrarily small, we get that

$$\|u^{(k)}\|^2_{H^1(0,1)} + \gamma \|u^{(k)}\|^2_{L^2(0,1)} \leq \hat{C}^{(k)} \left(\|(L_s u + \gamma u)^{(k)}\|^2_{L^2(0,1)} + \|u\|^2_{H^k(0,1)} \right)$$
$$+ \hat{C}_1^{(k)} \left| \overline{u^{(k)}(1)} \, u^{(k+1)}(1) \right| + \hat{C}_2^{(k)} \gamma |u^{(k)}(1)|^2.$$

Consider a smooth cut-off function χ such that $0 \leq \chi(x) \leq 1$ for all $x \in [0,1]$, $\chi \equiv 1$ in $[0, c/2]$, and $\operatorname{supp}(\chi) \subset [0,c]$, $0 < c < 1$. Hence

$$\|(\chi u)^{(k)}\|^2_{H^1(0,1)} + \gamma \|(\chi u)^{(k)}\|^2_{L^2(0,1)} \leq \hat{C}^{(k)} \left(\|[(L_s + \gamma)(\chi u)]^{(k)}\|^2_{L^2(0,1)} + \|\chi u\|^2_{H^k(0,1)} \right).$$

By adding $\|\chi u\|^2_{H^{k-1}(0,1)}$ and $\gamma \|(\chi u)^{(i)}\|^2_{L^2(0,1)}$ for every $i \in \{0, \ldots, k-1\}$, we deduce that

$$\|\chi u\|^2_{H^{k+1}(0,1)} + \gamma \|(\chi u)\|^2_{H^k(0,1)} \leq \hat{C}_3^{(k)} \left(\|(L_s + \gamma)(\chi u)\|^2_{H^k(0,1)} + \|(\chi u)\|^2_{H^k(0,1)} \right),$$

where we used the shifted Killing estimate for lower derivatives and the fact that

$$\gamma \|(\chi u)^{(i)}\|^2_{L^2(0,1)} \leq E((\chi u)^{(i)}).$$

Taking sufficiently large γ, we get that

$$\|\chi u\|^2_{H^{k+1}(0,1)} \leq \hat{C}_3^{(k)} \|(L_s + \gamma)(\chi u)\|^2_{H^k(0,1)}.$$

The right-hand side of this equation can be adjusted as follows:

$$\begin{aligned}\|(L_s+\gamma)(\chi u)\|^2_{H^k(0,1)} &\le \sum_{i=0}^{k}\left(\|(L_s u+\gamma u)^{(i)}\|^2_{L^2(0,c/2)} + \|[(L_s+\gamma)(\chi u)]^{(i)}\|^2_{L^2(c/2,1)}\right)\\ &\le \|(L_s+\gamma)u\|^2_{H^k(0,1)} + C_1^{(k)}\|u\|^2_{H^{k+1}(c/2,1)} + C_2^{(k)}\gamma\|u\|^2_{H^k(0,1))}.\end{aligned}$$

Next, we deal with the boundary term from the Killing estimate for $u\in C^\infty_{\mathrm{bc}}[0,1]$ without a cutoff function. We first consider $k=1$ where we obtained

$$\left|\overline{u'(1)}\,u''(1)\right|.$$

From the definition of L_s, we have

$$x(2-x)u'' = -(L_s+\gamma)u - 2(1-x)u' - 2s(1-x)^2u' + 2s(1-x)u + [s^2(1+(1-x)^2)+\gamma]u.$$

Since $u(1)=0$, we get

$$u''(1) = -(L_s u+\gamma u)(1),$$

and

$$\left|\overline{u'(1)}\,u''(1)\right| = \left|\overline{u'(1)}\,(L_s u+\gamma u)(1)\right| \le K^{(1)}\|u'\|_{H^1(0,1)}\|(L_s+\gamma)u\|_{H^1(0,1)}.$$

In the case $k=1$, we want to obtain

$$\|u\|^2_{H^2(0,1)} \le C\|(L_s+\gamma)u\|^2_{H^1(0,1)}.$$

Since we will get $\|u'\|_{H^1(0,1)}$ on the right-hand side, we add a small $\hat\varepsilon>0$ to get

$$\|u'\|_{H^1(0,1)}\|(L_s+\gamma)u\|_{H^1(0,1)} \le \hat\varepsilon\|u'\|^2_{H^1(0,1)} + \frac{1}{4\hat\varepsilon}\|(L_s+\gamma)u\|^2_{H^1(0,1)},$$

so we can eventually subtract the term $\|u'\|_{H^1(0,1)}$. This gives us

$$\begin{aligned}E(u') &\le \varepsilon\|(\hat L_{s-s_1}u+\gamma u)'\|^2_{L^2(0,1)} + \tilde\varepsilon\|u''\|^2_{L^2(0,1)} + C^{(1)}\|u\|^2_{H^1(0,1)}\\ &\quad+\frac{1}{\Re(s)-s_1}\Re\left[(\bar s-s_1)(\overline{u'(1)})(u''(1))\right]\\ &\le (\varepsilon+\tilde\varepsilon+\hat\varepsilon)C_3^{(1)}\|u'\|^2_{H^1(0,1)} + C_4^{(1)}\left(\|(L_s+\gamma)u\|^2_{H^1(0,1)} + \|u\|^2_{H^1(0,1)}\right).\end{aligned}$$

Hence

$$\begin{aligned}\|u\|^2_{H^2(c/2,1)} + \gamma\|u\|^2_{H^1(0,1)} &\le C_5^{(1)}(E(u')+E(u))\\ &\le \delta^{(1)}C_6^{(1)}\|u\|^2_{H^2(0,1)} + C_7^{(1)}\|(L_s+\gamma)u\|^2_{H^1(0,1)} + C_8^{(1)}\|u\|^2_{H^1(0,1)},\end{aligned}$$

where $\delta^{(1)}>0$ can be arbitrarily small. Putting it altogether, we have

$$\begin{aligned}\|u\|^2_{H^2(0,1)} + \gamma\|u\|^2_{H^1(0,1)} &= \|\chi u+(1-\chi)u\|^2_{H^2(0,1)} + \gamma\|u\|^2_{H^1(0,1)}\\ &\le 2\|\chi u\|^2_{H^2(0,1)} + 2\|(1-\chi)u\|^2_{H^2(c/2,1)} + \gamma\|u\|^2_{H^1(0,1)}\\ &\le 6\|\chi u\|^2_{H^2(0,1)} + 4\|u\|^2_{H^2(c/2,1)} + \gamma\|u\|^2_{H^1(0,1)}\\ &\le 6\,\hat C_3^{(1)}\|(L_s+\gamma)(\chi u)\|^2_{H^1(0,1)} + 4\|u\|^2_{H^2(c/2,1)} + \gamma\|u\|^2_{H^1(0,1)}\\ &\le C_9^{(1)}\|(L_s+\gamma)u\|^2_{H^1(0,1)} + C_{10}^{(1)}\left(\|u\|^2_{H^2(c/2,1)} + \gamma\|u\|^2_{H^1(0,1)}\right)\\ &\le \delta^{(1)}C_{11}^{(1)}\|u\|^2_{H^2(0,1)} + C_{12}^{(1)}\left(\|(L_s+\gamma)u\|^2_{H^1(0,1)} + \|u\|^2_{H^1(0,1)}\right).\end{aligned}$$

If we take sufficiently large γ (if it is not already) and consider $\delta^{(1)} < 1/C_{11}^{(1)}$, we deduce that

$$\|u\|_{H^2(0,1)}^2 \le C\|(L_s+\gamma)u\|_{H^1(0,1)}^2.$$

The inequality holds for all $u \in C_{\mathrm{bc}}^{\infty}[0,1]$, and therefore for all $u \in \mathrm{Dom}^2(L_s)$. This implies that the operator $L_s+\gamma$ defined on $\mathrm{Dom}^2(L_s)$ is injective for sufficiently large γ and

$$\mathrm{Dom}^2(L_s) \subset H^2(0,1)$$

if $\Re(s) > -3/2$.

