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A non-symmetric space-time coupling of finite and boundary element methods for a parabolic-elliptic interface problem

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Preface

As a Master student in Technomathematics with a specialization in “Numerics”, my goal is to apply mathematical methods to engineering problems. With this thesis i got the opportunity to apply the knowledge gained from my studies at Graz University of Technology to a research topic, namely coupling of finite and boundary element methods, especially in this case for a space-time setting. The idea for this thesis came from an application in engineering, namely on an electric motor.

Furthermore I would like to express my deepest gratitude to my supervisor Assoc. Prof. Dr. Günther Of at the Technical University of Graz for his support throughout the production of the master thesis and the implementation of the numerical part.

Contents

Introduction

The coupling of finite and boundary element methods should combine the best of both numerical schemes [26]. The FEM is for example of advantage if we have a non linear partial differential equation or some complicated material behavior. The BEM is for example advantageous, if we consider an exterior problem with a linear homogeneous partial differential equation with constant coefficients. In our case, we consider as a model problem the free space transmission problem for a space-time cylinder $Q := \Omega \times (0,T) \subset \mathbb{R}^{n+1}$ for $n = 2, 3$, where Ω is a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$ with the solution u_i of the interior and the solution u_e of the exterior domain:

$$\begin{aligned} \partial_t u_i(x,t) - \Delta_x u_i(x,t) &= f(x,t) && \text{for } (x,t) \in Q \\ -\Delta_x u_e(x,t) &= 0 && \text{for } (x,t) \in Q^{ext} := \mathbb{R}^n \backslash \overline{\Omega} \times (0,T) \\ u_i(x,0) &= 0 && \text{for } x \in \Omega \end{aligned} \tag{0.1}$$

with transmission conditions

$$u_i(x,t) = u_e(x,t) \quad \frac{\partial}{\partial n_x} u_i(x,t) = \frac{\partial}{\partial n_x} u_e(x,t), \quad \text{for } (x,t) \in \Sigma := \Gamma \times (0,T) \tag{0.2}$$

and with the radiation boundary condition for an unknown function $a(t) : [0,T] \to \mathbb{R}$, see [3, Remark 4],[6, p. 2 and Remark 2]

$$u_e(x,t) = a(t)\log(|x|) + \mathcal{O}(|x|^{-1}) \quad \text{as } |x| \to \infty, \tag{0.3}$$

which is determined in the solving process. The term $a(t)$ is of the form

$$a(t) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_\Gamma \partial_n u_e(x,t) ds_x & \text{for } n = 2 \\ 0 & \text{for } n = 3, \end{cases} \tag{0.4}$$

the second term $O(|x|^{-1})$ is due to the far field behavior (see [19, Lem. 6.21]) and we neglect constants (see here for example [19, Sect. 7.5]). For more information see for example [4, 11, 12]. Due to this radiation condition, we do not need any solvability condition. Also we assume for the two-dimensional space setting the scaling of the domain $\operatorname{diam}(\Omega) < 1$ [19, Thm. 6.23].

The stability of a non-symmetric coupling for polygonal Lipschitz interfaces regarding an elliptic-elliptic interface equation was first proved in 2009 by Sayas [15] where he showed this for standard finite and boundary element methods. He showed the results for a homogeneous Yukawa equation in the exterior domain in order to avoid annoyances regarding a stabilization term, the lack of ellipticity of the single layer boundary integral operator in the two dimensional setting and the far field behavior. In 2011, O. Steinbach proofed in [20] the ellipticity of the bilinear form, where he considered a transmission problem of a Poisson equation inside the domain and the Laplace equation in the exterior domain. He gave two proofs for the Johnson–Nédelec coupling of finite and boundary element methods [8] for general Lipschitz interfaces. In [13], O. Steinbach and G. Of extended the ellitpcity estimate of [20] with sufficient and necessary conditions. In [3], T. Führer and colleagues also considered the same problem with another stabilization term, regarding the Laplace equation they considered in contrast to O. Steinbach the indirect single layer potential ansatz. Also they stated and tested an adaptive scheme for this method. Older publications were not for Lipschitz interfaces, as they are built on the compactness of the double layer boundary integral operator.

A non-symmetric coupling of a parabolic-elliptic interface problem was firstly discussed in 1987 by MacCamy and Suri [11] for a interface problem of the Laplace equation in the exterior domain coupled with the heat equation in the interior domain. As mentioned in [4], "this problem describes time-dependent eddy currents in two-dimensional electrodynamics". Their analysis applies to smooth interfaces only, as it relies on the compactness of the double layer boundary integral operator. In 1990, this problem was further discussed by Costabel, Ervin and Stephan [4], where they considered a symmetric coupling. Furthermore they showed convergence for a semidiscrete and fully discretized Galerkin scheme for Lipschitz domains. In the fully discretized scheme they applied a Crank-Nicolson method. In 2018, Egger, Erath and Schorr [6] considered the non-symmetric coupling method by MacCamy and Suri, where they established well-posedness of the formulation for problems with non-smooth interfaces. They proofed the unique solvability in the analytic setting with the help of the Gårding inequality. They prove the stability of a semi-discrete scheme considering a conforming spatial discretization. Also they proved quasi-optimal error estimates for the conforming Galerkin approximations in space. Finally they show the well-posedness of a related variant of the implicit Euler-method. The numerical results fulfilled the theory. In 1997, Mund and Stephan [12] discussed an adaptive scheme of the symmetric coupling of the parabolic-elliptic interface problem.

The goal of this thesis is to derive and analyze a non-symmetric space-time formulation for the model problem of a parabolic-elliptic interface equation, i.e. space and time are considered together in a space time cylinder and proving unique solvability in this setting. We consider a one-equation coupling of finite and boundary element methods. If we compare to [6], we consider a more general situation for the same problem. The goal is to prove unique solvability with the help of an inf-sup condition for a more general space-time discretization and to prove a surjectivity statement in order to conclude unique solvability in the continuous as well as in the discrete setting.

Firstly, we will consider a related quasistatic problem of (0.1)-(0.4) in Chapt. 1, which will be an important tool. In Chapt. 2 we will state the weak space-time formulation of the model problem (0.1)-(0.4). We will specifically proof the unique solvability of the weak formulation in Sect. 2.1. Later in Sect. 2.2 we will state a discrete space-time variational formulation and prove its unique solvability. In Sect. 2.3, we discuss the equivalent system of linear equations and in Sect. 2.4 the integration methods in order to assemble the matrices with respect to the boundary element method. Finally in Chapt. 3, we discuss some numerical tests in order to illustrate the theoretical results. In particular, we test the numerical examples for different kind of grids.

1 A quasistatic coupled problem

We consider a related quasistatic problem as an auxiliary problem similar to [21], where O. Steinbach discusses the unique solvability of a space time formulation of the Dirichlet boundary value problem of the heat equation. Regarding the model problem (0.1) we consider the following problem with z_i the solution of the interior and z_e the solution of the exterior domain

$$\begin{aligned} -\Delta_x z_i(t) &= f(t) \quad \text{in } Q \\ -\Delta_x z_e(t) &= 0 \qquad \text{in } Q^{ext} \end{aligned} \tag{1.1}$$

with the transmission conditions

$$z_i(x,t) = z_e(x,t), \quad \frac{\partial}{\partial n_x} z_i(x,t) = \frac{\partial}{\partial n_x} z_e(x,t) \quad \text{for } (x,t) \in \Sigma := \Gamma \times (0,T) \tag{1.2}$$

and with the radiation boundary condition for an unknown function $a(t) : [0,T] \to \mathbb{R}$, see [3, Remark 4],[6, p. 2 and Remark 2], which is determined in the solving process (see (0.4)):

$$z_e(x,t) = a(t)\log(|x|) + \mathcal{O}(|x|^{-1}) \quad \text{as } |x| \to \infty. \tag{1.3}$$

The term $a(t)$ is of the form

$$a(t) = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_\Gamma \partial_n u_e(x,t) ds_x & \text{for } n = 2 \\ 0 & \text{for } n = 3. \end{cases} \tag{1.4}$$

Furthermore we assume $f \in L^2(0,T,\widetilde{H}^{-1}(\Omega))$. Also we assume for the two dimensional space setting the scaling of the domain $\operatorname{diam}(\Omega) < 1$ [19, Thm. 6.23].

1.1 A space time variational formulation

For the differential equation inside the domain we derive the weak formulation from Green's first formula

$$\int_Q (-\Delta_x z_i) v \; d(x,t) = \int_Q \nabla_x z_i \cdot \nabla_x v \; d(x,t) - \int_\Sigma w_z \gamma_0^{int} v d(s_x,t) = \langle f, v \rangle_Q, \tag{1.5}$$

where we consider the trace operator γ_0^{int} as the continuous extension of $\gamma_0^{int} v(x) := \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} v(\widetilde{x})$, w_z denotes the Neumann datum and the integral for the right-hand side is

$$\langle f, v \rangle_Q := \int_0^T \langle f(t), v(t) \rangle_\Omega dt$$

where $\langle \cdot, \cdot \rangle_\Omega$ denotes the duality pairing for $\widetilde{H}^{-1}(\Omega)$ and $H^1(\Omega)$.

For the exterior domain, we firstly discuss the boundary integral operators and equation in order to derive a solution for the exterior domain. The solution in the exterior domain can be described by using the (stationary) representation formula [19, p. 182] and be adapted for the time-dependent problem

$$u(x,t) = -\int_\Gamma U^*(x,y) \gamma_1^{ext} z_e(y,t) ds_y + \int_\Gamma \gamma_1^{ext} U^*(x,y) \gamma_0^{ext} z_e(y,t) ds_y \tag{1.6}$$

with the fundamental solution $U^*(x,y)$ of the Laplacian:

$$U^*(x,y) = -\frac{1}{2\pi} \log |x-y| \quad \text{in the two-dimensional space setting for } n=2 \text{ and}$$
$$U^*(x,y) = \frac{1}{4\pi} \frac{1}{|x-y|} \quad \text{in the three-dimensional space for setting } n=3.$$

Here γ_1^{ext} and γ_0^{ext} denote the exterior conormal derivative and trace operator, respectively.

To find a solution for the time-dependent exterior Laplace equation, we consider the direct formulation by the weakly singular boundary integral equation (BIE). As a starting point we consider [19, Sect. 7.1, Sect. 7.5]. The first boundary integral equation is according to the stationary setting

$$(V \gamma_1^{ext} z_e)(x,t) = (-\frac{1}{2} I + K) \gamma_0^{ext} z_e(t,x), \tag{1.7}$$

where K is the double layer boundary integral operator and V the single layer boundary integral operator. For $v \in L^2(0,T; L^\infty(\Gamma))$, $w \in L^2(0,T; L^\infty(\Gamma))$ we have the integral presentation of both operators:

$$(Vw)(x,t) = \int_\Gamma U^*(x,y) w(y,t) ds_y,$$
$$(Kv)(x,t) = \int_\Gamma \gamma_1^{int} U^*(x,y) v(y,t) ds_y.$$

Later we will make use of the second boundary integral equation according to the stationary setting:

$$\gamma_1^{ext} z_e = -D \gamma_0^{ext} z_e + (\frac{1}{2} I - K') \gamma_1^{ext} z_e, \tag{1.8}$$

where D denotes the hypersingular boundary and K' is the adjoined double layer boundary integral operator.

Regarding the interior domain, we consider the paper [21], where O. Steinbach discusses the related spaces for the Dirichlet problem of the heat equation. Therefore we consider the following two spaces [17]:

$$\begin{aligned} Y &:= L^2(0,T,H^1(\Omega)), \\ X_B &:= L^2(0,T,H^{-\frac{1}{2}}(\Gamma)). \end{aligned} \tag{1.9}$$

For w_z in (1.5) we use the space X_B. Related to $L^2(0,T;H^{\frac{1}{2}}(\Gamma))$ and $L^2(0,T;H^{-\frac{1}{2}}(\Gamma))$, we consider the dual pairing and norms for the boundary [17, Sect. 10.1]:

$$\langle v,w\rangle_\Sigma := \int_0^T \langle v(\cdot,t),w(\cdot,t)\rangle_\Gamma dt$$

$$\|v\|_{L^2(0,T;H^s(\Gamma))} := \sqrt{\int_0^T \|v(t)\|^2_{H^s(\Gamma)}dt}.$$

In X_B we will make use of an equivalent norm to $\|\cdot\|_{X_B}$, defined by

$$\|\tau\|_V^2 := \int_0^T \langle (V\tau)(t),\tau(t)\rangle_\Gamma dt \quad \text{for } \tau \in X_B. \tag{1.10}$$

We can easily check the equivalence of the norms. From the boundedness of the single layer boundary integral operator [19, Lem. 6.6], where c_2^V is for the boundedness constant, we can conclude for $\tau \in X_B$:

$$\begin{aligned} \|\tau\|_V^2 &= \int_0^T \langle (V\tau)(t),\tau(t)\rangle_\Gamma dt \leq \int_0^T \|(V\tau)(t)\|_{H^{1/2}(\Gamma)}\|\tau(t)\|_{H^{-1/2}(\Gamma)}dt \\ &\leq c_2^V \int_0^T \|\tau(t)\|_{H^{-1/2}(\Gamma)}\cdot\|\tau(t)\|_{H^{-1/2}(\Gamma)}dt = c_2^V\|\tau\|_{L^2(0,T,H^{-1/2}(\Gamma))}. \end{aligned}$$

From the ellipticity of V [19, Thm. 6.22, Thm. 6.23], where c_1^V is for the ellipticity constant, we conclude:

$$\|\tau\|_V^2 = \int_0^T \langle (V\tau)(t),\tau(t)\rangle_\Gamma \geq c_1^V \int_0^T \|\tau(t)\|^2_{H^{-1/2}(\Gamma)}dt = c_1^V\|\tau\|^2_{L^2(0,T,H^{-1/2}(\Gamma))}.$$

All in one, we have an equivalent norm for X_B

$$c_1^V\|\tau\|^2_{L^2(0,T,H^{-1/2}(\Gamma))} \leq \|\tau\|_V^2 \leq c_2^V\|\tau\|^2_{L^2(0,T,H^{-1/2}(\Gamma))}. \tag{1.11}$$

The weak formulation of the exterior boundary integral equation (1.7), where w_z is equal to the Neumann data since the transmission condition (1.2) holds, is:

$$\text{Find } w_z \in X_B: \quad \langle Vw_z,\tau\rangle_\Sigma + \langle(\frac{1}{2}I-K)z_e,\tau\rangle_\Sigma = 0 \quad \forall\tau\in X_B. \tag{1.12}$$