Moving to a higher regularity, for $k=2$, we obtained the boundary term

$$\left|\overline{u''(1)}\, u^{(3)}(1)\right|.$$

Using $(L_s u + \gamma u)'(x)$, we get

$$u^{(3)}(1) = -(L_s u + \gamma u)'(1) + (2 + s^2 + \gamma)u'(1).$$

Hence, using the appropriate shift, we get

$$\left|\overline{u''(1)}\, u^{(3)}(1)\right| \le \hat{\varepsilon} K_1^{(2)}\|u''\|_{H^1(0,1)}^2 + K_2^{(2)}\|(L_s u+\gamma u)'\|_{H^1(0,1)}^2 + (1+\gamma)K_3^{(2)}\|u'\|_{H^1(0,1)}^2.$$

However, in the case $k=2$, we want to obtain $\|u\|_{H^2(0,1)}^2$ without a prefactor dependent on γ on the right-hand side. In order to do that, we use the other boundary condition and the fact that the prefactors of terms with $L_s+\gamma$ can depend on γ. We have that

$$\begin{aligned}\left|\overline{u''(1)}\, u^{(3)}(1)\right| &\le \left|\overline{u''(1)}\,(L_s u+\gamma u)'(1)\right| + \left|(2+s^2)\overline{u''(1)}\,u'(1)\right| + \left|\gamma\, \overline{u''(1)}\,u'(1)\right| \\ &\le \left|\overline{u''(1)}\,(L_s u+\gamma u)'(1)\right| + \left|(2+s^2)\overline{u''(1)}\,u'(1)\right| + \left|\gamma\, \overline{(L_s u+\gamma u)(1)}\,u'(1)\right|.\end{aligned}$$

This gives us

$$\begin{aligned}\left|\overline{u''(1)}\, u^{(3)}(1)\right| \le\ & \hat{\varepsilon} K_1^{(2)}\|u''\|_{H^1(0,1)}^2 + K_2^{(2)}\|(L_s u+\gamma u)'\|_{H^1(0,1)}^2 \\ &+ K_3^{(2)}\|u'\|_{H^1(0,1)}^2 + \gamma^2 K_4^{(2)}\|(L_s+\gamma)u\|_{H^1(0,1)}^2,\end{aligned}$$

and thus

$$\begin{aligned}\|u\|_{H^3(c/2,1)}^2 + \gamma\|u\|_{H^2(0,1)}^2 &\le C_5^{(2)}(E(u'') + E(u') + E(u)) \\ &\le \delta^{(2)} C_6^{(2)}\|u\|_{H^3(0,1)}^2 + C_{7,\gamma}^{(2)}\|(L_s+\gamma)u\|_{H^2(0,1)}^2 + C_8^{(2)}\|u\|_{H^2(0,1)}^2,\end{aligned}$$

where $C_{7,\gamma}^{(2)}$ is a positive constant dependent on γ. Nevertheless, if we use a similar approach as in the case $k=1$ and take sufficiently large γ (if it is not already), we deduce that

$$\|u\|_{H^3(0,1)}^2 \le C\|(L_s+\gamma)u\|_{H^2(0,1)}^2.$$

The inequality holds for all $u \in \mathrm{Dom}^3(L_s)$, which implies that for $\Re(s) > -5/2$, the operator $L_s+\gamma$ defined on $\mathrm{Dom}^3(L_s)$ is injective for sufficiently large γ and

$$\mathrm{Dom}^3(L_s) \subset H^3(0,1).$$

For general $k \geq 3$, we have the boundary condition

$$\begin{aligned} u^{(k+1)}(1) &= -(L_s u + \gamma u)^{(k-1)}(1) + (k(k-1) + s^2 + \gamma) u^{(k-1)}(1) \\ &\quad - 2(k-1)^2 u^{(k-2)}(1) + s^2 (k-1)(k-2) u^{(k-3)}(1). \end{aligned}$$

In order to estimate the boundary term

$$\left| \overline{u^{(k)}(1)}\, u^{(k+1)}(1) \right|, \tag{2.30}$$

we use the same steps as before. Again, we should be careful with the prefactor γ in the boundary condition. In the previous case, we plugged another boundary condition to solve this issue. For general k, this leads to terms with powers of γ, which could potentially be a problem. We show how to deal with this.

We want to prove that

$$\|u\|^2_{H^{k+1}(c/2,1)} + \gamma \|u\|^2_{H^k(0,1)} \leq \delta^{(k)} C_6^{(k)} \|u\|^2_{H^{k+1}(0,1)} + C_{7,\gamma}^{(k)} \|(L_s + \gamma) u\|^2_{H^k(0,1)} + C_8^{(k)} \|u\|^2_{H^k(0,1)}.$$

Since this is true for $k = 1, 2$, we may assume that

$$\gamma \|u\|^2_{H^i(0,1)} \leq \delta^{(i)} C_6^{(i)} \|u\|^2_{H^{i+1}(0,1)} + C_{7,\gamma}^{(i)} \|(L_s + \gamma) u\|^2_{H^i(0,1)} + C_8^{(i)} \|u\|^2_{H^i(0,1)} \tag{2.31}$$

holds for all $i \in \{1, \ldots, k-1\}$. Using the boundary condition for $u^{(k+1)}(1)$ in the boundary term (2.30), we obtain the problematic term

$$\gamma \left| \overline{u^{(k)}(1)}\, u^{(k-1)}(1) \right|.$$

If we plug the boundary condition for $\overline{u^{(k)}(1)}$, we have to deal with

$$\gamma^2 \left| \overline{u^{(k-2)}(1)}\, u^{(k-1)}(1) \right| + \gamma \left| \overline{u^{(k-3)}(1)}\, u^{(k-1)}(1) \right| + \gamma \left| \overline{u^{(k-4)}(1)}\, u^{(k-1)}(1) \right|. \tag{2.32}$$

It follows from (2.31) that

$$\begin{aligned} \gamma^2 \|u\|^2_{H^{k-2}(0,1)} &\leq \gamma \big(\delta^{(k-2)} C_6^{(k-2)} \|u\|^2_{H^{k-1}(0,1)} + C_{7,\gamma}^{(k-2)} \|(L_s + \gamma) u\|^2_{H^{k-2}(0,1)} \\ &\quad + C_8^{(k-2)} \|u\|^2_{H^{k-2}(0,1)} \big) \\ &\leq \delta^{(k-2)} C_6^{(k-2)} \big(\delta^{(k-1)} C_6^{(k-1)} \|u\|^2_{H^k(0,1)} + C_{7,\gamma}^{(k-1)} \|(L_s + \gamma) u\|^2_{H^{k-1}(0,1)} \\ &\quad + C_8^{(k-1)} \|u\|^2_{H^{k-1}(0,1)} \big) + \gamma C_{7,\gamma}^{(k-2)} \|(L_s + \gamma) u\|^2_{H^{k-2}(0,1)} \\ &\quad + C_8^{(k-2)} \big(\delta^{(k-2)} C_6^{(k-2)} \|u\|^2_{H^{k-1}(0,1)} + C_{7,\gamma}^{(k-2)} \|(L_s + \gamma) u\|^2_{H^{k-2}(0,1)} \\ &\quad + C_8^{(k-2)} \|u\|^2_{H^{k-2}(0,1)} \big) \\ &\leq \tilde{\delta}^{(k-2)} \tilde{C}_6^{(k-2)} \|u\|^2_{H^k(0,1)} + \tilde{C}_{7,\gamma}^{(k-2)} \|(L_s + \gamma) u\|^2_{H^{k-1}(0,1)} + \tilde{C}_8^{(k-2)} \|u\|^2_{H^{k-1}(0,1)}. \end{aligned}$$

By using this, we can estimate the last two terms in (2.32), thus we are left with the term

$$\gamma^2 \left| \overline{u^{(k-2)}(1)}\, u^{(k-1)}(1) \right|.$$

Here, we plug the boundary condition for $u^{(k-1)}(1)$ and replicate the arguments above. This gives us

$$\gamma^3 \left| \overline{u^{(k-2)}(1)}\, u^{(k-3)}(1) \right|.$$

Repeating the steps again, we eventually get

$$\gamma^{k-2}\left|\overline{u''(1)}\,u^{(3)}(1)\right|$$

if k is even, or

$$\gamma^{k-2}\left|\overline{u^{(3)}(1)}\,u''(1)\right|$$

if k is odd. We now use the boundary condition for $u''(1)$ and put the prefactor γ^{k-2} together with $L_s u + \gamma u$.

Altogether, we obtain that

$$\begin{aligned}\|u\|^2_{H^{k+1}(c/2,1)} + \gamma\|u\|^2_{H^k(0,1)} &\leq C_5^{(k)} \sum_{i=0}^{k} E(u^{(i)}) \\ &\leq \delta^{(k)} C_6^{(k)} \|u\|^2_{H^{k+1}(0,1)} + C_{7,\gamma}^{(k)} \|(L_s+\gamma)u\|^2_{H^k(0,1)} + C_8^{(k)} \|u\|^2_{H^k(0,1)}.\end{aligned}$$

Thus taking sufficiently large γ, we deduce that for all $u \in \mathrm{Dom}^{k+1}(L_s)$,

$$\|u\|^2_{H^{k+1}(0,1)} \leq C\|(L_s+\gamma)u\|^2_{H^k(0,1)}. \tag{2.33}$$

It follows from this inequality that for $\Re(s) > -1/2-k$, $\mathrm{Dom}^{k+1}(L_s) \subset H^{k+1}(0,1)$ and $L_s+\gamma$ is injective for sufficiently large γ.

2.6 Fredholm theorem

In Sections 2.2 and 2.4, we have shown that for every $s \in \mathbb{C}$ with $\Re(s) > 1/2$, there exists $\gamma > 0$ such that $L_s + \gamma$ is bijective and the equation (2.9) holds. We now use these results together with the steps from [24, Thm. 4.9] to prove Theorem 2.1 for $\Re(s) > 1/2$. The bijectivity implies that the resolvent $(L_s+\gamma)^{-1}$ exists and from the equation (2.9), we get that $(L_s+\gamma)^{-1}$ is a bounded map from $L^2(0,1)$ to $H^1(0,1)$. Since $H^1(0,1)$ is compactly imbedded in $L^2(0,1)$ [2, Thm. 6.3], the resolvent $(L_s+\gamma)^{-1} \in \mathscr{B}(L^2(0,1))$ is compact. If we fix a bounded domain $\Omega \subset \{s \in \mathbb{C} : \Re(s) > 1/2\}$, then there exists $\lambda \in \mathbb{R}$ such that for all $s \in \Omega$, $(L_s+\lambda)^{-1}$ is compact.