If we couple this equation with (1.5) and make use of the transmission conditions (1.2) for $z = z_i = z_e$, we get the variational formulation

$$\text{Find } (z, w_z) \in Y \times X_B : \quad d((z, w_z), (v, \tau)) = \langle f, v \rangle_Q \quad \forall (v, \tau) \in Y \times X_B \tag{1.13}$$

with the bilinear form

$$d((z, w_z), (v, \tau)) := \langle \nabla_x z, \nabla_x v \rangle_{L^2(Q)} - \langle w_z, v \rangle_\Sigma + \langle V w_z, \tau \rangle_\Sigma + \langle (\frac{1}{2} I - K) z, \tau \rangle_\Sigma. \tag{1.14}$$

1.2 An equivalent variational formulation

The goal is to prove ellipticity and boundedness estimates for the bilinear form d. In the static case, O. Steinbach and G. Of [13] proved the stability of a one-equation coupling of finite and boundary element methods regarding a free space transmission problem of a Poisson equation with variable coefficients inside the domain and the Laplace equation in the exterior domain. They make use of a special stabilization term. In our case, we make another choice similar to the one in [3] for the static case. In total, we will transfer the mentioned analysis from the static case to our quasi-static problem (1.13). Accordingly we define a modified bilinear form of our quasistatic problem (1.1):

$$\begin{aligned} \widehat{d}((z, w_z),(v, \tau)) := {} & \langle \nabla_x z, \nabla_x v \rangle_{L^2(Q)} - \langle w_z, v \rangle_\Sigma + \langle V w_z, \tau \rangle_\Sigma + \langle (\frac{1}{2} I - K) z, \tau \rangle_\Sigma \\ & + \int_0^T \Big[\langle (\frac{1}{2} I - K) z(t) + V w_z(t), 1 \rangle_\Gamma \langle (\frac{1}{2} I - K) v(t) + V \tau(t), 1 \rangle_\Gamma \Big] dt. \end{aligned} \tag{1.15}$$

The equivalent variational formulation reads:

$$\text{Find } (z, w_z) \in Y \times X_B : \quad \widehat{d}((z, w_z), (v, \tau)) = \langle f, v \rangle_Q \quad \forall (v, \tau) \in Y \times X_B. \tag{1.16}$$

Our goal is to prove that both variational formulations are equivalent, which we state in the following lemma similar to [3, Thm. 13].

Lemma 1.1. *(i) Let $(z, w_z) \in Y \times X_B$ be a solution of (1.13), then (z, w_z) is a solution of (1.16).*
(ii) Any solution $(z, w_z) \in Y \times X_B$ of (1.16) is a solution of (1.13).

Proof. (i) Let $(z, w_z) \in Y \times X_B$ be a solution of (1.13). If we choose $(v, \tau) = (0, 1_x \cdot q(t))$ with $q(t) \in L^2(0, T)$ arbitrary and $1_x \in H^{-1/2}(\Gamma)$ we get from (1.13)

$$\int_0^T (\langle V w_z(t), 1_x \rangle_\Gamma + \langle (\frac{1}{2} I - K) z(t), 1_x \rangle_\Gamma) \cdot q(t) dt = 0 \quad \forall q(t) \in L^2(0, T)$$

and therefore

$$\langle V w_z(t), 1\rangle_\Gamma + \langle(\frac{1}{2}I - K)z, 1\rangle_\Gamma = 0 \quad \text{for almost every } t \in (0,T).$$

As a direct consequence we get that (z, w_z) is a solution of (1.16), as the additional term in $\widehat{d}$ vanishes.
(ii) Let $(z, w_z) \in Y \times X_B$ be a solution of (1.16). If we choose $(v,\tau) = (0, 1_x \cdot q(t))$ with $q(t) \in L^2(0,T)$ arbitrary and $1_x \in H^{-1/2}(\Gamma)$ we get from (1.16)

$$\int_0^T \big[\langle(\frac{1}{2}I - K)z(t) + V w_z(t), 1\rangle_\Gamma \langle V1, 1\rangle_\Gamma + \langle V w_z, 1\rangle_\Gamma + \langle(\frac{1}{2}I - K)z(t), 1\rangle_\Gamma\big] \cdot q(t)dt = 0 \quad \forall q \in L^2(0,T).$$

which is equivalent to

$$\int_0^T \Big[\langle(\frac{1}{2}I - K)z(t) + V w_z(t), 1\rangle_\Gamma (\langle V1, 1\rangle_\Gamma + 1)\Big] \cdot q(t)dt = 0 \quad \forall q \in L^2(0,T).$$

As $(\langle V1, 1\rangle_\Gamma + 1) > 0$, we conclude:

$$\langle(\frac{1}{2}I - K)z(t), 1\rangle_\Gamma + \langle V w_z(t), 1\rangle_\Gamma = 0 \quad \text{for almost every } t \in (0,T).$$

As a direct consequence (z, w_z) is a solution of (1.13). □

The modified variational formulation (1.16) motivates us to introduce a new norm for $(v,\tau) \in Y \times X_B$:

$$\|(v,\tau)\|^2_{Y \times X_B, f} := \int_0^T \Big[\|\nabla_x v\|^2_{L^2(\Omega)} + [f(v,\tau)]^2\Big] dt + \|\tau\|^2_V \tag{1.17}$$

where we make the following choice for f:

$$f(v,\tau) = \langle(\frac{1}{2}I - K)v(t) + V\tau(t), 1\rangle_\Gamma.$$

Our next goal is to prove that (1.17) defines an equivalent norm in $Y \times X_B$. Therefor, we will firstly prove the norm equivalence in the static case. In order not to have any inconvenience with the time dependent case we define the following norm only in the static case where V is the single layer boundary integral operator and $\tau \in H^{-1/2}(\Gamma)$:

$$\|\tau\|^2_{V_{stat}} := \langle V\tau, \tau\rangle_\Gamma.$$

Of course this is an equivalent norm in $H^{-1/2}(\Gamma)$. The proof of the following lemma mainly follows the idea of the proof of the norm equivalence theorem of Sobolev (see for example [19, Thm. 2.6] and [3, Lem. 10]).

Lemma 1.2. *Let the following two norms for $(v,\tau) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ be given:*

$$\|(v,\tau)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} := \{\|v\|^2_{H^1(\Omega)} + \|\tau\|^2_{V_{stat}}\}^{1/2}$$
$$\|(v,\tau)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma),f} := \{\|\nabla v\|^2_{L^2(\Omega)} + \|\tau\|^2_{V_{stat}} + |f(v,\tau)|^2\}^{1/2}$$

with

$$f(v,\tau) = \langle(\frac{1}{2}I - K)v + V\tau, 1\rangle_\Gamma.$$

Then both norms are equivalent.

Proof. The proof is splitted into two parts. In the first part we show that f is a bounded and linear functional and that we can estimate the second norm by the first one.
(i) We firstly estimate $|f(v,\tau)|$ by splitting the terms and estimating them by the boundedness of the operators (see [19, Lem. 6.6, Lem. 6.8]) and the trace theorem:

$$\begin{aligned}|f(v,\tau)|^2 &= |\langle v, 1\rangle_\Gamma - \langle(\frac{1}{2}I + K)v, 1\rangle_\Gamma + \langle V\tau, 1\rangle_\Gamma|^2\\ &\le c\Big(\langle v, 1\rangle_\Gamma^2 + \langle(\frac{1}{2}I + K)v, 1\rangle_\Gamma^2 + \langle V\tau, 1\rangle_\Gamma^2\Big)\\ &\le \tilde{c}(\|v\|^2_{H^1(\Omega)} + \|\tau\|^2_{V_{stat}}) \le \hat{c}\|(v,\tau)\|^2_{H^1(\Omega)\times H^{-1/2}(\Gamma)}.\end{aligned}$$

Now it follows that f is a bounded and linear functional and we can estimate the second norm by the first one.
(ii) For the second part we first make an short observation: Let $\beta \in \mathbb{R}$ arbitrary, then we get from $(\frac{1}{2}I + K)1_\Gamma = 0$:

$$|f(\beta 1_\Omega, 0)|^2 = 0 \Leftrightarrow |\langle(\frac{1}{2} - K)\beta\gamma_0^{int}1_\Omega, 1\rangle_\Gamma|^2 = 0 \Leftrightarrow |\beta\langle 1, 1\rangle_\Gamma|^2 = 0 \Rightarrow \beta = 0. \quad (1.18)$$

The proof is by contradiction. Let us assume that

$$\nexists c_0 \text{ s.t. } \|(v,\tau)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} \le c_0\|(v,\tau)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma),f}.$$

Then there exists a sequence $\{(v_n,\tau_n)\}_{n\in\mathbb{N}} \subset H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$n \le \frac{\|(v_n,\tau_n)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)}}{\|(v_n,\tau_n)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma),f}}.$$

Therefore we can now define the normalized sequence

$$(\overline{v}_n, \overline{\tau}_n) := \frac{(v_n,\tau_n)}{\|(v_n,\tau_n)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)}},$$

and we get by plugging the sequence into the norms:

$$\|(\overline{v}_n, \overline{\tau}_n)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} = 1,$$

$$\|(\overline{v}_n, \overline{\tau}_n)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma),f} = \frac{\|(v_n, \tau_n)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma),f}}{\|(v_n, \tau_n)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)}} \leq \frac{1}{n} \xrightarrow{n\to\infty} 0.$$

From the norm definition we see

$$\lim_{n\to\infty} |f(\overline{v}_n, \overline{\tau}_n)|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} (\|\nabla \overline{v}_n\|^2_{L^2(\Omega)} + \|\overline{\tau}_n\|^2_{V_{stat}}) = 0.$$

Since $\{\overline{v}_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\Omega)$ and $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ [24] there exists a subsequence $\{\overline{v}_{n'}\}_{n'\in\mathbb{N}} \subset \{\overline{v}_n\}_{n\in\mathbb{N}}$ converging in $L^2(\Omega)$, i.e.:

$$\overline{v} = \lim_{n'\to\infty} \overline{v}_{n'} \in L^2(\Omega).$$

Next, we show convergence in $H^1(\Omega)$. Due to the reason that we have

$$\lim_{n'\to\infty} \|\nabla \overline{v}_{n'}\|_{L^2(\Omega)} = 0,$$

there exists a sequence of functions $(\overline{v}_{n'})_{n'\in\mathbb{N}}$ such that each weak derivative of $\overline{v}_{n'}$ is converging to 0 in $L^2(\Omega)$. Now we can directly get that $\overline{v} \in H^1(\Omega)$ and that

$$\|\nabla \overline{v}\|_{L^2(\Omega)} = \lim_{n'\to\infty} \|\nabla \overline{v}_{n'}\|_{L^2(\Omega)} = 0.$$

Thus we have $\overline{v} \in H^1(\Omega)$ and $\|\nabla \overline{v}\|_{L^2(\Omega)} = 0$. Therefore $\overline{v}$ must be constant and by linearity and boundedness of f we have:

$$0 \leq |f(\overline{v}, 0)| = \lim_{n'\to\infty} |f(\overline{v}_{n'}, 0)| = 0.$$

By (1.18) we get that $\overline{v} = 0$. As

$$\lim_{n\to\infty} \|\overline{\tau}_n\|_{V_{stat}} = 0$$

we can directly follow that

$$\overline{\tau} := \lim_{n\to\infty} \overline{\tau}_n = 0.$$

Overall, we found $(\overline{v}, \overline{\tau}) = (0, 0)$, which is a contradiction to

$$\|(\overline{v}, \overline{\tau})\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} = 1$$

since

$$\|(\overline{v}, \overline{\tau})\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} = \lim_{n\to\infty} \|(\overline{v}_n, \overline{\tau}_n)\|_{H^1(\Omega)\times H^{-1/2}(\Gamma)} = 1.$$

□

As a direct consequence of Lemma 1.2 we get the following lemma.

Lemma 1.3. *The modified norm (1.17) is equivalent to the original one in the following sense*

$$c\|(v, \tau)\|_{Y\times X_B,f} \leq \|(v, \tau)\|_{Y\times X_B} \leq \widehat{c}\ \|(v, \tau)\|_{Y\times X_B,f}$$

where $c, \widehat{c}$ *are constants.*

1.3 Unique solvability of the variational formulation

Similar to (1.10) we define the following norm for $u \in L^2(0,T;H^{1/2}(\Gamma))$:

$$\|u\|^2_{V^{-1}} := \int_0^T \langle V^{-1}u(t), u(t)\rangle_\Gamma dt. \tag{1.19}$$

The goal is to prove the ellipticity of the modified bilinear form (1.17). In order to do that we state and proof some more estimates such that we able to prove the main result of this section. Since we know from the Lemma of Lax–Milgram [19, Thm. 3.4] that

$$\|V^{-1}u(t)\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \frac{1}{c_1^V}\|u(t)\|_{H^{\frac{1}{2}}(\Gamma)}$$

we can conclude

$$\int_0^T \langle V^{-1}u(t), u(t)\rangle_\Gamma dt \leq \int_0^T \Big[\|V^{-1}u(t)\|_{H^{-\frac{1}{2}}(\Gamma)}\|u(t)\|_{H^{\frac{1}{2}}(\Gamma)}\Big] dt \leq \frac{1}{c_1^V}\|u\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma)}.$$

From [19, Lem. 3.5] we know that

$$\langle V^{-1}u(t), u(t)\rangle_\Gamma \geq \frac{1}{c_2^V}\|u\|^2_{H^{\frac{1}{2}}(\Gamma)}$$

and therefore

$$\|u\|^2_{V^{-1}} \geq \frac{1}{c_2^V}\|u\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}.$$

Summarized, $\|u\|_{V^{-1}}$ is an equivalent norm in $L^2(0,T;H^{\frac{1}{2}}(\Gamma))$.

We find the interior Dirichlet to Neumann map from the boundary integral equations and then define the non-symmetric and symmetric representation of the interior Steklov–Poincaré operator S^{int} [19, Sect. 6.6.3]:

$$\gamma_1^{int}u = V^{-1}(\frac{1}{2}I + K)\gamma_0^{int}u = S^{int}\gamma_0^{int}u$$
$$\gamma_1^{int}u = [D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)]\gamma_0^{int}u = S^{int}\gamma_0^{int}u.$$

The following lemma states two important estimates of the Steklov–Poincaré operator in order to prove ellipticity of the modified bilinear form (1.15).