Note that the operator L_s has uniform domain for all $s \in \mathbb{C}$ with $\Re(s) > -1/2$:

$$\mathrm{Dom}(L_s) = \mathrm{Dom}(L_0) = \left\{u \in H^1(0,1) \mid u \in H^2_{\mathrm{loc}}(0,1],\, \mathscr{L}_0 u \in L^2(0,1), u(1) = 0\right\}.$$

Indeed, if we consider $u \in \mathrm{Dom}(L_s)$, it follows from (2.9) that $u \in H^1(0,1)$. Using the same argument as in (2.22), the boundary condition $u(1) = 0$ is preserved when we take the closure of $C^\infty_{\mathrm{bc}}[0,1]$. Finally, we have

$$\mathscr{L}_0 u = \mathscr{L}_s u + 2s(1-x)^2 u' - 2s(1-x)\,u - s^2(1+(1-x)^2)\,u \in L^2(0,1).$$

On the other hand, taking $u \in \mathrm{Dom}(L_0)$, there exists a sequence $\{u_n\}_{n=1}^\infty \subset C^\infty_{\mathrm{bc}}[0,1]$ such that

$$u_n \xrightarrow{L^2} u, \quad L_0 u_n \xrightarrow{L^2} L_0 u.$$

The equation (2.9) then implies that

$$u_n \xrightarrow{H^1} u.$$

Altogether, we get that

$$L_s u_n = L_0 u_n - 2s(1-x)^2 u_n' + 2s(1-x)\, u_n + s^2(1+(1-x)^2)\, u_n$$

converges to

$$L_0 u - 2s(1-x)^2 u' + 2s(1-x)\, u + s^2(1+(1-x)^2) u \in L^2(0,1)$$

in the L^2 norm. Since L_s is closed, $u \in \mathrm{Dom}(L_s)$.

Now, let us define

$$B(s) \coloneqq \lambda (L_s + \lambda)^{-1}.$$

We show that it is **analytic** in Ω [20, Def. 8.50], that is,

$$\lim_{s' \to s} \frac{B(s') - B(s)}{s' - s}$$

exists for all $s \in \Omega$. Using

$$\begin{aligned}(s+s')(L_{s'}+\lambda)^{-1} - 2s(L_s+\lambda)^{-1} &= (L_{s'}+\lambda)^{-1}[(s'+s)(L_s+\lambda) - 2s(L_{s'}+\lambda)](L_s+\lambda)^{-1} \\ &= (s'-s)(L_{s'}+\lambda)^{-1}\Bigg[L_0 + 2s(1-x)^2\frac{\mathrm{d}}{\mathrm{d}x} - 2s(1-x) \\ &\qquad - (2s's + s^2)(1+(1-x)^2) + \lambda\Bigg](L_s+\lambda)^{-1},\end{aligned}$$

we get that for every $u \in L^2(0,1)$,

$$\lim_{s' \to s} \|(s+s')(L_{s'}+\lambda)^{-1}u - 2s(L_s+\lambda)^{-1}u\|_{L^2(0,1)} = 0. \tag{2.34}$$

This follows from the fact that for a fixed $u \in L^2(0,1)$,

$$\left\|(L_{s'}+\lambda)^{-1}\left[L_0 + 2s(1-x)^2\frac{\mathrm{d}}{\mathrm{d}x} - 2s(1-x) - (2s's+s^2)(1+(1-x)^2) + \lambda\right](L_s+\lambda)^{-1}u\right\|_{L^2(0,1)}$$

is uniformly bounded in the complex parameters $s, s' \in \Omega$, where

$$L_0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}$$

are interpreted as bounded operators from the Banach space $\mathrm{Dom}(L_0)$ to $L^2(0,1)$ and from $H^1(0,1)$ to $L^2(0,1)$ respectively. Since

$$(L_{s'}+\lambda)^{-1} - (L_s+\lambda)^{-1} = (L_{s'}+\lambda)^{-1}(L_s - L_{s'})(L_s+\lambda)^{-1},$$

where

$$L_s - L_{s'} = 2(s'-s)(1-x)^2\frac{\mathrm{d}}{\mathrm{d}x} - 2(s'-s)(1-x) + (s^2 - s'^2)(1+(1-x)^2),$$

we deduce that for a fixed $u \in L^2$,

$$\left\|(L_{s'}+\lambda)^{-1}\left[2(1-x)^2\frac{\mathrm{d}}{\mathrm{d}x} - 2(1-x) - (s+s')(1+(1-x)^2)\right](L_s+\lambda)^{-1}u\right\|_{L^2(0,1)}$$

and

$$\left\| \frac{B(s') - B(s)}{s' - s} u + \lambda (L_s + \lambda)^{-1} \left[2(1-x)^2 \frac{\mathrm{d}}{\mathrm{d}x} - 2(1-x) + 2s(1 + (1-x)^2) \right] (L_s + \lambda)^{-1} u \right\|_{L^2(0,1)}$$

are also uniformly bounded in the complex parameters $s, s' \in \Omega$. These results, together with the equation (2.34), imply

$$\lim_{s' \to s} \frac{B(s') - B(s)}{s' - s} = -\lambda (L_s + \lambda)^{-1} \left(2(1-x)^2 \frac{\mathrm{d}}{\mathrm{d}x} - 2(1-x) + 2s(1 + (1-x)^2) \right) (L_s + \lambda)^{-1}$$

in the operator norm of $\mathscr{B}(L^2(0,1))$. Thus, the mapping $s \mapsto B(s)$ is analytic in Ω and we can apply the *Analytic Fredholm theorem* [20, Thm. 8.92].

Either

(i) $(1 - B(s))^{-1}$ exists for no $s \in \Omega$,
or

(ii) $(1 - B(s))^{-1}$ exists for all $s \in \Omega \setminus \Lambda^1_{QNF,\Omega}$ where $\Lambda^1_{QNF,\Omega}$ is a discrete subset of Ω. Moreover, the function $s \mapsto (1 - B(s))^{-1}$ is analytic in $\Omega \setminus \Lambda^1_{QNF,\Omega}$, and for $s \in \Lambda^1_{QNF,\Omega}$, $B(s)u = u$ has a finite-dimensional family of solutions.

It follows from $1 - B(s)$ being a Fredholm operator that the cokernel of this operator is also finite, and its dimension is equal to the dimension of the kernel.

An operator T with closed range is called a **Fredholm** operator if $\mathrm{nul}(T)$ and $\mathrm{def}(T)$ are finite. The index of T is defined by

$$\mathrm{ind}(T) := \mathrm{nul}(T) - \mathrm{def}(T).$$

Since the identity operator is a Fredholm operator of index 0, we get from the *second stability theorem* [14, Thm. IV.5.26] that $1 - B(s)$ has the same properties, and therefore

$$\dim \mathrm{Ker}(1 - B(s)) = \dim \mathrm{coKer}(1 - B(s)) < \infty.$$

In the sense of the proof of [10, Thm. 6.2.4], we can apply the results of $1 - B(s)$ to L_s since, for any $f \in L^2(0,1)$, u solves the equation

$$L_s u = f$$

if and only if

$$(1 - B(s))u = (L_s + \lambda)^{-1} f.$$

In order to obtain the second possibility in the Analytic Fredholm theorem, they use in [24, Thm. 4.9] the result from the semigroup theory that for $\mathbb{R} \ni s > M$, $(\mathcal{A} - s)^{-1}$ exists. It follows from [24, Lem. 4.2] that L_s^{-1} then also exists. Note that we can enlarge Ω in such a way that it always contains some $s > M$. Thus, we get for any $s \in \mathbb{C}$ with $\Re(s) > 1/2$, either

(i) $\hat{L}_s^{-1}$ exists as a bounded map from $L^2(0,1)$ to $H^1(0,1)$,
or

(ii) there exists a finite-dimensional family of solutions to $L_s u = 0$.

Possibility (ii) is obtained only for $s \in \Lambda^1_{QNF}$, where Λ^1_{QNF} is a discrete set of points that accumulate only at infinity. The function $s \mapsto L_s$ is meromorphic on $\{s : \Re(s) > 1/2\}$, with poles at Λ^1_{QNF}. This proves Theorem 2.1 for $\Re(s) > 1/2$.

There is also an alternative way of proving Theorem 2.1. Let Ω_0 be a bounded domain such that $\overline{\Omega_0} \subseteq \{s \in \mathbb{C} : \Re(s) > -1/2\}$ and $0 \in \Omega_0$. We have for all $u \in \mathrm{Dom}(L_0)$,

$$\|L_s u\|_{L^2(0,1)} \leq \|L_0 u\|_{L^2(0,1)} + 2 \sup_{s\in\Omega_0} |s| \|u\|_{H^1(0,1)} + 2 \sup_{s\in\Omega_0} |s|^2 \|u\|_{L^2(0,1)},$$

and for all $v \in L^2(0,1)$,

$$\lim_{s'\to s} \frac{\langle v, L_{s'}u\rangle_{L^2(0,1)} - \langle v, L_s u\rangle_{L^2(0,1)}}{s'-s} = \langle v, (-2(1-x)^2)u' + 2s(1+(1-x)^2)u\rangle_{L^2(0,1)}.$$

This implies that $L_s u$ is **holomorphic** for $s \in \Omega_0$ for all $u \in \mathrm{Dom}(L_0)$ [14, Sec. VII.1], that is, it is differentiable at each $s \in \Omega_0$. Satisfying the conditions of uniform domain and holomorphicity of $L_s u$, the family of operators L_s is then **holomorphic of type** (A) in Ω_0 [14, Sec. VII.2]. It follows that L_s is holomorphic in Ω_0 and thus continuous in s in the generalised sense [14, Sec. IV.2]. The generalised convergence of closed unbounded operators is defined by the "distance" between their graphs.