Lemma 1.4. *([13, Lem. 2.1]) For all $v \in L^2(0,T;H^{1/2}(\Gamma))$ there holds the inequalities:*

$$\frac{1}{c_K}\|(\frac{1}{2}I + K)v\|^2_{V^{-1}} \leq \langle S^{int}v, v\rangle_\Sigma \leq \frac{1}{1-c_K}\|(\frac{1}{2}I + K)v\|^2_{V^{-1}}.$$

Proof. The proof is similar to the one of the static case. We have to take into account the additional time integration only. Firstly we assume $v(t) \in H_*^{1/2}(\Gamma) := \{w \in H^{\frac{1}{2}}(\Gamma) : \langle w, t_{eq} \rangle_\Gamma = 0\}$ where $t_{eq} = V^{-1}1$ defines the natural density ([19, p. 144]). The lower estimate follows by using the symmetric representation of the interior Steklov–Poincaré operator, the fact that $KV = VK'$ and contraction properties from [19, Sect. 6.6.4], where we have in the static case:

$$\langle Dv(t), v(t) \rangle_\Gamma \geq c_K(1 - c_K)\langle V^{-1}v(t), v(t) \rangle_\Gamma$$
$$\langle V^{-1}(\frac{1}{2}I + K)v(t), (\frac{1}{2}I + K)v(t) \rangle_\Gamma^{1/2} \leq c_K \langle V^{-1}v(t), v(t) \rangle_\Gamma^{1/2}.$$

If we now plug everything in, we get:

$$\begin{aligned}
\langle S^{int}v, v \rangle_\Sigma &= \langle Dv, v \rangle_\Sigma + \langle (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)v, v \rangle_\Sigma \\
&= \langle Dv, v \rangle_\Sigma + \langle V^{-1}(\frac{1}{2}I + K)v, (\frac{1}{2}I + K)v \rangle_\Sigma \\
&= \langle Dv, v \rangle_\Sigma + \|(\frac{1}{2}I + K)v\|_{V^{-1}}^2 \\
&= \int_0^T \langle Dv(t), v(t) \rangle_\Gamma dt + \|(\frac{1}{2}I + K)v\|_{V^{-1}}^2 \\
&\geq c_K(1 - c_K)\|v\|_{V^{-1}}^2 + \|(\frac{1}{2}I + K)v\|_{V^{-1}}^2 \\
&\geq (\frac{1}{c_K}(1 - c_K) + 1)\|(\frac{1}{2}I + K)v\|_{V^{-1}}^2 = \frac{1}{c_K}\|(\frac{1}{2}I + K)v\|_{V^{-1}}^2.
\end{aligned}$$

The upper estimate follows by using the non-symmetric version of the interior Steklov–Poincaré operator and the lower estimate of ([19, Thm. 6.26]):

$$\begin{aligned}
\langle S^{int}v, v \rangle_\Sigma = \langle V^{-1}(\frac{1}{2}I + K)v, v \rangle_\Sigma &\leq \|(\frac{1}{2}I + K)v\|_{V^{-1}}\|v\|_{V^{-1}} \\
&\leq \frac{1}{1 - c_k}\|(\frac{1}{2}I + K)v\|_{V^{-1}}^2.
\end{aligned}$$

According to the definition of $H_*^{\frac{1}{2}}(\Gamma)$ we can split $v(t) = \tilde{v}(t) + \alpha(t)1_\Gamma$ with $\tilde{v}(t) \in H_*^{\frac{1}{2}}(\Gamma)$ and $\alpha(t) = \frac{\langle v(t), t_{eq} \rangle_\Gamma}{\langle 1, t_{eq} \rangle_\Gamma}$ [19, Sect. 6.6.2]. As the kernel in the static case $\ker S_{stat}^{int} = \ker\,(\frac{1}{2}I + K_{stat}) = \mathrm{span}\{1_\Gamma\}$ and $D1_\Gamma = 0$ the assertion also holds for $\tilde{v}(t) \in H^{1/2}(\Gamma)$ and therefore for $v \in L^2(0, T; H^{1/2}(\Gamma))$. □

Now we are able to state and proof the ellipticity estimate for the modified bilinear form (1.15). The proof mainly follows the idea of [13] and [20], where Steinbach and Of proofed it for the static case, but with another stabilization parameter from [3] and the additional time integration.

Theorem 1.5. *The modified bilinear form (1.15) is* $L^2(0,T;H^1(\Omega))\times L^2(0,T;H^{-1/2}(\Gamma))$ *elliptic in the following sense:*

$$\widehat{d}((v,\tau),(v,\tau)) \geq c_{el}\|(v,\tau)\|^2_{Y\times X_B,f}$$

for all $(v,t)\in L^2(0,T;H^1(\Omega))\times L^2(0,T;H^{-1/2}(\Gamma))$ *with* c_{el} *the ellipticity constant*

$$c_{el} := 1-\frac{1}{2}c_K$$

where $c_K\in[0,1)$.

Proof. First we rewrite the bilinear form (1.15):

$$\begin{aligned}\widehat{d}((v,\tau);(v,\tau)) = &\int_0^T \|\nabla_x v(t)\|_{L^2(\Omega)}dt + \int_0^T |f(v(t),\tau(t))|^2 dt + \langle V\tau,\tau\rangle_\Sigma \\ &- \Big\langle\Big(\frac{1}{2}I+K\Big)v,\tau\Big\rangle_\Sigma.\end{aligned}$$

Now we consider a splitting $v = v_\Sigma + \tilde{v}$, where v_Σ is the harmonic extension of the function v restricted to the boundary Γ and therefore to the lateral boundary Σ, moreover we have $\tilde{v}\in L^2(0,T;H^1_0(\Omega))$. In particular we have

$$\int_Q \nabla_x v_\Sigma \cdot \nabla_x \tilde{v} d(x,t) = 0.$$

Furthermore we can conclude

$$\begin{aligned}\int_Q \nabla_x v\cdot\nabla_x v d(x,t) &= \int_Q \nabla_x(v_\Sigma+\tilde{v})\cdot\nabla_x(v_\Sigma+\tilde{v})d(x,t)\\ &= \int_Q \nabla_x v_\Sigma\cdot\nabla_x v_\Sigma d(x,t) + \int_Q \nabla_x\tilde{v}\cdot\nabla_x\tilde{v}d(x,t)\\ \int_Q \nabla_x v_\Sigma\cdot\nabla_x v_\Sigma d(x,t) &= \int_Q[-\Delta v_\Sigma]v_\Sigma d(x,t) + \int_\Sigma \frac{\partial}{\partial n_x}v_\Sigma v_\Sigma d(s_x,t)\\ &= \langle S^{int}v_\Sigma, v_\Sigma\rangle_\Sigma = \langle S^{int}v,v\rangle_\Sigma\end{aligned}$$

and rewrite

$$\begin{aligned}\widehat{d}((v,\tau);(v,\tau)) = &\left[\int_0^T \|\nabla_x\tilde{v}(t)\|^2_{L^2(\Omega)}dt + \langle S^{int}v,v\rangle_\Sigma + \int_0^T |f(v(t),\tau(t))|^2 dt\right]\\ &+ \langle V\tau,\tau\rangle_\Sigma - \Big\langle\Big(\frac{1}{2}I+K\Big)v,\tau\Big\rangle_\Sigma.\end{aligned}$$

Now we estimate the terms related to S^{int} and the boundary integral operator by using the lower estimate of Lemma 1.4, a worst case estimate by the Cauchy Schwarz inequality:

$$\begin{aligned}
\langle S^{int} v, v\rangle_\Sigma + \|\tau\|_V^2 - \Big\langle \Big(\frac{1}{2}I + K\Big)v, \tau\Big\rangle_\Sigma & \\
&\geq \langle S^{int} v, v\rangle_\Sigma + \|\tau\|_V^2 - \Big\|\Big(\frac{1}{2}I + K\Big)v\Big\|_{V^{-1}} \|\tau\|_V \\
&\geq \langle S^{int} v, v\rangle_\Sigma + \|\tau\|_V^2 - \sqrt{c_K \langle S^{int} v, v\rangle_\Sigma} \|\tau\|_V \\
&= \begin{pmatrix} \|\tau\|_V \\ \langle S^{int} v, v\rangle_\Sigma^{1/2} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2}\sqrt{c_K} \\ -\frac{1}{2}\sqrt{c_K} & 1 \end{pmatrix} \begin{pmatrix} \|\tau\|_V \\ \langle S^{int} v, v\rangle_\Sigma^{1/2} \end{pmatrix} \\
&\geq (1 - \frac{1}{2} c_K)\big[\|\tau\|_V^2 + \langle S^{int} v, v\rangle_\Sigma\big],
\end{aligned}$$

where we have used the smallest eigenvalue in the last estimate. We obtain for the whole bilinear form:

$$\begin{aligned}
\widehat{d}((v,\tau),(v,\tau)) \geq \Bigg[\int_0^T \|\nabla_x \tilde{v}(t)\|_{L^2(\Omega)}^2 dt + \int_0^T |f(v(t),\tau(t))|^2 dt\Bigg] & \\
+ (1 - \frac{1}{2} c_K)\big[\|\tau\|_V^2 + \langle S^{int} v, v\rangle_\Sigma\big]&.
\end{aligned}$$

By rewriting

$$\langle S^{int} v, v\rangle_\Sigma = \int_0^T \|\nabla_x v_\Sigma\|_{L^2(\Omega)}^2 dt$$

the assertion follows.

□

Now we proof the boundedness of the modified bilinear form (1.15).

Theorem 1.6. *The modified bilinear form (1.15) is $L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^{-1/2}(\Gamma))$ bounded satisfying:*

$$|\widehat{d}((z,w_z),(v,\tau))| \leq c_b \big[\|(z,w_z)\|_{Y\times X_{B,f}} \|(v,\tau)\|_{Y\times X_{B,f}}\big].$$

Proof. We start this proof by estimating the bilinear form for each term:

$$\begin{aligned}
|\widehat{d}((z,w_z),(v,\tau))| \leq & \Big(\int_0^T \|\nabla_x z(t)\|_{L^2(\Omega)}^2 dt\Big)^{1/2} \Big(\int_0^T \|\nabla_x v(t)\|_{L^2(\Omega)}^2 dt\Big)^{1/2} + |\langle w_z, v\rangle_\Sigma| \\
&+ \sqrt{\int_0^T |f(z(t),w_z(t))|^2 dt} \sqrt{\int_0^T |f(v(t),\tau(t))|^2 dt} \\
&+ |\langle V w_z, \tau\rangle_\Sigma| + |\langle (\frac{1}{2} I - K) z, \tau\rangle_\Sigma|.
\end{aligned}$$

Now we are estimating each term, for the first and third term we get:

$$\Big(\int_0^T \|\nabla_x z\|^2_{L^2(\Omega)}dt\Big)^{1/2}\Big(\int_0^T \|\nabla_x v\|^2_{L^2(\Omega)}dt\Big)^{1/2} + \sqrt{\int_0^T |f(z,w_z)|^2 dt}\sqrt{\int_0^T |f(v,\tau)|^2 dt}$$
$$= \begin{pmatrix} (\int_0^T \|\nabla_x z\|^2_{L^2(\Omega)}dt)^{1/2} \\ (\int_0^T |f(z,w_z)^2 dt)^{1/2} \end{pmatrix} \cdot \begin{pmatrix} (\int_0^T \|\nabla_x v\|^2_{L^2(\Omega)}dt)^{1/2} \\ (\int_0^T |f(v,\tau)^2 dt)^{1/2} \end{pmatrix}$$
$$\le \Big(\int_0^T (\|\nabla_x z\|^2_{L^2(\Omega)} + |f(z,w_z)|^2)dt\Big)^{1/2}\Big(\int_0^T (\|\nabla_x v\|^2_{L^2(\Omega)} + |f(v,\tau)|^2)dt\Big)^{1/2}.$$

For the second term we estimate due to the fact that $\|\cdot\|_V$ is an equivalent norm in $L^2(0,T,H^{-1/2}(\Gamma))$ and by the trace theorem:

$$|\langle w_z, v\rangle_\Sigma| \le \|w_z\|_{L^2(0,T;H^{-1/2}(\Gamma))}\|\gamma_0^{int} v\|_{L^2(0,T;H^{1/2}(\Gamma))} \le c\|w_z\|_V\|v\|_Y.$$

We estimate the fourth term by the Cauchy Schwarz inequality:

$$|\langle V w_z, \tau\rangle_\Sigma| \le c\|w_z\|_V\|\tau\|_V.$$

For the last term we get by using the boundedness of K, the trace theorem and due to the fact that $\|\cdot\|_V$ is an equivalent norm in $L^2(0,T,H^{-1/2}(\Gamma))$:

$$|\langle(\frac{1}{2}I - K)z, \tau\rangle_\Sigma| \le |\langle z,\tau\rangle_\Sigma| + |\langle\frac{1}{2}I + K)z,\tau\rangle_\Sigma| \le \hat{c}\|z\|_Y\|\tau\|_V.$$

If we put everything together we get by applying the Cauchy Schwarz inequality

$$|\widehat{d}((z,w_z),(v,\tau))| \le c\Big(\|w_z\|_V\|v\|_Y + \|w_z\|_V\|\tau\|_V + \|z\|_Y\|\tau\|_V$$
$$+ \Big(\int_0^T (\|\nabla_x z\|^2_{L^2(\Omega)} + |f(z,w_z)|^2)dt\Big)^{1/2}\Big(\int_0^T (\|\nabla_x v\|^2_{L^2(\Omega)} + |f(v,\tau)|^2)dt\Big)^{1/2}\Big)$$
$$\le c_b\big[\|(z,w_z)\|_{(Y,f)\times X_B}\|(v,\tau)\|_{(Y,f)\times X_B}\big].$$

□

By the Lemma of Lax–Milgram [19, Thm. 3.4] we get that the modified variational formulation (1.16) is well posed and therefore the equivalent original variational formulation (1.5) is uniquely solvable. The last two theorems will later be of importance.

1.4 Adjoint variational formulation

Now we briefly look at the adjoint problem of the modified variational formulation (1.13) of the quasistatic problem:

$$\text{Find } (\widehat{z},\widehat{w}) \in Y \times X_B: \quad \widehat{d}((v,\tau),(\widehat{z},\widehat{w})) = \langle f, v\rangle_Q + \langle g,\tau\rangle_\Sigma \quad \forall (v,\tau) \in Y \times X_B, \tag{1.20}$$

where we assume some arbitrary $f \in L^2(0,T,\widetilde{H}^{-1}(\Omega))$ and $g \in L^2(0,T,H^{1/2}(\Gamma))$. As we have ellipticity and boundedness of $\widehat{d}$, we can directly state the unique solvability and stability of the adjoint problem (1.20).