Returning to the equation (2.9), we deduce that there exist positive constants γ_0 and C such that

$$\|u\|^2_{H^1(0,1)} \leq C\|(L_s + \gamma_0)u\|^2_{L^2(0,1)} \tag{2.35}$$

holds for all $s \in \Omega_0$. We can easily see that the family of operators $L_s + \gamma_0$ is also holomorphic of type (A) in Ω_0. Since [8, Thm. III.10.1] implies

$$\dim \mathrm{Ker}(L_s^* + \gamma_0) \leq 2,$$

the family of operators $L_s + \gamma_0$ consists of injective operators with closed range and finite-dimensional cokernel, and thus is a holomorphic family of Fredholm operators. It then follows from [14, Thm. IV.5.17] and the generalised convergence that it has a locally constant index. Since we know that L_0 is selfadjoint, the equation (2.35) implies its surjectivity, and therefore $\mathrm{ind}(L_0 + \gamma_0) = 0$. Hence we conclude that $\mathrm{ind}(L_s + \gamma_0) = 0$ and $\gamma_0 \in \rho(L_s)$ for all $s \in \Omega_0$. Altogether we have that $(L_s + \gamma_0)^{-1} \in \mathscr{B}(L^2(0,1))$ is compact for all $s \in \Omega_0$.

At this point, we could repeat the steps above and use the Analytic Fredholm theorem, in this case for $\Re(s) > -1/2$. However, applying [14, Thm. VII.1.10] to the family of operators L_s in Ω_0, we immediately get a similar result. To obtain the possibility of finite singular points, we show that $0 \in \rho(L_0)$. The general solution of the equation $\mathscr{L}_0 u = 0$ is

$$u(x) = c_1 + c_2 \ln\left(\frac{2-x}{x}\right), \quad c_1, c_2 \in \mathbb{C}.$$

In order to belong to the domain of L_0, it must be in $H^1(0,1)$ and $u(1) = 0$. In this case, the conditions are not satisfied for any non-zero solution since

$$\ln\left(\frac{2-x}{x}\right) \notin H^1(0,1).$$

On the other hand, let $f \in L^2(0,1)$ be an arbitrary function and we define

$$u_f(x) := \int_x^1 \frac{1}{t(2-t)} \int_0^t f(s)\, ds\, dt.$$

We see that u_f is a solution of the equation $\mathscr{L}_0 u_f = f \in L^2(0,1)$, and it satisfies $u_f(1) = 0$ and $u_f \in H^2_{\text{loc}}(0,1]$. It follows from Lemma A.1 and the inequality

$$|u_f(x)|^2 \leq \left(\int_0^1 \left| \frac{1}{t(2-t)} \int_0^t f(s)\, ds \right| dt \right)^2 \leq \int_0^1 \left| \frac{1}{t(2-t)} \int_0^t f(s)\, ds \right|^2 dt$$

that $u_f \in H^1(0,1)$, and therefore, $u_f \in \text{Dom}(L_0)$.

Note that if 1 is not an eigenvalue of the compact operator $\gamma_0(L_s + \gamma_0)^{-1} \in \mathscr{B}(L^2(0,1))$, then it belongs to the resolvent set. Thus we have proved Theorem 2.1.

Corollary 2.7. *The compact operator $L_s^{-1} \in \mathscr{B}(L^2(0,1))$ exists for all $s \in \mathbb{C}$ with $\Re(s) > -1/2$.*

Proof. The general solution of the equation $\mathscr{L}_s u = 0$ for $s \neq 0$ is

$$u(x) = e^{sx} \left[c_1 + c_2 \left(\frac{2-x}{x} \right)^s \right],$$

where $c_1, c_2 \in \mathbb{C}$. Since $x^{-s} \notin H^1(0,1)$ for $\Re(s) \geq -1/2$, only the trivial solution satisfies the boundary condition $u(1) = 0$. The claim follows from Theorem 2.1 and the fact that $0 \in \rho(L_0)$. □

Chapter 3

Pseudospectrum and quasinormal modes

The pseudospectrum is a tool for analysing the instability of the spectrum arising from the non-normality of a linear operator. It provides valuable insight in various fields, such as numerical solutions of differential equations, hydrodynamic stability, and non-Hermitian quantum mechanics. The pseudospectrum quantifies how far a given operator deviates from being normal. While it does not yield exact results, it offers approximations whose bounds can be tight enough to serve as reliable indicators. Since quasinormal modes have been defined as eigenvalues of non-selfadjoint operators, their potential instability has recently been investigated using pseudospectral analysis.

3.1 Pseudospectrum

In this section, we primarily follow the book [9]. For simplicity, we define the pseudospectrum on a Hilbert space, but it can be defined accordingly on a Banach space. Let $\mathcal{A}$ be a closed, densely defined operator on a Hilbert space $\mathcal{H}$, and let $\varepsilon > 0$. The **ε-pseudospectrum** (or simply **pseudospectrum**) $\sigma_\varepsilon(\mathcal{A})$ of $\mathcal{A}$ is the set of all $z \in \mathbb{C}$ such that

$$\left\|(\mathcal{A} - z)^{-1}\right\| > \varepsilon^{-1}.$$

Some sources define the pseudospectrum using a non-strict inequality. A discussion of the differences between these definitions can be found in [22]. One potential issue with the non-strict version is the phenomenon of "jumping pseudospectra", as discussed in [9], which may lead to discontinuities in the pseudospectral sets. We adopt the standard convention that $\|(\mathcal{A}-z)^{-1}\| = \infty$ whenever $z \in \sigma(\mathcal{A})$, which is consistent with the resolvent bound

$$\left\|(\mathcal{A} - z)^{-1}\right\| \geq \frac{1}{\operatorname{dist}(z, \sigma(\mathcal{A}))}. \tag{3.1}$$

The pseudospectrum admits several equivalent characterisations. We refer to [9] for detailed proofs of these equivalences. One particularly important interpretation, relevant to spectral instability, is based on norm-bounded perturbations. A complex number z belongs to $\sigma_\varepsilon(\mathcal{A})$ if and only if $z \in \sigma(\mathcal{A} + V)$ for some $V \in \mathscr{B}(\mathcal{H})$ with $\|V\| < \varepsilon$. Another equivalent formulation involves pseudoeigenvectors. $\sigma_\varepsilon(\mathcal{A})$ is the set of $z \in \mathbb{C}$ such that $z \in \sigma(\mathcal{A})$ or

$$\|(\mathcal{A} - z)u\| < \varepsilon \tag{3.2}$$

for some $u \in \mathrm{Dom}(\mathcal{A})$ with $\|u\| = 1$. If the equation (3.2) holds for some u and ε, then z is an ε-**pseudoeigenvalue** and u is a corresponding ε-**pseudoeigenvector** (or ε-**pseudomode**, or ε-**pseudoeigenfunction**). Surprisingly, pseudoeigenfunctions could be smooth and compactly supported [17].

The pseudospectrum has the following properties:

1. $\sigma_\varepsilon(\mathcal{A})$ is a non-empty open set of $\mathbb{C}$, and any bounded connected component of $\sigma_\varepsilon(\mathcal{A})$ has a non-empty intersection with $\sigma(\mathcal{A})$. If the spectrum of $\mathcal{A}$ is empty, then $\sigma_\varepsilon(\mathcal{A})$ is unbounded.

2. Pseudospectra are strictly nested supersets of the spectrum, that is,

$$\bigcap_{\varepsilon>0} \sigma_\varepsilon(\mathcal{A}) = \sigma(\mathcal{A}).$$

3. For any $\delta > 0$

$$\sigma_{\varepsilon+\delta}(\mathcal{A}) \supseteq \sigma_\varepsilon(\mathcal{A}) + B_\delta,$$

where B_δ is the open disk of the radius δ.

The last property is related to the lower estimate of the pseudospectrum. Using the equation (3.1), we get

$$\{z \in \mathbb{C} \mid \mathrm{dist}(z, \sigma(\mathcal{A})) < \varepsilon\} \subseteq \sigma_\varepsilon(\mathcal{A}),$$

that is, the pseudospectrum always contains the ε-neighbourhood of the spectrum.

The upper estimate of the pseudospectrum can be obtained by considering the numerical range. The **numerical range** of $\mathcal{A}$ is the set

$$W(\mathcal{A}) := \{\langle u, \mathcal{A}u\rangle \mid u \in \mathrm{Dom}(\mathcal{A}),\ \|u\| = 1\}.$$

Then, we get

$$\sigma_\varepsilon(\mathcal{A}) \subseteq \{z \in \mathbb{C} \mid \mathrm{dist}(z, \overline{W(\mathcal{A})}) < \varepsilon\}, \tag{3.3}$$

that is, the pseudospectrum is contained in the ε-neighbourhood of the closure of the numerical range. For an unbounded operator $\mathcal{A}$, (3.3) holds if $\mathbb{C} \setminus \overline{W(\mathcal{A})}$ is connected and contains at least one point from the resolvent set.

Considering the adjoint operator $\mathcal{A}^*$, for any $z \in \mathbb{C}$ and $\varepsilon > 0$, we have

$$\left\|(\mathcal{A} - z)^{-1}\right\| = \left\|(\mathcal{A}^* - \bar{z})^{-1}\right\|$$

and $z \in \sigma_\varepsilon(\mathcal{A}) \iff \overline{z} \in \sigma_\varepsilon(\mathcal{A}^*)$. Additionally, if $u \in \mathrm{Dom}(H)$ is an ε-pseudoeigenvector of $\mathcal{A}$ corresponding to the ε-pseudoeigenvalue $z \notin \sigma(\mathcal{A})$, then $\mathcal{A}^*$ has an ε-pseudoeigenvector $v \in \mathrm{Dom}(\mathcal{A}^*)$ corresponding to the ε-pseudoeigenvalue $\bar{z}$. If $\mathcal{A}$ is a normal operator, the inequality in the equation (3.1) becomes an equality, which gives

$$\{z \in \mathbb{C} \mid \mathrm{dist}(z, \sigma(\mathcal{A})) < \varepsilon\} = \sigma_\varepsilon(\mathcal{A}).$$

Thus the pseudospectrum of normal operators does not give us any additional information that is not already given by the spectrum. We see that the normal operators posses the spectral stability, which is important for example in quantum mechanics, where the observables are represented by selfadjoint operators and the eigenvalues are possible measurement values. On the other hand, non-normal operators (usually referred to as non-selfadjoint operators in this context) usually suffer from the spectral instability, that is, small perturbations can drastically change the spectrum.

An important application of the pseudospectrum is in the study of time-dependent dynamical systems. Traditional spectral analysis, which focuses on eigenvalues, can be misleading in this context, as eigenvalues reflect only the asymptotic (late time) behaviour and fail to capture transient dynamics. These transient effects may arise due to the non-normality of the underlying operator. In linear dynamical systems, the pseudospectrum provides a valuable framework for assessing transient growth induced by non-normality.

We consider the first order time-dependent equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathcal{A}u.$$

If $\mathcal{A}$ is independent of t, the solution can be expressed as

$$u(t) = e^{t\mathcal{A}}u(0).$$

In the case of $\mathcal{A}$ being a matrix or bounded linear operator, $e^{t\mathcal{A}}$ is defined by a convergent power series. For a densely defined closed unbounded linear operator, the theory of C_0 semigroups is used. The norm of $e^{t\mathcal{A}}$ as a function of t gives us information about the maximal growth rate of the solution $u(t)$ in time.

Next, we define **spectral**, ε-**pseudospectral** and **numerical abscissa** of $\mathcal{A}$ as

$$\alpha(\mathcal{A}) := \sup_{z\in\sigma(\mathcal{A})} \Re(z), \qquad \alpha_\varepsilon(\mathcal{A}) := \sup_{z\in\sigma_\varepsilon(\mathcal{A})} \Re(z), \qquad \omega(\mathcal{A}) := \sup_{z\in W(\mathcal{A})} \Re(z),$$

respectively. The initial growth is determined by the numerical range. For a bounded operator $\mathcal{A}$, there exist $\gamma \in \mathbb{R}$ and $M \geq 1$ such that

$$\left\|e^{t\mathcal{A}}\right\| \leq Me^{t\gamma}, \quad \forall t \geq 0.$$

Moreover,

$$\left\|e^{t\mathcal{A}}\right\| \leq e^{t\omega(\mathcal{A})}, \quad \forall t \geq 0,$$

$$\left\|e^{t\mathcal{A}}\right\| = e^{t\omega(\mathcal{A})} + o(t), \text{ as } t \to 0,$$

and

$$\omega(\mathcal{A}) = \frac{\mathrm{d}}{\mathrm{d}t}\left\|e^{t\mathcal{A}}\right\|\Big|_{t=0} = \lim_{\varepsilon\to\infty}(\alpha_\varepsilon(\mathcal{A}) - \varepsilon).$$

In particular, $\left\|e^{t\mathcal{A}}\right\| \leq 1$ for all $t \geq 0$ (i.e. $\mathcal{A}$ is **contractive**) if and only if $\omega(\mathcal{A}) \leq 0$. This shows that the numerical abscissa can give us the exact answer about the initial slope of $\left\|e^{t\mathcal{A}}\right\|$ and can be related to the pseudospectral abscissa by $\varepsilon \to \infty$. For a unbounded operator $\mathcal{A}$, we have the following statement. If $\omega(\mathcal{A}) = \infty$, then either $\mathcal{A}$ does not generate a C_0 semigroup, or it generates but does not satisfy $\left\|e^{t\mathcal{A}}\right\| \leq e^{t\gamma}$ for any $\gamma \in \mathbb{R}$. On the other hand, suppose that $\omega(\mathcal{A})$ and $\alpha(\mathcal{A})$ are both finite. Then $\mathcal{A}$ generates a C_0 semigroup with $\left\|e^{t\mathcal{A}}\right\| \leq e^{t\gamma}$ for a $\gamma \in \mathbb{R}$, and all the properties above hold.

On the other hand, the late time behaviour is governed by the spectrum, and the spectral abscissa gives us the lower bound of $\left\|e^{t\mathcal{A}}\right\|$, that is,

$$e^{t\alpha(\mathcal{A})} \leq \left\|e^{t\mathcal{A}}\right\|, \quad \forall t \geq 0.$$

Moreover, if $\mathcal{A}$ is a matrix or a bounded operator,

$$\lim_{t\to\infty} t^{-1}\log\left\|e^{t\mathcal{A}}\right\| = \alpha(\mathcal{A}). \tag{3.4}$$

To get a sharper lower bound of $\left\|e^{t\mathcal{A}}\right\|$, especially for the transient growth, the pseudospectral abscissa is used. For any closed densely defined linear operator $\mathcal{A}$ generating a C_0 semigroup, we have

$$\sup_{t\geq 0}\left\|e^{t\mathcal{A}}\right\| \geq \frac{\alpha_\varepsilon(\mathcal{A})}{\varepsilon}.$$

If we define the **Kreiss constant** of $\mathcal{A}$ as

$$\mathcal{K}(\mathcal{A}) \equiv \sup_{\varepsilon>0}\frac{\alpha_\varepsilon(\mathcal{A})}{\varepsilon} = \sup_{\Re(z)>0}\left(\Re(z)\|(\mathcal{A}-z)^{-1}\|\right),$$

we get

$$\sup_{t\geq 0}\left\|e^{t\mathcal{A}}\right\| \geq \mathcal{K}(\mathcal{A}).$$

In order to have an estimate for the late time behaviour of unbounded operators similar to the equation (3.4), we define the **growth bound** of $\mathcal{A}$:

$$\omega_0(\mathcal{A}) \coloneqq \lim_{t\to\infty} t^{-1}\log\left\|e^{t\mathcal{A}}\right\|.$$

Such limit always exists. For unbounded operators generating C_0 semigroups, we have only the inequality

$$\omega_0(\mathcal{A}) \geq \alpha(\mathcal{A}).$$

The pseudospectrum can be used to better estimate the growth bound. Let $\mathcal{A}$ be a closed densely defined linear operator that generates a C_0 semigroup. Then

$$\omega_0(\mathcal{A}) = \lim_{\varepsilon\to 0}\alpha_\varepsilon(\mathcal{A}),$$

and the following conditions are equivalent:

$$\begin{aligned}
&\omega_0(\mathcal{A}) = \omega(\mathcal{A}),\\
&\left\|e^{t\mathcal{A}}\right\| = e^{t\omega_0(\mathcal{A})}, \quad \forall t>0,\\
&\alpha_\varepsilon(\mathcal{A}) = \omega_0(\mathcal{A}) + \epsilon, \quad \forall \varepsilon > 0.
\end{aligned}$$

3.2 Instability of quasinormal modes

In [13], the pseudospectrum was introduced as a tool for analysing the instability of quasinormal modes. As discussed in Chapter 1, the hyperboloidal approach to wave propagation allows quasinormal modes to be defined as the eigenvalues of a non-selfadjoint operator, specifically, the infinitesimal generator $\mathcal{A}$. This formulation naturally raises the possibility of spectral instability, making the pseudospectrum a suitable framework for investigating the theoretical sensitivity of quasinormal modes under general perturbations. The authors of [13] also present numerical techniques for computing the pseudospectrum of $\mathcal{A}$, offering a practical foundation for such studies.

We should emphasise that the definition of the pseudospectrum is norm-dependent. The notion of what constitutes a "small" or "large" perturbation varies with the choice of the norm.