The adjoint variational formulation is similar to the non-symmetric coupling of the stationary problem with the indirect single layer potential ansatz of (1.1) (for more details see [3]). There we have the ansatz

$$\begin{aligned} z(x,t) &= (\widetilde{V}\widehat{w})(x,t) \quad \text{in } Q^{ext} \\ (V\widehat{w})(x,t) &= \gamma_0^{ext} z(x,t) \quad \text{on } \Sigma \end{aligned}$$

where $\widehat{w}$ is a density function. Furthermore, we can rewrite the Neumann transmission condition by applying the trace operator using the representation from [19, Sect. 6.3]:

$$\gamma_1^{int} z(t) = \gamma_1^{ext} z(t) = \gamma_1^{ext}(\widetilde{V}\widehat{w})(t) = (-\frac{1}{2}I + K')\widehat{w}(t).$$

Thus we get the coupled problem with the indirect ansatz regarding the BEM part:

$$\begin{aligned} \int_Q \nabla_x z \cdot \nabla_x v d(x,t) + \langle (\frac{1}{2}I - K')\widehat{w}, v\rangle_\Sigma &= \int_Q f v d(x,t) \\ \gamma_0^{ext} z(t) &= V w_z(t). \end{aligned} \tag{1.21}$$

Remark 1.1. The stabilization [3, Thm. 8] for the indirect approach differs from the one of the direct approach. Thus the stabilized bilinear form related to (1.21) does not agree with $\widehat{d}((v,\tau),(\widehat{z},\widehat{w}))$.

2 A non-symmetric space-time coupling for a parabolic-elliptic interface problem

2.1 Variational formulation and uniqueness in the analytic setting

Our goal now is to derive a variational formulation of the model problem (0.1)-(0.3) and to prove unique solvability of the formulation. Furthermore we assume $f \in L^2(0,T,\widetilde{H}^{-1}(\Omega))$ in (0.1). Firstly, we reconsider the spaces defined in (1.9) and define one more space [22]:

$$X := \{u \in Y : \partial_t u \in Y', u(x,0) = 0 \text{ for } x \in \Omega\}.$$

We may proceed as in Chapt. 1 but with the additional term of the temporal derivative. The variational formulation for the model problem reads as follows: Find $u_i \in X$ such that

$$\int_0^T \int_\Omega (\partial_t u_i(x,t)v(x,t) + \nabla_x u_i(x,t) \cdot \nabla_x v(x,t))dxdt - \langle w, v\rangle_\Sigma = \langle f, v\rangle_Q \quad \forall v \in Y, \tag{2.1}$$

where the conormal derivative w is the unique solution [19, Sec. 7.1] of the variational formulation of the exterior weakly singular boundary integral equation (for definition see (1.7)):

$$\text{Find } w \in X_B\text{:} \quad \langle Vw, \tau\rangle_\Sigma + \langle (\frac{1}{2}I - K)u_e, \tau\rangle_\Sigma = 0 \quad \text{for all } \tau \in X_B. \tag{2.2}$$

Using the transmission condition $u_e = u_i$ (0.2) we get the coupled variational formulation

$$\text{Find } (u, w) \in X \times X_B : \quad a((u,w),(v,\tau)) = \langle f, v\rangle_Q \quad \forall (v,\tau) \in Y \times X_B \tag{2.3}$$

with the bilinear form of the coupled problem:

$$a((u,w),(v,\tau)) := \int_0^T \int_\Omega (\partial_t u(x,t)v(x,t) + \nabla_x u(x,t) \cdot \nabla_x v(x,t))dxdt - \langle w, v\rangle_\Sigma + \langle Vw, \tau\rangle_\Sigma + \langle (\frac{1}{2}I - K)u, \tau\rangle_\Sigma.$$

For ease of presentation we have substituted u_i by u, i.e. we have omitted the index.

Similar as in Chapt. 1 we make use of a special stabilization term. Again we make a similar choice to the one in [3] and therefore the same one as in the quasistatic case, see equation (1.15). Now we can state the modified variational formulation

$$\text{Find } (u,w) \in X \times X_B: \quad \widehat{a}((u,w),(v,\tau)) = \langle f, v\rangle_Q \quad \forall (v,\tau) \in Y \times X_B. \tag{2.4}$$

with the modified bilinear form:

$$\widehat{a}((u,w),(v,\tau)) := \langle \partial_t u, v\rangle_Q + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)} - \langle w, v\rangle_\Sigma + \langle Vw, \tau\rangle_\Sigma + \\ \langle (\frac{1}{2}I - K)u, \tau\rangle_\Sigma + \int_0^T \Big[\langle (\frac{1}{2}I - K)u(t) + Vw(t), 1\rangle_\Gamma \langle (\frac{1}{2}I - K)v(t) + V\tau(t), 1\rangle_\Gamma\Big]\, dt. \tag{2.5}$$

Next, we show that both variational formulations are equivalent.

Lemma 2.1. *(i) Let $(u,w) \in X \times X_B$ be a solution of (2.3), then (u,w) is a solution of (2.4).*
(ii) Every solution $(u,w) \in X \times X_B$ of (2.4) is a solution of (2.3).

Proof. The proof follows the same lines as in the proof of Lemma 1.1 since in (i) and (ii) the term $\langle \partial_t u, v\rangle_\Sigma$ vanishes for the specific choice $(v,\tau) = (0, 1_x \cdot q(t))$. □

We equip the spaces with the norms:

$$\begin{aligned} \|u\|_X^2 &= \|u\|_Y^2 + \|\partial_t u\|_{Y'}^2 \\ \|(v,\tau)\|_{Y\times X_B,f}^2 &:= \int_0^T \Big[\|\nabla_x v\|_{L^2(\Omega)}^2 + [f(v,\tau)]^2\Big]\, dt + \|\tau\|_V^2 \\ \|(u,w)\|_{X\times X_B,f}^2 &:= \int_0^T \Big[\|\nabla_x u\|_{L^2(\Omega)}^2 + [f(u,w)]^2\Big]\, dt + \|\partial_t u\|_{Y'}^2 + \|w\|_V^2, \end{aligned} \tag{2.6}$$

where we make the following choice for f

$$f(v,\tau) = \langle (\frac{1}{2}I - K)v(t) + V\tau(t), 1\rangle_\Gamma.$$

Due to Lemma 1.3, $\|(\cdot,\cdot)\|_{Y\times X_B,f}$ is an equivalent norm in $Y \times X_B$. Accordingly, $\|(\cdot,\cdot)\|_{X\times X_B,f}$ is an equivalent norm in $X \times X_B$.

Lemma 2.2. *The modified bilinear form (2.5) is bounded in the following sense:*

$$|\widehat{a}((u,w),(v,\tau))| \le \max\{1, c_b\}\big[\|(u,w)\|_{X\times X_B,f}\|(v,\tau)\|_{Y\times X_B,f}\big].$$

Proof. In the proof we mainly apply the boundedness estimate of Theorem 1.6, where we proofed it for the modified bilinear form in the quasistatic case:

$$\begin{aligned}
|\widehat{a}((u,w),(v,\tau))| &\leq |\langle \partial_t u, v\rangle_Q| + |\widehat{d}((u,w),(v,\tau))| \\
&\leq \|\partial_t u\|_{Y'}\|v\|_Y + c_b\|(u,w)\|_{Y\times X_B,f}\|(v,\tau)\|_{Y\times X_B,f} \\
&= \begin{pmatrix} c_b\|(u,w)\|_{Y\times X_B,f} \\ \|\partial_t u\|_{Y'} \end{pmatrix} \cdot \begin{pmatrix} \|(v,\tau)\|_{Y\times X_B,f} \\ \|v\|_Y \end{pmatrix} \\
&\leq \max\{1,c_b\}\left[\|(u,w)\|_{X\times X_B,f}\|(v,\tau)\|_{Y\times X_B,f}\right].
\end{aligned}$$

□

Our goal now is to derive an equivalent norm for $\|\partial_t u\|_{Y'}$ which we will use in the proof of the injectivity of $\widehat{a}$ in Theorem 2.4. We consider the general related dual norm:

$$\|(h,g)\|_{(Y\times X_B,f)'} = \sup_{(0,0)\neq(v,\tau)\in(Y\times X_B)} \frac{\langle h,v\rangle_Q + \langle g,\tau\rangle_\Sigma}{\|(v,\tau)\|_{Y\times X_B,f}}.$$

If we set $h = \partial_t u$ and $g = 0$, we have the following norm (for detailed information see [2, Sect. 4]):

$$|(\partial_t u, 0)\|_{(Y\times X_B,f)'} = \|\partial_t u\|_{Y'} = \sup_{(0,0)\neq(v,\tau)\in(Y\times X_B)} \frac{\langle \partial_t u, v\rangle_Q}{\|(v,\tau)\|_{Y\times X_B,f}}. \tag{2.7}$$

In the spirit of [21], we can state the following equivalent norm by the help of the adjoint problem (1.20) of the quasistatic case.

Lemma 2.3. *For $u \in X$ and the weak solution $(z,w_z) \in Y \times X_B$ of the quasistatic problem*

$$\text{Find } (z,w_z) \in Y\times X_B: \quad \widehat{d}((v,\tau),(z,w_z)) = \langle \partial_t u, v\rangle_Q \quad \forall (v,\tau)\in Y\times X_B, \tag{2.8}$$

there holds the following "norm equivalence":

$$c_{el}\|(z,w_z)\|_{Y\times X_B,f} \leq \|\partial_t u\|_{Y'} \leq c_b\|(z,w_z)\|_{Y\times X_B,f}. \tag{2.9}$$

Proof. First we rewrite the norm of $\partial_t u$ by (2.7) and (2.8)

$$\begin{aligned}
\|\partial_t u\|_{Y'} &= \sup_{(0,0)\neq(v,\tau)\in(Y\times X_B)} \frac{\langle \partial_t u, v\rangle_Q}{\|(v,\tau)\|_{Y\times X_B,f}} \\
&= \sup_{(0,0)\neq(v,\tau)\in(Y\times X_B)} \frac{\widehat{d}((v,\tau),(z,w_z))}{\|(v,\tau)\|_{Y\times X_B,f}}.
\end{aligned}$$

For the lower estimate we use the ellipticity estimate of the bilinear form $\widehat{d}$ in Theorem 1.5 with the specific choice $v = z, \tau = w_z$:

$$\|\partial_t u\|_{Y'} \geq \frac{\widehat{d}((z, w_z), (z, w_z))}{\|(z, w_z)\|_{Y\times X_B,f}} \geq c_{el} \frac{\|(z, w_z)\|^2_{Y\times X_B,f}}{\|(z, w_z)\|_{Y\times X_B,f}} = c_{el}\|(z, w_z)\|_{Y\times X_B,f}.$$

For the upper estimate we use the boundedness estimate of $\widehat{d}$ in Theorem 1.6:

$$\|\partial_t u\|_{Y'} \leq c_b \sup_{(0,0)\neq(v,\tau)\in(Y\times X_B)} \frac{\|(v, \tau)\|_{Y\times X_B,f}\|(z, w_z)\|_{Y\times X_B,f}}{\|(v, \tau)\|_{Y\times X_B,f}} = c_b\|(z, w_z)\|_{Y\times X_B,f}.$$

□

Now our aim is to prove the unique solvability of the coupled variational formulation (2.4). The following theorem states the injectivity of the modified coupled bilinear form $\widehat{a}$. The proof mainly follows the idea of [21], where O. Steinbach proved the inf-sup condition for a space-time formulation of the Dirichlet boundary value problem of the heat equation. Later we prove the surjectivity in Theorem 2.6.

Theorem 2.4. *There exists a constant* $c_s > 0$ *such that for all* $(u, w) \in X \times X_B$

$$c_s\|(u, w)\|_{X\times X_B,f} \leq \sup_{(0,0)\neq(v,\tau)\in(Y\times X_B)} \frac{|\widehat{a}((u, w), (v, \tau))|}{\|(v, \tau)\|_{Y\times X_B,f}}. \tag{2.10}$$

Proof. The idea is to find a suitable $(\overline{v}, \overline{\tau})$. We will construct this by the help of the solution (z, w_z) of the adjoint problem (2.8). We see for the special choice $\tau = w$:

$$\langle \partial_t u, u\rangle_Q = \widehat{d}((u, w), (z, w_z)). \tag{2.11}$$

Now we make a specific choice for $\overline{v} = u + z$ and $\overline{\tau} = w + w_z$ where z and w_z are the solution of the quasistatic problem (2.8) and then we derive lower estimates to switch to the right norm. We rewrite the bilinear form and apply the ellipticity estimate of $\widehat{d}$, see Theorem 1.5:

$$\begin{aligned}\widehat{a}((u, w), (\overline{v}, \overline{\tau})) &= \widehat{a}((u, w), (u + z, w + w_z))\\ &= \langle \partial_t u, u\rangle_Q + \widehat{d}((u, w), (u, w)) + \langle \partial_t u, z\rangle_Q + \underbrace{\widehat{d}((u, w), (z, w_z))}_{\overset{(2.11)}{=}\langle \partial_t u, u\rangle_Q}\\ &\geq 2\langle \partial_t u, u\rangle_Q + c_{el}\|(u, w)\|^2_{Y\times X_B,f} + \langle \partial_t u, z\rangle_Q. \end{aligned} \tag{2.12}$$

We estimate the last term by the ellipticity of Theorem 1.5 and estimate (2.9):

$$\langle \partial_t u, z\rangle_Q \overset{(2.8)}{=} \widehat{d}((z, w_z), (z, w_z)) \geq c_{el}\|(z, w_z)\|^2_{Y\times X_B,f} \geq \frac{c_{el}}{c_b^2}\|\partial_t u\|^2_{Y'}. \tag{2.13}$$

The first term in (2.12) is non-negative as we get by the Lions–Magenes lemma [10] and the initial condition $u(x,0)=0$

$$2\langle\partial_t u,u\rangle_Q = 2\int_0^T \langle\partial_t u,u\rangle_\Omega dt = \int_0^T \frac{d}{dt}\int_\Omega [u(x,t)]^2 dxdt$$
$$= \int_\Omega [u(x,t)]^2 dx\Big|_0^T = \int_\Omega [u(x,T)]^2 dx \geq 0. \tag{2.14}$$