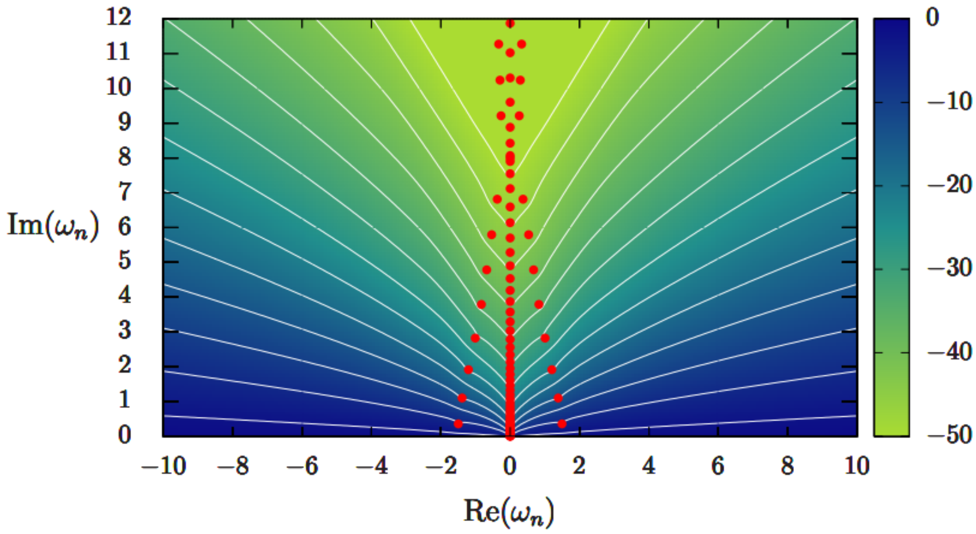

Figure 3.1: Numerical visualisation of the pseudospectrum of gravitational modes $l = 2$ in the Schwarzschild black hole space time. The quasinormal frequencies $\omega_n = -is_n$ are marked in red. White lines (contour lines) represent the points where the norms of the resolvents have the same value, that is boundaries of each ε-pseudospectrum. The colour scale on the right-side in the picture indicates the value of $\log_{10}\varepsilon$. The figure has been adapted from [13].

In physically meaningful contexts, it is crucial to adopt an appropriate *energy norm* which is derived from the energy content of the system. The energy norm proposed in [13] is constructed from the stress-energy tensor of the studied scalar field.

In [13], the pseudospectrum of the Schwarzschild black hole has been analysed in detail. The results suggest that the quasinormal overtones, that is, strongly damped modes, are spectrally unstable, with the instability increasing with the damping. In contrast, the fundamental quasinormal modes, which decay most slowly, appear to be stable. The pseudospectrum for gravitational modes with angular momentum number $l = 2$ is shown in Figure 3.1. It is important to note, however, that this method provides only a theoretical perspective on instability. For insight into whether physically relevant perturbations can actually trigger such instabilities, we refer the reader to the detailed discussion in [13].

The pseudospectrum was investigated in various spacetimes, for example Reissner–Nordström black hole spacetime [7], asymptotically de Sitter black hole spacetime [21] or asymptotically anti-de Sitter black hole spacetime [3], as well as in the case of compact horizonless objects [5]. In all cases, the pseudospectrum has a similar structure as in the case of the Schwarzshild black hole. In the case of horizonless compact objects, the transient effects in the early time were studied by evaluating the norm of $e^{t\mathcal{A}}$ and it was suggested that the operator is contractive.

Interestingly, in [26], they propose a different method to study the instability of the quasinormal modes. As we already mentioned, the problem of finding the eigenvalues of the operator $\mathcal{A}$ can be formulated as finding such s for which 0 is the eigenvalue of the Laplace-transformed operator $\hat{L}_s$. Considering de Sitter black hole spacetime, they define the "pseudospectrum" differently, using the norm of the operator $\hat{L}_s^{-1}$ instead of the resolvent $(\mathcal{A} - s)^{-1}$, that is, for $k \geq 0$

$$\tilde{\sigma}_\varepsilon^k(\mathcal{A}) \coloneqq \left\{ s \in U_k \mid \|\hat{L}_s^{-1}\|_k > \varepsilon^{-1} \right\},$$

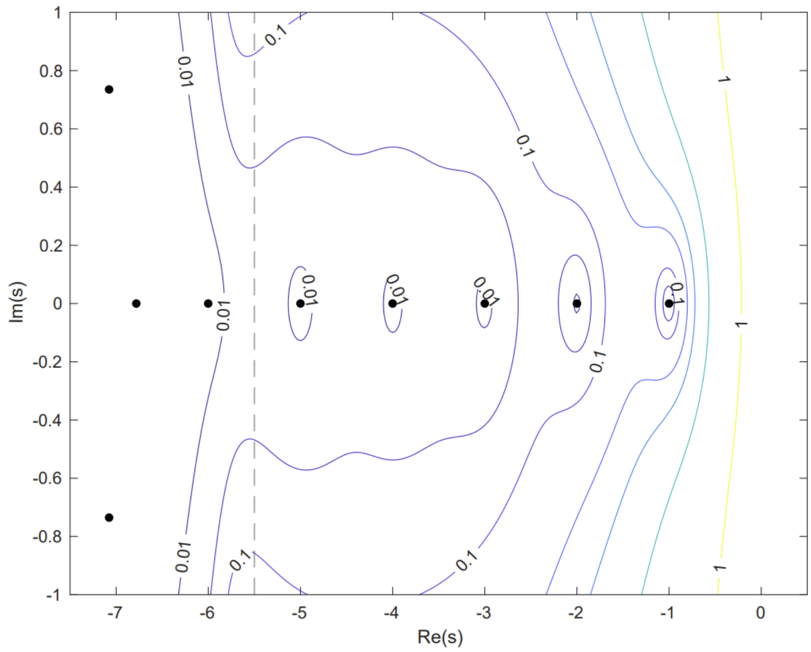

Figure 3.2: Numerical visualisation of the "pseudospectrum" for $k = 5$ in the de Sitter black hole spacetime. The black dots mark the quasinormal frequencies. The boundary of U_5 is marked by the dash line. The coloured lines (contour lines) represent the points where the norms of the resolvents have the same value, that is boundaries of each ε-"pseudospectrum". The number is the value of ε. The figure has been adapted from [26].

where U_k is a half-plane of $\mathbb{C}$ on which $\mathbf{H}^k$-quasinormal spectrum is defined, and $\|\cdot\|_k$ is the norm of the Banach space $\mathscr{B}(H^k(0,1))$. They argue that the advantage of this approach is that $\hat{L}_s^{-1}$ is compact and also not a operator matrix. On the other hand, we cannot use existing numerical libraries for computation. The numerical realisation of the "pseudospectrum" for $k = 5$ is shown in Figure 3.2

Conclusion

In this thesis, we explored the topic of quasinormal modes in black hole spectroscopy through the lens of spectral theory of linear operators. We began by introducing the definition of quasinormal modes and reviewing various methods for their computation. In the classical approach, quasinormal modes are extracted from the wave equation in standard coordinates; however, this method presents several challenges. Many of these can be addressed by adopting the approach in which the spatial slices of the spacetime foliation are hyperboloidal. In this setting, the evolution of quasinormal modes can be framed using the theory of C_0 semigroups. The wave equation governing the evolution can be recast as a first-order differential equation, with the infinitesimal generator $\mathcal{A}$. The quasinormal frequencies then appear as eigenvalues of this non-selfadjoint operator.

We illustrated this framework using the example of a (1+1)-dimensional asymptotically anti-de Sitter black hole spacetime, as studied in [24]. In that setting, energy estimates are used to define the semigroup, with the quasinormal modes appearing as smooth eigenfunctions of $\mathcal{A}$, and their frequencies as the corresponding eigenvalues. The spectral problem for the operator matrix $\mathcal{A}$ can be reduced to the invertibility of a second-order differential operator $\hat{L}_s$. This operator has the structure of a Laplace-transformed wave equation and can be obtained by the decomposition of the operator matrix.

In the second chapter, we investigated the second-order differential operator L_s, defined on $L^2(0,1)$, which is related to Laplace-transformed operator via $L_s = w(x)\hat{L}_s$, where $w(x) = (1+(1-x)^2)$ is the weight function. The main objective was to prove Theorem 2.1. We reviewed the proof of Lemma 1.1 from [24] in more detail, starting with the existence of a compact resolvent $(L_s+\gamma)^{-1}$ for some $\gamma \in \mathbb{R}$. Using energy estimates, we derived the inequality (2.9), which implies that for $\Re(s) > -1/2$, the domain of L_s is contained in $H^1(0,1)$, and $L_s+\gamma$ is injective for sufficiently large γ. In [24], the surjectivity of $L_s+\gamma$ was investigated by studying the injectivity of the adjoint operator, relying on the closed-range property implied by (2.9). From the equations (2.12) and (2.16), it was shown that for $\Re(s) \neq 1/2$, the domain of $L_s^\dagger$ lies in $H^1(0,1)$, and $L_s^\dagger+\gamma$ is injective for sufficiently large γ.