Before we finally prove the statement, we have one more estimate for $(\overline{v},\overline{\tau})$ using the estimate (2.9):

$$\|(\overline{v},\overline{\tau})\|_{Y\times X_B,f} = \left(\int_0^T\Big(\|\nabla_x(u+z)\|^2_{L^2(\Omega)} + [f(u,w)+f(z,w_z)]^2\Big)\,dt + \|w+w_z\|_V^2\right)^{1/2}$$
$$\leq \sqrt{2}\Big(\int_0^T \Big(\|\nabla_x u\|^2_{L^2(\Omega)} + [f(u,w)]^2\Big)\,dt + \|w\|_V^2 \tag{2.15}$$
$$+\int_0^T \Big(\|\nabla_x z\|^2_{L^2(\Omega)} + [f(z,w_z)]^2\Big)\,dt + \|w_z\|_V^2\Big)^{1/2}$$
$$= \sqrt{2}\big(\|(u,w)\|^2_{Y\times X_B,f} + \|(z,w_z)\|^2_{Y\times X_B,f}\big)$$
$$\leq \sqrt{2}(\|(u,w)\|^2_{Y\times X_B,f} + 1/c_{el}^2\|\partial_t u\|^2_{Y'})^{1/2}$$
$$\leq \sqrt{2}\max\{1,1/c_{el}\}\|(u,w)\|_{X\times X_B,f}. \tag{2.16}$$

Now we return to (2.12) and estimate by the two inequalities (2.14)and (2.13):

$$\widehat{a}((u,w),(\overline{v},\overline{\tau})) \geq 2\langle\partial_t u,u\rangle_Q + c_{el}\|(u,w)\|^2_{Y\times X_B,f} + \langle\partial_t u,z\rangle_Q$$
$$\geq c_{el}\|(u,w)\|^2_{Y\times X_B,f} + \frac{c_{el}}{c_b^2}\|\partial_t u\|^2_{Y'}$$
$$\geq c_{el}\min\{1,\frac{1}{c_b^2}\}\|(u,w)\|^2_{X\times X_B,f}.$$

With the inequality (2.15) we finally get

$$\widehat{a}((u,w),(\overline{v},\overline{\tau})) \geq c_s\|(u,w)\|_{X\times X_B,f}\|(\overline{v},\overline{\tau})\|_{Y\times X_B,f}$$

with

$$c_s = \frac{c_{el}\min\{1,\frac{1}{c_b^2}\}}{\sqrt{2}\max\{1,1/c_{el}\}}.$$

The statement follows by dividing by $\|(\overline{v},\overline{\tau})\|_{Y\times X_B,f}$ and taking the supremum. □

Of course, uniqueness follows from Theorem 2.4. Otherwise if our problem (2.4) had two solutions $u,\overline{u}$ and $w,\overline{w}$, we get $\widehat{a}(u-\overline{u},w-\overline{w};v,\tau)=0 \quad \forall(v,\tau)\in(Y\times$

X_B). Therefore $\|(u - \overline{u}, w - \overline{w})\|_{X \times V} \leq 0$ follows from (2.10). So injectivity follows immediately from the inf sup condition (2.10).

The next step is to prove the surjectivity of the modified bilinear form (2.5). But before we go on we define the exterior Dirichlet to Neumann map from the first boundary integral equation (1.7) and the corresponding Steklov Poincaré operator S^{ext}:

$$-\gamma_1^{ext} u = V^{-1}(\frac{1}{2}I - K)\gamma_0^{ext} u = S^{ext}\gamma_0^{ext} u.$$

With the second boundary integral equation (1.8) we find the symmetric representation of the exterior Steklov Poincaré operator:

$$S^{ext}\gamma_0^{ext} u = [D + (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K)]\gamma_0^{ext} u. \tag{2.17}$$

Before we go on with the theorem regarding the surjectivity of the bilinear form, we need to prove the following lemma, which is similar the Lions–Magenes lemma [10] and [23, Lem. 1.2] or [17, Thm. 10.9].

Lemma 2.5. *For a bounded symmetric bilinear form $b : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ there holds for $u \in H^1_{0,}(0, T; H^1(\Omega))$*

$$\int_0^T b(u(t), \partial_t u(t))dt = \frac{1}{2} b(u(T), u(T)). \tag{2.18}$$

Proof. From [25, p. 17, p. 24] and references therein we know that $C_0^\infty((0, T])$ is dense in $H^1_{0,}(0, T)$. The index "0," denotes zero initial conditions. Hence $C_0^\infty((0, T]; H^1(\Omega))$ is dense in $H^1_{0,}(0, T; H^1(\Omega)) := \{v \in H^1(0, T; H^1(\Omega)) | v(0) = 0\}$ with $H^1(0, T; H^1(\Omega)) := \{v \in L^2(0, T; H^1(\Omega)) | \partial_t v \in L^2(0, T; H^1(\Omega)\}$ [25, Chapt. 2]. Now we obtain a sequence of functions $(u_n)_{n\in\mathbb{N}}$ such that:

$$\lim_{n\to\infty} u_n = u \in H^1_{0,}(0, T; H^1(\Omega)). \tag{2.19}$$

Due to the fact that $(u_n)_{n\in\mathbb{N}}$ are smooth functions, we can obtain with the chain rule and the fact that b is a symmetric bilinear form:

$$\frac{d}{dt} b(u_n(t), u_n(t)) = 2b(u_n(t), \partial_t u_n(t)).$$

We can conclude with the fundamental theorem of calculus, the fact that $u_n(0) = 0$ and therefore $b(u_n(0), u_n(0)) = 0$:

$$b(u_n(T), u_n(T)) = b(u_n(T), u_n(T)) - b(u_n(0), u_n(0)) = 2\int_0^T b(u_n(t), \partial_t u_n(t))dt.$$

Hence (2.18) holds for smooth functions. What remains to show is that the assertion (2.18) also holds for $u \in H^1_{0,}(0, T; H^1(\Omega))$. Therefor we calculate the differences of both sides and show that they are converging to 0 by taking the limit. For the right-hand side we get by adding 0 and estimating the bilinear form with the boundedness constant c_2^B

$$\begin{aligned}
&|b(u(T),u(T)) - b(u_n(T),u_n(T))| \\
&\qquad = |b(u(T),u(T)-u_n(T)) - b(u_n(T),u(T)-u_n(T))| \\
&\qquad \leq c_2^B \|u(T)\|_{H^1(\Omega)} \|u(T)-u_n(T)\|_{H^1(\Omega)} + c_2^B \|u_n(T)\|_{H^1(\Omega)} \|u(T)-u_n(T)\|_{H^1(\Omega)},
\end{aligned}$$

since $u(t) \in H^1(\Omega)$ and $u_n(t) \in H^1(\Omega)$ for all $t \in (0,T)$ we get that both terms $\|u(T)\|_{H^1(\Omega)}$ and $\|u_n(T)\|_{H^1(\Omega)}$ are bounded. By the Sobolev embedding of $H^1((0,T))$ in $C^0([0,T])$ [1] we get the continuity of u in $t \in [0,T]$ and

$$\|u(T)-u_n(T)\|_{H^1(\Omega)} \leq \|u-u_n\|_{C([0,T];H^1(\Omega)} \leq c\|u-u_n\|_{H^1(0,T;H^1(\Omega))}.$$

Hence $b(u_n(T),u_n(T)) \to b(u(T),u(T))$ in $H^1(\Omega)$.

For the left-hand side we can conclude with the triangular inequality, the boundedness of the bilinear form and the Cauchy Schwarz inequality:

$$\begin{aligned}
&\Big| \int_0^T b(u(t),\partial_t u(t))dt - \int_0^T b(u_n(t),\partial_t u_n(t))dt \Big| \\
&\qquad \leq \int_0^T |b(u(t)-u_n(t),\partial_t u(t))|dt + \int_0^T |b(u_n(t),\partial_t(u_n(t)-u(t)))|dt \\
&\qquad \leq c_2^B \int_0^T (\|u(t)-u_n(t)\|_{H^1(\Omega)} \|\partial_t u(t)\|_{H^1(\Omega)})dt \\
&\qquad\quad + c_2^B \int_0^T (\|u_n(t)\|_{H^1(\Omega)} \|\partial_t(u_n(t)-u(t))\|_{H^1(\Omega)})dt \\
&\qquad \leq c_2^B \|u-u_n\|_{L^2(0,T,H^1(\Omega))} \|\partial_t u\|_{L^2(0,T,H^1(\Omega))} \\
&\qquad\quad + c_2^B \|\partial_t(u_n-u)\|_{L^2(0,T,H^1(\Omega))} \|u_n\|_{L^2(0,T,H^1(\Omega))} \\
&\qquad \to 0,
\end{aligned}$$

as $u_n \to u$, $\partial_t u_n \to \partial_t u$ in $L^2(0,T;H^1(\Omega))$. □

Remark 2.1. For Lemma 2.5 we consider two different cases for $u, v \in H^1(\Omega)$, namely

$$\begin{aligned}
b_1(v,w) &= \int_\Omega \nabla v \cdot \nabla w dx \quad \text{and} \\
b_2(v,w) &= \langle S^{ext}\gamma_0^{int} v, \gamma_0^{int} w\rangle_\Gamma,
\end{aligned}$$

The first bilinear form $b_1(v,w)$ is of course symmetric and bounded since

$$b_1(v,w) \leq \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}.$$

For the second bilinear form we firstly consider [19, Sect. 6.6.3], where the main properties of the boundary integral operators are given. Hence we know that S^{ext} is symmetric and elliptic. Regarding the boundedness of b_2, we rewrite it with the help of the fact that V^{-1} is bounded (by Lax Milgram), that S^{ext} is bounded and the trace theorem [19, Thm. 2.21]:

$$\begin{aligned}\langle S^{ext}\gamma_0^{int}v, \gamma_0^{int}w\rangle_\Gamma &\leq \|S^{ext}\gamma_0^{int}v\|_{H^{-1/2}(\Gamma)}\|\gamma_0^{int}w\|_{H^{1/2}(\Gamma)}\\ &\leq c_2^{ext}\|\gamma_0^{int}v\|_{H^{1/2}(\Gamma)}\|\gamma_0^{int}w\|_{H^{1/2}(\Gamma)}\\ &\leq c_2^{ext}c_T^2\|v\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)}.\end{aligned}$$

Therefore b_2 is bounded, of course it is also symmetric due to the symmetric representation (2.17) of S^{ext} .

The surjectivity of $\widehat{a}$ is stated in the following theorem. The proof mainly follows the idea of [22] where O. Steinbach and M. Zank proved also surjectivity for a Dirichlet boundary value problem of the heat equation.

Theorem 2.6. *For arbitrary $v \in Y$ and $\tau \in X_B$ with $(v,\tau) \neq (0,0)$, there exist $\overline{u} \in X$ and $\overline{w} \in X_B$ such that*

$$\widehat{a}((\overline{u},\overline{w}),(v,\tau)) > 0. \tag{2.20}$$

Proof. (i) Firstly we prove the general case, where we assume $v \neq 0$. We make the following choice similar to [22]

$$\overline{u}(t) = \int_0^t v(s)ds, \quad \overline{w} = -S^{ext}\overline{u} \tag{2.21}$$

where S^{ext} denotes the exterior Steklov Poincaré operator. Due to $u(x,0) = 0$ we have

$$\partial_t\overline{u} = v.$$

We plug (2.21) into the bilinear form (2.5) and use the non-symmetric representation of the Steklov Poincaré operator:

$$\begin{aligned}\widehat{a}((\overline{u},\overline{w}),(v,\tau)) &= \langle v,v\rangle_Q + \langle\nabla_x\overline{u},\nabla_x\partial_t\overline{u}\rangle_{L^2(Q)} + \langle S^{ext}\overline{u},\partial_t\overline{u}\rangle_\Sigma\\ &- \langle(\frac{1}{2}I-K)\overline{u},\tau\rangle_\Sigma + \langle(\frac{1}{2}I-K)\overline{u},\tau\rangle_\Sigma\\ &+ \int_0^T \Big[\langle(\frac{1}{2}I-K)\overline{u}(t) - (\frac{1}{2}I-K)\overline{u}(t)\rangle_\Gamma\langle(\frac{1}{2}I-K)v(t)+V\tau(t),1\rangle_\Gamma\Big]dt.\end{aligned}$$

The last three terms cancel immediately. For the first term we have

$$\langle v,v\rangle_Q = \int_0^T\int_\Omega v^2dxdt > 0 \quad \text{for } v \neq 0$$

and for the second term we get by applying Lemma 2.5 with $b(v,w) = \int_\Omega \nabla v \cdot \nabla w dx$, see Remark 2.1

$$\langle \nabla_x \overline{u}, \nabla_x \partial_t \overline{u} \rangle_{L^2(Q)} = \frac{1}{2} \int_\Omega |\nabla_x \overline{u}(x,T)|^2 dx \geq 0.$$

For the third term we apply also Lemma 2.5 with $b(v,w) = \langle S^{ext} v, w \rangle_\Gamma$, for more information see Remark 2.1, therefore we get

$$\int_0^T \langle S^{ext} \overline{u}, \partial_t \overline{u} \rangle_\Gamma dt = \frac{1}{2} \langle S^{ext} \overline{u}(T), \overline{u}(T) \rangle_\Gamma \geq 0.$$

(ii) Now we consider the case $v = 0$ and $\tau \neq 0$. By plugging in the special choice $\overline{u} = 0$ and $\overline{w} = \tau$, we get

$$\widehat{a}((0,\tau),(0,\tau)) = \langle V\tau, \tau \rangle_\Sigma + \int_0^T \langle V\tau(t), 1 \rangle_\Gamma^2 dt > 0$$

since $\langle V\tau, \tau \rangle_\Sigma > 0$ and $\langle V\tau(t), 1 \rangle_\Gamma^2 \geq 0$.

□

From Theorem 2.4 and 2.6 the unique solvability of the modified varational formulation (2.4) follows immediately. Therefore the original variational formulation (2.3) is also uniquely solvable since both formulations are equivalent, see Lemma 2.1.