However, we provided counterexamples showing that $L_s^\dagger$ is not the adjoint operator for all $s \in \mathbb{C}$. Specifically, in Corollary 2.4, we demonstrated that for $-1/2 < \Re(s) < 1/2$, the domain of the true adjoint is not contained in $H^1(0,1)$, revealing a subtle issue in the previous analysis. For $s = 0$, the operator L_0 is selfadjoint and corresponds to the Friedrichs extension. When $\Re(s) > 1/2$, the domain of L_s^* is contained in $H^1(0,1)$, and from equation (2.26) it follows that $L_s^*+\gamma$ is injective for sufficiently large γ. Altogether we concluded that for $\Re(s) > 1/2$, the resolvent $(L_s+\gamma)^{-1}$ exists and maps $L^2(0,1)$ into $H^1(0,1)$. In [24], the operator L_s is also considered on higher-regularity spaces $H^k(0,1)$, $k \geq 1$, allowing the properties of the resolvent to be extended to the full region $\Re(s) > -1/2$. This eventually leads to the proof of Lemma 1.1 in [24] for general k. We showed for completeness that considering L_s as an operator on $H^k(0,1)$

with domain $\mathrm{Dom}^{k+1}(L_s)$ leads to the inequality (2.33), implying that for $\Re(s) > -1/2 - k$, the domain $\mathrm{Dom}^{k+1}(L_s)$ is contained in $H^{k+1}(0,1)$ and $L_s + \gamma$ injective for sufficiently large γ.

Following [24, Thm. 4.9], we applied the Analytic Fredholm Theorem to prove Theorem 2.1 for $\Re(s) > 1/2$. For the whole region $\Re(s) > -1/2$, we used an alternative approach. Considering a certain bounded domain $\Omega_0 \subset \mathbb{C}$ and $\gamma_0 \in \mathbb{R}$, we showed that $L_s + \gamma_0$ forms a holomorphic family of Fredholm operators in Ω_0. Theorem 2.1 then follows from [14, Thm. VII.1.10]. Corollary 2.7 establishes that L_s^{-1} exists as a compact operator on $L^2(0,1)$ for all $s \in \mathbb{C}$ with $\Re(s) > -1/2$. Together with [24, Lem. 4.2], this justifies the definition of quasinormal modes as eigenvalues of $\mathcal{A}$ on the half-plane $\Re(s) > -1/2$.

Finally, we discussed the notion of the pseudospectrum and its significance in the context of quasinormal mode instability. After presenting its definition and key properties, we showed how the pseudospectrum can capture transient effects in time evolution by estimating the norm of the infinitesimal generator. We then reviewed the application of pseudospectral analysis to quasinormal mode instability in [13], where it is shown that quasinormal overtones exhibit spectral instability, increasing with greater damping. Conversely, the fundamental quasinormal modes appear stable. We also introduced an alternative definition of pseudospectrum for quasinormal modes proposed in [26].

In the future, we aim to develop a deeper understanding of the operator-theoretic framework of quasinormal modes, and to further incorporate pseudospectral analysis. With the increasing precision of gravitational wave observations, the field of black hole spectroscopy offers a fertile ground for research, promising continued theoretical development and new physical insights.

Bibliography

[1] ABBOTT, B. P., et al. Observation of Gravitational Waves from a Binary Black Hole Merger. Online. *Physical Review Letters*. 2016, vol. 116, no. 6. ISSN 0031-9007. Available at: `https://doi.org/10.1103/PhysRevLett.116.061102`. [cit. 2025-05-09].

[2] ADAMS, R. A. and FOURNIER, J. J. F. *Sobolev spaces*. 2nd ed. Pure and applied mathematics. Amsterdam: Academic Press, 2003. ISBN 0-12-044143-8.

[3] AREÁN, D.; FARIÑA, D. G. and LANDSTEINER, K. Pseudospectra of holographic quasinormal modes. Online. *Journal of High Energy Physics*. 2023, vol. 2023, no. 12. ISSN 1029-8479. Available at: `https://doi.org/10.1007/JHEP12(2023)187`. [cit. 2025-05-09].

[4] BERTI, E.; CARDOSO, V. and STARINETS, A. O. Quasinormal modes of black holes and black branes. Online. *Classical and Quantum Gravity*. 2009, vol. 26, no. 16. ISSN 0264-9381. Available at: `https://doi.org/10.1088/0264-9381/26/16/163001`. [cit. 2025-05-09].

[5] BOYANOV, V.; DESTOUNIS, K.; MACEDO, R. P.; CARDOSO, V. a JARAMILLO, J. L. Pseudospectrum of horizonless compact objects: A bootstrap instability mechanism. Online. *Physical Review D*. 2023, vol. 107, no. 6. ISSN 2470-0010. Available at: `https://doi.org/10.1103/PhysRevD.107.064012` [cit. 2025-05-09].

[6] DESTOUNIS, K. and DUQUE, F. Black-hole spectroscopy: quasinormal modes, ringdown stability and the pseudospectrum. Online. In: PAPANTONOPOULOS, E. and MAVROMATOS, N. (ed.). *Compact objects in the Universe*. Cham: Springer Nature Switzerland, 2024. p. 155-202. ISBN 978-3-031-55097-3. Available at: `https://doi.org/10.1007/978-3-031-55098-0_6`. [cit. 2025-05-09].

[7] DESTOUNIS, K.; MACEDO, R. P.; BERTI, E.; CARDOSO, V. and JARAMILLO, J. L. Pseudospectrum of Reissner-Nordström black holes: Quasinormal mode instability and universality. Online. *Physical Review D*. 2021, vol. 104, no. 8. ISSN 2470-0010. Avaible at: `https://doi.org/10.1103/PhysRevD.104.084091`. [cit. 2025-05-09].

[8] EDMUNDS, D. E. and EVANS, W. D. *Spectral Theory and Differential Operators*. 2nd ed. Oxford mathematical monographs. Oxford: Oxford University Press, 2018. ISBN 978-0-19-881205-0.

[9] EMBREE, M. and TREFETHEN, L. N. *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*. Princeton: Princeton University Press, 2005. ISBN 978-0-691-11946-5.

[10] EVANS, L. C. *Partial Differential Equations*. 2nd ed. Providence: American Mathematical Society, 2010. ISBN 978-0-8218-4974-3.

[11] GASPERÍN, E. and JARAMILLO, J. L. Energy scales and black hole pseudospectra: the structural role of the scalar product. Online. *Classical and Quantum Gravity*. 2022, vol. 39, no. 11. ISSN 0264-9381. Avaible at: `https://doi.org/10.1088/1361-6382/ac5054`. [cit. 2025-05-09].

[12] GOLDSTEIN, J. A. *Semigroups of Linear Operators and Applications*. Oxford Mathematical Monographs. New York: Oxford Univ.Press, 1985. ISBN 0-19-503540-2.

[13] JARAMILLO, J. L.; MACEDO, R. P. and SHEIKH, L. A. Pseudospectrum and black hole quasinormal mode instability. Online. *Physical Review X*. 2021, vol. 11, no. 3. ISSN 2160-3308. Available at: `https://doi.org/10.1103/PhysRevX.11.031003`. [cit. 2025-05-09].

[14] KATO, T. *Perturbation theory for linear operators*. 2nd ed. Grundlehren der mathematischen Wissenschaften. Berlin: Springer, 1980. ISBN 0-387-07558-5.

[15] KOKKOTAS, K. D. and SCHMIDT, B. G. Quasi-Normal Modes of Stars and Black Holes. Online. *Living Reviews in Relativity*. 1999, vol. 2, no. 1. ISSN 2367-3613. Available at: `https://doi.org/10.12942/lrr-1999-2`. [cit. 2025-05-09].

[16] KONOPLYA, R. A. and ZHIDENKO, A. Quasinormal modes of black holes: From astrophysics to string theory. Online. *Reviews of Modern Physics*. 2011, vol. 83, no. 3, pp. 793-836. ISSN 0034-6861. Available at: `https://doi.org/10.1103/RevModPhys.83.793`. [cit. 2025-05-09].

[17] KREJČIŘÍK, D.; SIEGL, P.; TATER, M. and VIOLA, J. Pseudospectra in non-Hermitian quantum mechanics. Online. *Journal of Mathematical Physics*. 2015, vol. 56, no. 10. ISSN 0022-2488. Available at: `https://doi.org/10.1063/1.4934378`. [cit. 2025-05-09].

[18] KREJČIŘÍK, D. *Geometrical aspects of spectral theory*. Geometrical aspects of spectral theory course lecture notes. Available at: `https://nsa.fjfi.cvut.cz/david/other/gspec.pdf`. [cit. 2025-05-09].

[19] MACEDO, R. P. and ZENGINOĞLU, A. Hyperboloidal Approach to Quasinormal Modes. Online. *Frontiers in Physics*. 2025, vol. 12. ISSN 2296-424X. Available at: `https://doi.org/10.3389/fphy.2024.1497601`. [cit. 2025-05-09].

[20] RENARDY, M. and ROGERS, R. *An introduction to partial differential equations*. 2nd ed. New York: Springer-Verlag, 2004. ISBN 0-387-00444-0.

[21] SARKAR, S.; RAHMAN, M. and CHAKRABORTY, S. Perturbing the perturbed: Stability of quasinormal modes in presence of a positive cosmological constant. Online. *Physical Review D*. 2023, vol. 108, no. 10. ISSN 2470-0010. Available at: `https://doi.org/10.1103/PhysRevD.108.104002`. [cit. 2025-05-09].

[22] SHARGORODSKY, E. On the definition of pseudospectra. Online. *Bulletin of the London Mathematical Society*. 2009, vol. 41, no. 3, pp. 524-534. ISSN 00246093. Available at: `https://doi.org/10.1112/blms/bdp031`. [cit. 2025-05-09].

[23] TRETTER, C. *Spectral theory of block operator matrices and applications*. London: Imperial College Press, 2008. ISBN 978-1-86094-768-1.

[24] WARNICK, C. M. On Quasinormal Modes of Asymptotically Anti-de Sitter Black Holes. Online. *Communications in Mathematical Physics*. 2015, vol. 333, no. 2, pp. 959-1035. ISSN 0010-3616. Available at: `https://doi.org/10.1007/s00220-014-2171-1`. [cit. 2025-05-09].

[25] WARNICK, C. M. The Massive Wave Equation in Asymptotically AdS Spacetimes. Online. *Communications in Mathematical Physics*. 2013, vol. 321, no. 1, pp. 85-111. ISSN 0010-3616. Available at: `https://doi.org/10.1007/s00220-013-1720-3`. [cit. 2025-05-09].

[26] WARNICK, C. M. (In)stability of de Sitter quasinormal mode spectra. Online. *Frontiers in Applied Mathematics and Statistics*. 2024, vol. 10. ISSN 2297-4687. Available at: `https://doi.org/10.3389/fams.2024.1472401`. [cit. 2025-05-09].

Appendix A

Supplementary proofs

Lemma A.1. *Let $g \in L^2(0,1)$. If we define*

$$h(x) := \frac{1}{x(2-x)} \int_0^x g(t)\,dt,$$

then $h \in L^2(0,1)$ and there exists a positive constant C such that

$$\|h\|_{L^2(0,1)} \leq C\|g\|_{L^2(0,1)}.$$

Proof. We define

$$I_\varepsilon := \int_\varepsilon^1 \frac{1}{x^2} \left| \int_0^x g(t)\,dt \right|^2 dx,$$

where $\varepsilon \in (0,1)$. If we integrate by parts, we get

$$I_\varepsilon = \left[-\frac{1}{x} \left| \int_0^x g(t)\,dt \right|^2 \right]_\varepsilon^1 + 2\Re \left[\int_\varepsilon^1 \frac{1}{x} \left(\int_0^x g(t)\,dt \right) \overline{g(x)}\,dx \right].$$

Using Cauchy–Schwarz inequality, the boundary term can be estimated by the norm of g:

$$\frac{1}{x} \left| \int_0^x g(t)\,dt \right|^2 \leq \frac{1}{x} \left(\int_0^x dx \right) \left(\int_0^x |g(x)|^2\,dx \right) \leq \|g\|^2_{L^2(0,1)}.$$

In the same way, we estimate the other term:

$$\begin{aligned} 2\Re \left[\int_\varepsilon^1 \frac{1}{x} \left(\int_0^x g(t)\,dt \right) \overline{g(x)}\,dx \right] &\leq 2\sqrt{\int_\varepsilon^1 \frac{1}{x^2} \left| \int_0^x g(t)\,dt \right|^2 dx}\ \sqrt{\int_\varepsilon^1 |g(x)|^2\,dx} \\ &\leq 2\sqrt{I_\varepsilon}\|g\|_{L^2(0,1)}. \end{aligned}$$

Altogether we have that for $\delta > 0$,

$$I_\varepsilon \leq 2\|g\|^2_{L^2(0,1)} + 2\sqrt{I_\varepsilon}\|g\|_{L^2(0,1)} \leq \delta I_\varepsilon + \left(2 + \frac{1}{\delta}\right) \|g\|^2_{L^2(0,1)}.$$

Considering $\delta < 1$, we obtain

$$\|h\|^2_{L^2(0,1)} \leq \lim_{\varepsilon \to 0^+} I_\varepsilon \leq C_\delta \|g\|^2_{L^2(0,1)},$$

where C_δ is a positive constant. □

Lemma A.2. *Let $u \in L^2(0,1)$ be a function such that $\mathscr{L}_0 u \in L^2(0,1)$. Then*

$$u \in H^1(0,1) \quad \Longleftrightarrow \quad \lim_{x \to 0^+} x(2-x)u'(x) = 0.$$

Proof. We define

$$g(x) \coloneqq -(x(2-x)u'(x))'.$$

If we integrate this, we get that for some $c \in \mathbb{C}$,

$$x(2-x)u'(x) = -\int_0^x g(t)\,dt + c.$$

This implies

$$\lim_{x \to 0^+} x(2-x)u'(x) = c$$

and

$$u'(x) = -\frac{1}{x(2-x)}\int_0^x g(t)\,dt + \frac{c}{x(2-x)}.$$

By Lemma A.1,

$$\frac{1}{x(2-x)}\int_0^x g(t)\,dt \in L^2(0,1).$$

Since

$$\frac{1}{x(2-x)} \notin L^2(0,1),$$

we get the desired result. ☐

Lemma A.3. *Let $g \in L^2(0,1)$ and $\Re(s) > 1/2$. Then*

$$x^{\bar{s}-1}\int_x^1 e^{2\bar{s}y}(2-y)^{\bar{s}-1}y^{-\bar{s}-1}\int_0^y g(t)\,dt\,dy \in L^2(0,1).$$

Proof. For $\varepsilon \in (0,1)$, we define

$$I_\varepsilon \coloneqq \int_\varepsilon^1 x^{2\Re(s)-2}\left|\int_x^1 e^{2\bar{s}y}(2-y)^{\bar{s}-1}y^{-\bar{s}-1}\int_0^y g(t)\,dt\,dy\right|^2 dx.$$

Integrating by parts, we get

$$\begin{aligned} I_\varepsilon = &\left[\frac{x^{2\Re(s)-1}}{2\Re(s)-1}\left|\int_x^1 e^{2\bar{s}y}(2-y)^{\bar{s}-1}y^{-\bar{s}-1}\int_0^y g(t)\,dt\,dy\right|^2\right]_\varepsilon^1 \\ &- \int_\varepsilon^1 \frac{2x^{2\Re(s)-1}}{2\Re(s)-1}\Re\Bigg\{\left(\int_x^1 e^{2\bar{s}y}(2-y)^{\bar{s}-1}y^{-\bar{s}-1}\int_0^y g(t)\,dt\,dy\right) \\ &\qquad \overline{e^{2\bar{s}x}(2-x)^{\bar{s}-1}x^{-\bar{s}-1}\int_0^x g(t)\,dt}\Bigg\}\,dx. \end{aligned}$$

If we set

$$h(x) \coloneqq \frac{1}{x(2-x)}\int_0^x g(t)\,dt$$

as in Lemma A.1, then by using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\left|\int_x^1 e^{2\bar{s}y}(2-y)^{\bar{s}-1}y^{-\bar{s}-1}\int_0^y g(t)\,dt\,dy\right|^2 \\
&\qquad \leq \|h\|_{L^2(0,1)}^2 \int_x^1 \left|e^{2\bar{s}y}(2-y)^{\bar{s}}y^{-\bar{s}}\right|^2\,dy \\
&\qquad \leq \|h\|_{L^2(0,1)}^2 \left\|e^{2\bar{s}x}(2-x)^{\bar{s}}\right\|_{L^\infty(0,1)}^2 \int_x^1 y^{-2\Re(s)}\,dy \\
&\qquad = \|h\|_{L^2(0,1)}^2 \left\|e^{2\bar{s}x}(2-x)^{\bar{s}}\right\|_{L^\infty(0,1)}^2 \left(-\frac{1}{2\Re(s)-1}\right)\left(1-x^{-2\Re(s)+1}\right) \\
&\qquad \leq \|h\|_{L^2(0,1)}^2 \left\|e^{2\bar{s}x}(2-x)^{\bar{s}}\right\|_{L^\infty(0,1)}^2 \left|\frac{2}{2\Re(s)-1}\right|\; x^{-2\Re(s)+1}.
\end{aligned}$$

This will be used to estimate the boundary terms. Next, we prepare an estimate for the other term:

$$\begin{aligned}
&\Re\left\{\left(\int_x^1 e^{2\bar{s}y}(2-y)^{\bar{s}-1}y^{-\bar{s}-1}\int_0^y g(t)\,dt\,dy\right)\overline{e^{2\bar{s}x}(2-x)^{\bar{s}-1}x^{-\bar{s}-1}\int_0^x g(t)\,dt}\right\} \\
&\qquad \leq \left|\int_x^1 e^{2\bar{s}y}(2-y)^{\bar{s}-1}y^{-\bar{s}-1}\int_0^y g(t)\,dt\,dy\right| \left\|e^{2\bar{s}x}(2-x)^{\bar{s}}\right\|_{L^\infty(0,1)}\; x^{-\Re(s)}\;|h(x)|.
\end{aligned}$$

Altogether, we have

$$\begin{aligned}
I_\varepsilon &\leq C_s\left(\|h\|_{L^2(0,1)}^2 + \int_\varepsilon^1 x^{\Re(s)-1}\left|\int_x^1 e^{2\bar{s}y}(2-y)^{\bar{s}-1}y^{-\bar{s}-1}\int_0^y g(t)\,dt\,dy\right| |h(x)|\,dx\right) \\
&\leq C_s\left(\|h\|_{L^2(0,1)}^2 + \sqrt{I_\varepsilon}\|h\|_{L^2(0,1)}^2\right) \\
&\leq \delta I_\varepsilon + C_{s,\delta}\|h\|_{L^2(0,1)}^2.
\end{aligned}$$

where $\delta > 0$ and C_s and $C_{s,\delta}$ are positive constants. Taking $\delta < 1$, we get that I_ε is bounded uniformly in ε by a multiple of $\|h\|_{L^2(0,1)}^2$. Applying Lemma A.1, we get the desired result. □