2.2 Discrete variational formulation for the coupled problem

We consider two different kind of decomposition of the FE and BE mesh. For ease of presentation we consider here only the two dimensional space setting, but the theoretical results can be extended for a dimension higher. In a first consideration, we assume a conforming decomposition of the FE mesh into tetrahedrons and a conforming triangulation of the lateral boundary regarding the BE mesh. Furthermore we assume that the boundary mesh is the trace of the volume mesh. For the discrete variational formulation related to (2.4) we use the spaces

$$X_h = Y_h = S_h^1(Q) \cap X \subset X \subset Y$$
$$X_{B,h} = S_h^0(\Sigma).$$

We consider the following discrete variational formulation: Find $(\hat{u}_h, \hat{w}_h) \in X_h \times X_{B,h}$:

$$a((\hat{u}_h, \hat{w}_h); (v_h, \tau_h)) = \langle f, v_h \rangle_Q \quad \forall (v_h, \tau_h) \in X_h \times X_{B,h} \tag{2.22}$$

and the modified variational formulation: Find $(u_h, w_h) \in X_h \times X_{B,h}$:

$$\widehat{a}((u_h, w_h); (v_h, \tau_h)) = \langle f, v_h \rangle_Q \quad \forall (v_h, \tau_h) \in X_h \times X_{B,h}. \tag{2.23}$$

Remark 2.2. The variational formulations (2.22) and (2.23) are in general not equivalent in the sense as in Lemma 2.1 since $1_x q(t) \notin S_h^0(\Sigma)$. In Lemma 2.7 we will discuss the similarity of both variational formulations for the special case of a tensor product space.

In a second case, we consider a tensor product mesh, where the time interval $(0, T)$ is decomposed via the time steps

$$0 = t_0 < t_1 < \cdots < t_{n_t} = T$$

where n_t denotes the number of time intervals $(t_{\ell-1}, t_\ell)$ for $\ell = 1, \cdots n_t$. Also we assume that the boundary mesh is the trace of the volume mesh. We denote the following spaces for the tensor product mesh:

$$\begin{aligned} X_h = Y_h &= S_{h_x}^1(\Omega) \times S_{h_t}^1(0,T) \cap X \\ X_{B,h} &= S_{h_x}^0(\Gamma) \times S_{h_t}^0(0,T). \end{aligned} \tag{2.24}$$

The solution (u_h, w_h) of (2.23) and $(\hat{u}_h, \hat{w}_h)$ are of the following form, where we assume $u_0(x) = 0$, we only state them for the modified case since the solution of the non modified variational formulation is of an equivalent form:

$$\begin{aligned} u_h(x,t) &= \sum_{\ell=0}^{n_t} u_\ell(x)\varphi_\ell(t) \quad \text{for } u_\ell \in S_h^1(\Omega), \\ w_h(x,t) &= \sum_{\ell=1}^{n_t} w_\ell(x)\psi_\ell(t) \quad \text{for } w_\ell \in S_h^0(\Gamma), \end{aligned}$$

with the piecewise linear basis functions

$$\varphi_\ell(t) = \begin{cases} 1 & \text{for } t = t_\ell, \\ 0 & \text{for } t = t_k \neq t_\ell, \\ \text{piecewise linear} & \text{else} \end{cases}$$

and the piecewise constant basis functions

$$\psi_\ell(t) = \begin{cases} 1 & \text{for } t \in (t_{\ell-1}, t_\ell), \\ 0 & \text{else.} \end{cases}$$

Lemma 2.7. *Under the assumptions that we consider a decomposition of the space-time domain Q in a tensor product mesh and with the discrete spaces (2.24), let $(\hat{u}_h, \hat{w}_h) \in X_h \times X_{B,h}$ be a solution of (2.22) and $(u_h, w_h) \in X_h \times X_{B,h}$ be a solution of (2.23). Then there hold*

$$\begin{aligned} 0 &= \langle V\hat{w}_\ell, 1_x\rangle_\Gamma + \langle (\frac{1}{2}I - K)\frac{\hat{u}_{\ell-1} + \hat{u}_\ell}{2}, 1_x\rangle_\Gamma, \\ 0 &= \langle V w_\ell, 1_x\rangle_\Gamma + \langle (\frac{1}{2}I - K)\frac{u_{\ell-1} + u_\ell}{2}, 1_x\rangle_\Gamma, \end{aligned}$$

i.e. the constraint is fulfilled in the midpoints $t = \frac{t_\ell + t_{\ell-1}}{2}$ of the time intervals.

Proof. Let $(\hat{u}_h, \hat{w}_h) \in X_h \times X_{B,h}$ be a solution of (2.22). If we choose $(v, \tau) = (0, 1_x\psi_\ell(t))$ with $1_x \in H^{-1/2}(\Gamma)$ and $\psi_\ell(t) \in S_h^0(0,T)$ for arbitrary $\ell \in \{1, \cdots, n_t\}$, we get from (2.22)

$$\int_{t_{\ell-1}}^{t_\ell} (\langle V\hat{w}_h(t), 1_x\rangle_\Gamma + \langle(\frac{1}{2}I - K)\hat{u}_h(t), 1_x\rangle_\Gamma)dt = 0.$$

Let $(u, w_h) \in X_h \times X_{B,h}$ be a solution of (2.23). If we choose $(v, \tau) = (0, 1_x\psi_\ell(t))$ with $1_x \in H^{-1/2}(\Gamma)$ and $\psi_\ell(t) \in S_h^0(0,T)$ for arbitrary $\ell \in \{1, \cdots, n_t\}$, we get from (2.23)

$$\int_{t_{\ell-1}}^{t_\ell} [\langle(\frac{1}{2}I - K)u_h(t) + Vw_h(t), 1_x\rangle_\Gamma \langle V1_x, 1_x\rangle_\Gamma + \langle Vw_h, 1_x\rangle_\Gamma + \langle(\frac{1}{2}I - K)u_h(t), 1_x\rangle_\Gamma]dt = 0,$$

which is equivalent to

$$(\langle V1_x, 1_x\rangle_\Gamma + 1)\int_{t_{\ell-1}}^{t_\ell} \Big[\langle Vw_h(t), 1_x\rangle_\Gamma + \langle(\frac{1}{2}I - K)u_h(t), 1_x\rangle_\Gamma\Big]dt = 0.$$

As $(\langle V1_x, 1_x\rangle_\Gamma + 1) > 0$ we get

$$\int_{t_{\ell-1}}^{t_\ell} \Big[\langle Vw_h(t), 1_x\rangle_\Gamma + \langle(\frac{1}{2}I - K)u_h(t), 1_x\rangle_\Gamma\Big]dt = 0. \tag{2.25}$$

Therefore we have the same constraint for both cases. We now only consider the second one. The first one is similar. On the time interval $(t_{\ell-1}, t_\ell)$ we use the local representations of $w_h(x,t)$ and $u_h(x,t)$:

$$\begin{aligned} w_h(x,t) &= w_\ell(x) \\ u_h(x,t) &= u_{\ell-1}(x)\frac{t_\ell - t}{t_\ell - t_{\ell-1}} + u_\ell(x)\frac{t - t_{\ell-1}}{t_\ell - t_{\ell-1}}. \end{aligned}$$

By plugging them into the constraint (2.25) we get by integration

$$(t_\ell - t_{\ell-1})\langle Vw_\ell, 1_x\rangle_\Gamma + \frac{t_\ell - t_{\ell-1}}{2}\langle(\frac{1}{2}I - K)(u_{\ell-1} + u_\ell), 1_x\rangle_\Gamma = 0.$$

By dividing by $t_\ell - t_{\ell-1}$ the constraint of the statement follows. Since

$$u_h(\frac{t_{\ell-1} + t_\ell}{2}) = \frac{u_{\ell-1} + u_\ell}{2}$$

the constraint is fulfilled in the midpoints of the intervals. □

Remark 2.3. For the tensor product setting with

$$X_h = Y_h = (S^1_{h_x}(\Omega) \times S^1_{h_t}(0,T)) \cap X$$
$$X_{B,h} = S^0_{h_x}(\Gamma) \times S^1_{h_t}(0,T)$$

we can proof the equivalence of the variational formulations (2.22) and (2.23). Unfortunately, such spaces are not suitable for a general space-time mesh.

Motivated by [21, p. 9] we now want to derive a discrete norm. In the proof of the inf-sup condition (2.10) we made use of the "norm equivalence" (2.9)

$$c_{el}\|(z, w_z)\|_{Y\times X_B,f} \le \|\partial_t u\|_{Y'} \le c_b\|(z, w_z)\|_{Y\times X_B,f}$$

where $(z, w_z) \in Y \times X_B$ is the solution of the adjoint problem (2.8). Moreover (z, w_z) was essential for our choice of the test functions in the proof of Theorem 2.4. For the discrete inf-sup condition we would like to find discrete counterparts to (z, w_z). We consider the discrete adjoint problem: Find $(z_h, w_{z,h}) \in Y_h \times X_{B,h}$:

$$\widehat{d}((v_h, \tau_h), (z_h, w_{z,h})) = \langle \partial_t u, v_h\rangle_Q \quad \forall (v_h, \tau_h) \in Y_h \times X_{B,h}. \tag{2.26}$$

Later we will replace u by u_h. Motivated by (2.9), we define the discrete norm

$$\|(u, w)\|_{X_h\times X_B,f} = \left(\|(u, w)\|^2_{Y\times X_B,f} + \|(z_h, w_{z,h})\|^2_{Y\times X_B,f}\right)^{1/2} \tag{2.27}$$

where $(z_h, w_{z,h})$ is the solution of (2.26). We now state and proof two lemmas regarding the discrete norm, which will be of importance later.

Lemma 2.8. *For $(z_h, w_{z,h})$ the solution of (2.23) there holds the following inequality:*

$$\|(z_h, w_{z,h})\|_{Y\times X_B,f} \le \frac{c_b}{c_{el}}\|(z, w_z)\|_{Y\times X_B,f} \le \frac{c_b}{c_{el}^2}\|\partial_t u\|_{Y'}.$$

Proof. By the ellipticity estimate from Theorem 1.5, equation (2.26) for the special choice $(v_h, \tau_h) = (z_h, w_{z,h})$, (2.8) for $Y_h \subset Y$ and $X_B \subset X_{B,h}$ with $(v, \tau) = (z_h, w_{z,h})$ and the boundedness estimate of Theorem 1.6 we get

$$\begin{aligned}\|(z_h, w_{z,h})\|^2_{Y\times X_B,f} &\le \frac{1}{c_{el}}\widehat{d}((z_h, w_{z,h}), (z_h, w_{z,h})) = \frac{1}{c_{el}}\langle \partial_t u, z_h\rangle_Q \\ &= \frac{1}{c_{el}}\widehat{d}((z_h, w_{z,h}), (z, w_z)) \le \frac{c_b}{c_{el}}\|(z_h, w_{z,h})\|_{Y\times X_B,f}\|(z, w_z)\|_{Y\times X_B,f}.\end{aligned}$$

The first part of the inequality follows now by dividing the term $\|(z_h, w_{z,h})\|_{Y\times X_B,f}$ on both ends. The last part follows by applying the lower estimate of the "norm equivalence" (2.9). □

Lemma 2.9. $\|(\cdot,\cdot)\|_{X_h\times X_B,f} : X \times X_B \to \mathbb{R}^+_0$ *defines a mesh dependent norm.*

Proof. We check the properties for the definition of a norm. The norm is bounded, since Lemma 2.8 holds.
(i) Firstly we check the definiteness:

$$\|(u,w)\|^2_{X_h\times X_B,f} = \|(u,w)\|^2_{Y\times X_B,f} + \|(z_h,w_{z,h})\|^2_{Y\times X_B,f} = 0,$$

since both parts are greater or equal than zero, each part is zero. From the first term it follows directly that $u=0$ and $w=0$ since $\|(\cdot,\cdot)\|_{Y\times X_B,f}$ defines a norm. $(z_h,w_{z,h})$ is the solution of the adjoint problem (2.26) with $\partial_t u = 0$ in the right-hand side. Thus we conclude by the unique solvability of the variational formulation of the quasistatic problem (1.16) that $z_h = 0$ and $w_{z,h}=0$.
(ii) To check the absolute homogeneity of the norm, we reconsider the adjoint problem (2.26) and multiply both sides with α, therefore the problem is of the following form

$$\widehat{d}((v_h,\tau_h),(\alpha z_h,\alpha w_{z,h})) = \langle \partial_t(\alpha u), v_h\rangle_Q.$$

Hence $(\alpha z_h,\alpha w_{z,h})$ is the solution of the modified problem and now we can rewrite the norm

$$\begin{aligned}\|(\alpha u,\alpha w)\|_{X_h\times X_B,f} &= \left(\|(\alpha u,\alpha w)\|^2_{Y\times X_B,f} + \|(\alpha z_h,\alpha w_{z,h})\|^2_{Y\times X_B,f}\right)^{1/2}\\ &= \left(|\alpha|^2\|(u,w)\|^2_{Y\times X_B,f} + |\alpha|^2\|(z_h,w_{z,h})\|^2_{Y\times X_B,f}\right)^{1/2}\\ &= |\alpha|\|(u,w)\|_{X_h\times X_B,f},\end{aligned}$$

the equality follows since $\|(\cdot,\cdot)\|_{Y\times X_B,f}$ defines per se a norm.
(iii) We check the triangular inequality. In a first step we rewrite the norm

$$\|(u+\widehat{u},w+\widehat{w})\|^2_{X_h\times X_B,f} = \|(u+\widehat{u},w+\widehat{w})\|^2_{Y\times X_B,f} + \|(\widetilde{z}_h,\widetilde{w}_{z,h})\|^2_{Y\times X_B,f},$$

where $(\widetilde{z}_h,\widetilde{w}_{z,h})$ is the modified solution of the variational formulation (2.26) with:

$$\widehat{d}((v_h,\tau_h),(\widetilde{z}_h,\widetilde{w}_{z,h})) = \langle \partial_t(u+\widehat{u}), v_h\rangle_Q.$$

Starting from the solution of the two problems

$$\begin{aligned}\widehat{d}((v_h,\tau_h),(z_h,w_{z,h})) &= \langle \partial_t u, v_h\rangle_Q\\ \widehat{d}((v_h,\tau_h),(\widehat{z}_h,\widehat{w}_{z,h})) &= \langle \partial_t \widehat{u}, v_h\rangle_Q,\end{aligned}$$

we conclude $(\widetilde{z}_h,\widetilde{w}_{z,h}) = (z_h+\widehat{z}_h, w_{z,h}+\widehat{w}_{z,h})$. Due to this conclusion, we further estimate since $\|(\cdot,\cdot)\|_{Y\times X_B,f}$ defines a norm:

$$\|(\widetilde{z}_h,\widetilde{w}_{z,h})\|_{Y\times X_B,f} \le \|(z_h,w_{z,h})\|_{Y\times X_B,f} + \|(\widehat{z}_h,\widehat{w}_{z,h})\|_{Y\times X_B,f}.$$

Hence we can now make the final estimate by applying the triangular inequality for the Euclidean norm:

$$\begin{aligned}
\|(u+\widehat{u}, w+\widehat{w})\|_{X_h\times X_B,f} &\leq \Big((\|(u,w)\|_{Y\times X_B,f} + \|(\widehat{u},\widehat{w})\|_{Y\times X_B,f})^2 \\
&\quad + (\|(z_h,w_{z,h})\|_{Y\times X_B,f} + \|(\widehat{z}_h,\widehat{w}_{z,h})\|_{Y\times X_B,f})^2 \Big)^{1/2} \\
&= \left\| \begin{pmatrix} \|(u,w)\|_{Y\times X_B,f} + \|(\widehat{u},\widehat{w})\|_{Y\times X_B,f} \\ \|(z_h,w_{z,h})\|_{Y\times X_B,f} + \|(\widehat{z}_h,\widehat{w}_{z,h})\|_{Y\times X_B,f} \end{pmatrix} \right\|_2 \\
&= \left\| \begin{pmatrix} \|(u,w)\|_{Y\times X_B,f} \\ \|(z_h,w_{z,h})\|_{Y\times X_B,f} \end{pmatrix} + \begin{pmatrix} \|(\widehat{u},\widehat{w})\|_{Y\times X_B,f} \\ \|(\widehat{z}_h,\widehat{w}_{z,h})\|_{Y\times X_B,f} \end{pmatrix} \right\|_2 \\
&\leq \left\| \begin{pmatrix} \|(u,w)\|_{Y\times X_B,f} \\ \|(z_h,w_{z,h})\|_{Y\times X_B,f} \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} \|(\widehat{u},\widehat{w})\|_{Y\times X_B,f} \\ \|(\widehat{z}_h,\widehat{w}_{z,h})\|_{Y\times X_B,f} \end{pmatrix} \right\|_2 \\
&= \|(u,w)\|_{X_h\times X_B,f} + \|(\widehat{u},\widehat{w})\|_{X_h,X_B,f}.
\end{aligned}$$

□

Now we can state the discrete inf-sup condition.

Theorem 2.10. *There exist a constant $\tilde{c}_s > 0$ such that*

$$\tilde{c}_s \|(u_h,w_h)\|_{X_h\times X_B,f} \leq \sup_{(0,0)\neq(v_h,\tau_h)\in X_h\times X_{B,h}} \frac{|\widehat{a}((u_h,w_h),(v_h,\tau_h))|}{\|(v_h,\tau_h)\|_{Y\times X_B,f}} \quad \forall u_h \in X_h, w_h \in X_{B,h}. \tag{2.28}$$

Proof. As in the proof of Theorem 2.4 we consider the adjoint formulation of the quasistatic problem similar to (2.11): Find $(z_h, w_{z,h}) \in Y_h \times X_{B,h}$ such that:

$$\widehat{d}((v_h,\tau_h),(z_h,w_{z,h})) = \langle \partial_t u_h, v_h\rangle_Q \quad \forall (v_h,\tau_h) \in Y_h \times X_{B,h}. \tag{2.29}$$

We now repeat the steps in order to show (2.15) but make some adjustments to the discrete norm (2.27). We have for the special choice $(v_h,\tau_h) = (u_h,w_h)$:

$$\langle \partial_t u_h, u_h\rangle_Q = \widehat{d}((u_h,w_h),(z_h,w_{z,h})).$$

We make a specific choice $\overline{v}_h = u_h + z_h$ and $\overline{w}_h = w_h + w_{z,h}$ where $(z_h, w_{z,h})$ denotes the solution of (2.29). Therefore we get the same result as in (2.12):

$$\begin{aligned}
&\widehat{a}((u_h,w_h),(\overline{v}_h,\overline{\tau}_h)) = \widehat{a}((u_h,w_h),(u_h+z_h, w+w_{z,h})) \\
&\quad = \langle \partial_t u_h, u_h\rangle_Q + \widehat{d}((u_h,w_h),(u_h,w_h)) + \langle \partial_t u_h, z_h\rangle_Q + \underbrace{\widehat{d}((u_h,w_h),(z_h,w_{z,h}))}_{=\langle \partial_t u_h, u_h\rangle_Q} \\
&\quad \geq 2\langle \partial_t u_h, u_h\rangle_Q + c_{el}\|(u_h,w_h)\|^2_{Y\times X_B,f} + \langle \partial_t u_h, z_h\rangle_Q.
\end{aligned} \tag{2.30}$$

We estimate the last term by the ellipticity of Theorem 1.5:

$$\langle \partial_t u_h, z_h \rangle_Q = \widehat{d}((z_h, w_{z,h}), (z_h, w_{z,h})) \geq c_{el} \|(z_h, w_{z,h})\|^2_{Y \times X_B, f}. \tag{2.31}$$

Furthermore we have due to the Lions–Magenes lemma [10] and the initial condition $u_h(x, 0) = 0$:

$$\begin{aligned} 2\langle \partial_t u_h, u_h \rangle_Q &= 2 \int_0^T \langle \partial_t u_h, u_h \rangle_\Omega dt = \int_0^T \frac{d}{dt} \int_\Omega [u_h(x,t)]^2 dx dt \\ &= \int_\Omega [u_h(x,t)]^2 dx \Big|_0^T = \int_\Omega [u_h(x,T)]^2 dx \geq 0. \end{aligned}$$

Different to the proof of Theorem 2.4, we stick to the discrete norm (2.27) in order to prove a similar statement as (2.15):

$$\begin{aligned} \|(\overline{v}_h, \overline{\tau}_h)\|_{Y \times X_B, f} &= \Big(\int_0^T \|\nabla_x (u_h + z_h)\|^2_{L^2(\Omega)} + [f(u_h, w_h) + f(z_h, w_{z,h})]^2 dt \\ &\qquad + \|w_h + w_{z,h}\|^2_V \Big)^{1/2} \\ &\leq \sqrt{2} \Big(\int_0^T \Big(\|\nabla_x u_h\|^2_{L^2(\Omega)} + [f(u_h, w_h)]^2 \Big) dt + \|w_h\|^2_V \\ &\qquad + \int_0^T \Big(\|\nabla_x z_h\|^2_{L^2(\Omega)} + [f(z_h, w_{z,h})]^2 \Big) dt + \|w_{z,h}\|^2_V \Big)^{1/2} \\ &= \sqrt{2} \left(\int_0^T \Big(\|\nabla_x u_h\|^2_{L^2(\Omega)} + [f(u_h, w_h)]^2 \Big) dt + \|w_h\|^2_V + \|(z_h, w_{z,h})\|^2_{Y \times X_B, f} \right)^{1/2} \\ &= \sqrt{2} \|(u_h, w_h)\|_{X_h \times X_B, f}. \end{aligned} \tag{2.32}$$

In the last step we rewrite equation (2.30) and apply estimate (2.31):

$$\begin{aligned} a((u_h, w_h), (\overline{v}_h, \overline{\tau}_h)) &\geq 2\langle \partial_t u_h, u_h \rangle_Q + c_{el} \|(u_h, w_h)\|^2_{Y \times X_B, f} + \langle \partial_t u_h, z_h \rangle_Q \\ &\geq c_{el} \|(u_h, w_h)\|^2_{Y \times X_B, f} + c_{el} \|(z_h, w_{z,h})\|^2_{Y \times X_B, f} \\ &\overset{(2.32)}{\geq} \tilde{c}_s \|(u_h, w_h)\|_{X_h \times X_B, f} \|(\overline{v}_h, \overline{\tau}_h)\|_{Y \times X_B, f} \end{aligned}$$

with $\tilde{c}_s$ defined as:

$$\tilde{c}_s = \frac{c_{el}}{\sqrt{2}}.$$

The statement follows by dividing by $\|(\overline{v}_h, \overline{\tau}_h)\|_{Y \times X_B, f}$ and taking the supremum as in the analytic setting. □

From Theorem 2.10 injectivity follows directly with the same argument as in the analytic setting. Since we have a finite dimensional problem and a quadratic system we get surjectivity since injectivity and surjectivity are now equivalent. Our problem is therefore indeed uniquely solvable. Similar to [21, Thm. 3.2] and [19, Thm. 8.4] we can state the following quasi optimal error estimate.

Theorem 2.11. *Let $(u,w) \in X \times X_B$ and $(u_h, w_h) \in X_h \times X_{B,h}$ be the unique solutions of the variational formulations (2.4) and (2.23), respectively. Then there holds the error estimate:*

$$\|(u-u_h, w-w_h)\|_{X_h \times X_B,f} \leq \left(1 + \frac{\max\{1, c_b\}}{\tilde{c}_s}\right) \inf_{(v_h,\tau_h)\in X_h\times X_B} \|(u-v_h, w-\tau_h)\|_{X\times X_B,f}.$$

Proof. For every $(u,w) \in X \times X_B$, we can define a Galerkin projection similar to [19, Thm. 8.4], where we get that $G_h(u,w) = (u_h, w_h)$ is the solution of the variational formulation

$$\widehat{a}(G_h(u,w), (v_h,\tau_h)) = \widehat{a}((u,w),(v_h,\tau_h)) \quad \forall (v_h,\tau_h) \in X_h \times X_{B,h}$$

and we can also observe that $G_h(v_h,\tau_h) = (v_h,\tau_h)$ for all $(v_h,\tau_h) \in X_h \times X_{B,h}$. From the Galerkin orthogonality

$$\widehat{a}((u_h - u, w_h - w),(v_h,\tau_h)) = 0 \quad \forall (v_h,\tau_h) \in X_h \times X_{B,h}$$

we get from the discrete inf-sup condition (Theorem 2.10) and the boundedness of the bilinear form $\widehat{a}$ (Lemma 2.2):

$$\begin{aligned} \tilde{c}_s \|(u_h,w_h)\|_{X_h\times X_B,f} &\leq \sup_{(0,0)\neq(v_h,\tau_h)\in X_h\times X_{B,h}} \frac{|\widehat{a}((u_h,w_h),(v_h,\tau_h))|}{\|(v_h,\tau_h)\|_{Y\times X_B,f}} \\ &= \sup_{(0,0)\neq(v_h,\tau_h)\in X_h\times X_{B,h}} \frac{|\widehat{a}((u,w),(v_h,\tau_h))|}{\|(v_h,\tau_h)\|_{Y\times X_B,f}} \\ &\leq \max\{1,c_b\}\|(u,w)\|_{X\times X_B,f}. \end{aligned}$$

Thus the Galerkin projection is bounded:

$$\|G_h(u,w)\|_{X_h\times X_B,f} \leq \frac{\max\{1,c_b\}}{\tilde{c}_s}\|(u,w)\|_{X\times X_B,f}.$$

By the triangular inequality and the previous estimate we get for the error

$$\begin{aligned} \|(u-u_h, w-w_h)\|_{X_h\times X_B,f} &= \|(u - v_h + v_h - u_h, w - \tau_h + \tau_h - w_h)\|_{X_h\times X_B,f} \\ &\leq \|(u-v_h, w-\tau_h)\|_{X_h\times X_B,f} + \|(u_h - v_h, w_h - \tau_h)\|_{X_h\times X_B,f} \\ &= \|(u-v_h, w-\tau_h)\|_{X_h\times X_B,f} + \|G_h(u-v_h, w-\tau_h)\|_{X_h\times X_B,f} \\ &\leq \left(1 + \frac{\max\{1,c_b\}}{\tilde{c}_s}\right)\|(u-v_h, w-\tau_h)\|_{X\times X_B,f}, \end{aligned}$$

using Lemma 2.8. The statement follows by taking the infimum. □

Similar to Lemma 1.3, it is only necessary to consider $\|(\cdot,\cdot)\|_{X_h\times X_B}$ since this an equivalent norm to $\|(\cdot,\cdot)\|_{X_h\times X_B,f}$. Therefore we can state the following Corollary, where we conclude a error estimate, with the help of Theorem 2.11.

Corollary 2.12. *Under the assumptions $u\in H^2(Q), \partial_t u \in H^2(Q)$ and $w\in H^1(\Sigma)$ we get the following error estimate:*

$$\|(u-u_h, w-w_h)\|_{X_h\times X_B} \leq \hat{c}\|(u-u_h, w-w_h)\|_{X_h\times X_B,f} \leq ch\left(\|u\|_{H^2(Q)} + |w|_{H^1(\Sigma)}\right).$$

Proof. We apply Theorem 2.11, norm equivalences and check the approximation properties:

$$\begin{aligned}
&\inf_{(v_h,\tau_h)\in X_h\times X_B} \|(u-v_h, w-\tau_h)\|_{X\times X_B} \\
&\qquad = \inf_{(v_h,\tau_h)\in X_h\times X_B} \left(\|u-v_h\|_Y^2 + \|\partial_t(u-v_h)\|_{Y'}^2 + \|w-\tau_h\|_V^2\right)^{1/2} \\
&\qquad \leq \inf_{v_h\in X_h} \left(\|u-v_h\|_Y + \|\partial_t(u-v_h)\|_{Y'}\right) + \inf_{\tau_h\in X_B} \|w-\tau_h\|_V \\
&\qquad \leq \inf_{v_h\in X_h} \left(\|u-v_h\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t(u-v_h)\|_{L^2(Q)}\right) + c\inf_{\tau_h\in X_B}\|w-\tau_h\|_{X_B} \\
&\qquad \leq \widehat{c}\inf_{v_h\in X_h}\|u-v_h\|_{H^1(Q)} + c\inf_{\tau_h\in X_B}\|w-\tau_h\|_{X_B} \\
&\qquad \leq c_1 h\|u\|_{H^2(Q)} + c_2 h|w|_{H^1(\Sigma)},
\end{aligned}$$

where we used approximation results from [19, Sect. 11] for the FEM part, while we estimate the norm and then apply the approximation property [19, p. 238] for the BEM part:

$$\begin{aligned}
\inf_{\tau_h\in X_B}\|w-\tau_h\|_{X_B} &= \inf_{\tau_h\in X_B}\sqrt{\int_0^T \|w(t)-\tau_h(t)\|_{H^{-1/2}(\Gamma)}^2 dt} \\
&\leq \inf_{\tau_h\in X_B}\sqrt{\int_0^T \|w(t)-\tau_h(t)\|_{L^2(\Gamma)}^2 dt} \\
&= \inf_{\tau_h\in X_B}\|w-\tau_h\|_{L^2(\Sigma)} \leq ch|w|_{H^1(\Sigma)}.
\end{aligned}$$

The assumptions for the estimate are $u\in H^2(Q), \partial_t u\in H^2(Q)$ and $w\in H^1(\Sigma)$. □

2.3 System of linear equations for the coupled problem

We now consider a finite dimensional quadratic problem related to (2.23) with an injective modified bilinear form. Due to that fact, we know that surjectivity and injectivity are indeed equivalent. So our system of linear equations is indeed uniquely solvable. Here we consider regarding the FE mesh piecewise linear, globally continuous functions, i.e. $S_h^1(Q) = \text{span}\{\varphi_i\}_{i=1}^M$, with the basis function

$$\varphi_i(x,t) = \begin{cases} 1 & \text{for } (x,t) = (x_i,t_i), \\ 0 & \text{for } (x,t) = (x_k,t_k) \neq (x_i,t_i), \\ \text{piecewise linear} & \text{else,} \end{cases}$$

where (x_i,t_i) is a vertex of the tetrahedral FE mesh. Regarding the elements on the boundary, i.e. the BE mesh, we consider the piecewise constant functions, i.e. $S_h^0(\Sigma) = \text{span}\{\psi_k\}_{k=1}^N$, with the basis functions

$$\psi_k(x,t) = \begin{cases} 1 & \text{for } (x,t) \in \tau_k, \\ 0 & \text{else,} \end{cases} \tag{2.33}$$

where τ_k denotes a boundary element. Here we denote M as the number of nodes of the FE mesh and N the number of elements on the boundary. If we considered the modified discrete variational formulation (2.23), several modifications would be necessary for the system of linear equations and the assembling of the matrices for the stabilization term would be quite challenging. For the sake of convenience, we consider the corresponding system of linear equations for the discrete variational formulation (2.22) without stabilization, which looks as follows

$$\begin{pmatrix} A_h & -M_h^T \\ \frac{1}{2}M_h - K_h & V_h \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{w} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ 0 \end{pmatrix}, \tag{2.34}$$

where the single blocks are given by

$$\begin{aligned} A_h[j,i] &= \int_Q (\nabla_x \varphi_i \cdot \nabla_x \varphi_j + \partial_t \varphi_i \varphi_j)\, dxdt \\ M_h[\ell,i] &= \langle \varphi_{i|_\Sigma}, \psi_\ell \rangle_\Sigma \\ K_h[\ell,i] &= \langle K\varphi_{i|_\Sigma}, \psi_\ell \rangle_\Sigma \\ V_h[\ell,k] &= \langle V\psi_k, \psi_\ell \rangle_\Sigma \\ f[i] &= \langle f, \varphi_i \rangle_Q \end{aligned}$$

for $i,j = 1,...,M, k,\ell = 1,...,N$. If we reorder the degree of freedoms regarding the FE mesh, the system 2.34 of linear equations has the following form

$$\begin{pmatrix} A_{II} & A_{I\Sigma} & \\ A_{\Sigma I} & A_{\Sigma\Sigma} & -M_h^T \\ & \frac{1}{2}M_h - K_h & V_h \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{u}_\Sigma \\ \underline{w} \end{pmatrix} = \begin{pmatrix} \underline{f}_I \\ \underline{f}_\Sigma \\ 0 \end{pmatrix}, \tag{2.35}$$

where $\underline{u}_I$ is the vector of the coefficients related to the inner degrees of freedom and $\underline{u}_\Sigma$ is the vector of coefficients related to the degrees of freedom at the interface $\Sigma = \Gamma \times (0,T)$.

Remark 2.4. If we would consider non-homogeneous initial conditions, there needs to be done a modification to the right-hand side, namely

$$\begin{aligned}\widehat{\underline{f}}_I &= \underline{f}_I - A_{IIN}\underline{u}_{IN}\\ \widehat{\underline{f}}_\Sigma &= \underline{f}_\Sigma - A_{\Sigma IN}\underline{u}_{IN},\end{aligned}$$

where $\underline{u}_{IN}$ is the vector of coefficients related to the degrees of freedom at $t = 0$. For the implementation, we simply consider for this vector an interpolation since the initial data is known a priori.

We now consider a Schur complement system by eliminating the inner degrees of freedom $\underline{u}_I$. Hence we get by eliminating the first line of (2.35):

$$\underline{u}_I = (A_{II})^{-1}\left(\underline{f}_I - A_{I\Sigma}\underline{u}_\Sigma\right).$$

By plugging that into the second line of the system of linear equation (2.35), we get for the first term in the second line of (2.36):

$$A_{\Sigma I}\underline{u}_I = A_{\Sigma I}(A_{II})^{-1}\underline{f}_I - A_{\Sigma I}(A_{II})^{-1}A_{I\Sigma}\underline{u}_\Sigma.$$

Therefore we can define the Schur complement matrix as

$$S^{FEM} = A_{\Sigma\Sigma} - A_{\Sigma I}(A_{II})^{-1}A_{I\Sigma}$$

and the system of linear equations in the reduced form

$$\begin{pmatrix} S^{FEM} & -M_h^T \\ \frac{1}{2}M_h - K_h & V_h \end{pmatrix}\begin{pmatrix} \underline{u}_\Sigma \\ \underline{w} \end{pmatrix} = \begin{pmatrix} \underline{f}_\Sigma - A_{\Sigma I}(A_{II})^{-1}\underline{f}_I \\ 0 \end{pmatrix}. \tag{2.36}$$

To implement the system of linear equations as Schur complement system, we need a good realization of the inverse matrix. We use the package Pardiso [14, 5, 7] for the realization of the matrix $(A_{II})^{-1}$. The system of linear equations (2.36) is solved by a GMRES solver without any preconditioning. To derive more general numerical examples we consider the following modification of the problem.

Remark 2.5. If we consider jump terms in the transmission conditions (0.2) (similar to [6]), namely

$$\begin{aligned} u_i(x,t) - u_e(x,t) &= \hat{u}_0(x,t) \quad \text{for } (x,t) \in \Sigma \\ \frac{\partial}{\partial n_x}u_i(x,t) - \frac{\partial}{\partial n_x}u_e(x,t) &= \phi_0(x,t) \quad \text{for } (x,t) \in \Sigma, \end{aligned}$$

small adjustments of the right-hand side of the system of linear equations (2.36) will be necessary. As S^{FEM} stays the same, we mention it only for the reduced system of linear equations (2.36). There we have the following form by plugging in the transmission conditions where we couple both variational formulations (2.1) and (2.2) into (2.3):

$$\begin{pmatrix} S^{FEM} & -M_h^T \\ \frac{1}{2}M_h - K_h & V_h \end{pmatrix} \begin{pmatrix} \underline{u}_\Sigma \\ \underline{w} \end{pmatrix} = \begin{pmatrix} \underline{f}_\Sigma - A_{\Sigma I}(A_{II})^{-1}\underline{f}_I + \underline{\phi}_0 \\ (\frac{1}{2}M_h - K_h)\underline{\hat{u}}_0 \end{pmatrix}$$

with

$$\phi_0[i] = \langle \phi_0, \varphi_i \rangle_\Sigma \quad \text{for } i = 1, \cdots N$$

and the vector $\underline{\hat{u}}_0$ of the interpolation of the jump term $\hat{u}_0$ into $S_h^1(Q)|_\Sigma$. Regarding the vector $\underline{\phi}_0$ for the conormal derivative we apply a numerical integration scheme.

This allows us to consider more general numerical examples in Chapter 3 as we do not need the continuity of the solution at the interface while satisfying the radiation in the exterior domain easily and we can test many different solutions in the interior domain.

2.4 Integration methods for the boundary integral operators

In this section, we consider the realization of the boundary integrals matrices V_h and K_h from (2.34) for a general decomposition of the BE mesh, i.e. we do not consider a tensor product mesh. The main results for these integration methods were already given in [9]. In order not to rewrite the main parts, we mainly give the results and statements from there to discuss how the matrices are assembled. We restrict the presentation to the two-dimensional space setting.

As mentioned in the beginning of Sect. 2.2 we consider a conforming triangulation of the lateral boundary $\Sigma = \cup_{i=1}^N \overline{T_i}$, where T_i are the triangles of the BE mesh. For the system of linear equations (2.36), it is necessary to assemble the Galerkin matrix V_h and the double layer matrix K_h. We discuss the integration methods respectively in the Subsect. 2.4.1 and 2.4.2. The mass matrix M_h can be assembled exactly.

2.4.1 Integration methods for the single layer Galerkin matrix

The matrix entries of V_h look as follows:

$$V_h[j,i] = \int_0^T \int_\Gamma \varphi_j^0(t,x) \int_\Gamma U^*(x,y)\varphi_i^0(t,y) ds_y ds_x dt = \int_{t_1}^{t_2} \int_{\tau_j(t)} \int_{\tau_i(t)} U^*(x,y) ds_y ds_x dt, \tag{2.37}$$

where φ_j^0 are the piecewise constant functions in (2.33). The line segments $\tau_j(t)$ and $\tau_i(t)$ represent the spatial component of the triangles T_j and T_i at an explicit moment

t in time. The interval (t_1, t_2) is the temporal support that both triangles T_j and T_i share. In most cases, it is an empty interval and the matrix entry is 0, i.e. the time intervals of T_i and T_j do not overlap. In the non-zero case, the element $\tau_j(t)$ for $t \in (t_1, t_2)$ is a line segment. The same holds also for the second element. To evaluate these matrix entries, we consider exact integration in space [18] and numerical Gauss quadrature in time. To shorten the representation we define

$$f(t) = \int_{\tau_j(t)} \int_{\tau_i(t)} U^*(x, y) ds_y ds_x. \tag{2.38}$$

First we substitute t such that we have an integral on the interval $(0, 1)$ and then we can apply a Gauss quadrature rule, where g corresponds to the number of Gaussian points:

$$V_h[j, i] = (t_2 - t_1) \int_0^1 f(t_1 + s(t_2 - t_1)) ds \approx (t_2 - t_1) \sum_{k=1}^{g} f(t_1 + s_k(t_2 - t_1)) w_k. \tag{2.39}$$

Here w_k are the weights of the Gauss quadrature rule and s_k are the Gauss nodes on the interval $(0, 1)$. Since we have an implementation for the time-independent Laplace equation we can compute the spatial part $f(t)$ exactly [18] . The next step in the implementation is to find the two line segments $\tau_j(t)$ and $\tau_i(t)$ for the spatial part, as we cut the two triangles at an explicit time and then we can evaluate the spatial part.

From the tests in [9], we noticed that we have to split the integrals for specific situations. In such situations, $f(t)$ is not smooth. As we use Gauss quadrature, we have to take care of the vertices of the triangles, which are located in the middle of the time interval (t_1, t_2). Depending on the setting of two triangles T_1 and T_2 we have to consider 1, 2 or 3 intervals in our splitting. In our implementation we have a vector **value** with up to 4 values. Namely we have **value**$[1] = \max\{\min T_1, \min T_2\}$, **value**$[4] = \min\{\max T_1, \max T_2\}$ with respect to the temporal component only. The other two entries are reserved for the other vertices of the triangles, which could be between **value**[1] and **value**[4] with respect to the temporal component. They are only filled, if there is a vertex in between, otherwise the entry is ignored. In the end we have a quadrature method, where we split the integrals in this specific way and use numerical Gauss integration for each part. Based on our observation we use two Gaussian points for each part for the evaluation of the matrix entries for our numerical tests in Chapt. 3.

2.4.2 Integration methods for the double layer matrix

The evaluation of the matrix entries of the double layer boundary integral operator is similar to those of the Galerkin matrix V_h in (2.37), but we have to keep in mind that we have to use piecewise linear ansatz functions:

$$K_h[i,k] = \sum_{\ell:T_\ell\in\mathrm{supp}\varphi_k^1} \int_{T_i}\int_{T_\ell} \frac{\partial}{\partial n_y} U^*(x,y)\varphi_k^1(y,t)ds_y ds_x dt$$
$$= \sum_{\ell:T_\ell\in\mathrm{supp}\varphi_k^1} \int_{t_1}^{t_2}\int_{\tau_i(t)}\int_{\tau_\ell(t)} \frac{\partial}{\partial n_y} U^*(x,y)\varphi_k^1(y,t)ds_y ds_x dt.$$

For the piecewise linear and continuous form function, which are similar to the FE-form functions in $\mathbb{R}^2$, we consider a local representation in a dimension lower at specific times t_j of the temporal Gauss quadrature. We also have to keep in mind that the interval (t_1, t_2) may be empty or we have to cut the triangles for integration. We end up in line segment $\tau_i(t_j)$ and $\tau_\ell(t_j)$ for both triangles. The form function for the triangle T_ℓ can be described in a two dimensional setting by

$$\varphi_k^1(\tilde{y},t) = \alpha_k(t)\varphi_{\ell_1}^{1,2d}(\tilde{y}) + \beta_k(t)\varphi_{\ell_2}^{1,2d}(\tilde{y}) \quad \text{for } \tilde{y}\in\tau_\ell(t),$$

where $\varphi_{\ell_1}^{1,2d}$ and $\varphi_{\ell_2}^{1,2d}$ are the nodal basis functions of the line segment $\tau_\ell(t)$ related to virtual nodes y_{ℓ_1} and y_{ℓ_2} with indices ℓ_1 and ℓ_2. These nodes are the intersection points of the edges of T_ℓ with the plane $t = t_j$. The coefficients α_k and β_k are the evaluations of φ_k^1, i.e.

$$\alpha_k(t) = \varphi_k^1(y_{\ell_1}, t), \quad \beta_k(t) = \varphi_k^1(y_{\ell_2}, t).$$

Thus the evaluation of the matrix entries looks as follows:

$$K_h[i,k] = \sum_{\ell:T_\ell\in\mathrm{supp}\varphi_k^1} \int_{t_1}^{t_2} (\alpha_k(t)K_{i,\ell}^{2d}[1](t) + \beta_k(t)K_{i,\ell}^{2d}[2](t))dt,$$

where $K_{i,\ell}^{2d}$ are the exact evaluations of the double layer potential matrix entries in space [18] only:

$$K_{i,\ell}^{2d}[1](t) = \int_{\tau_i(t)}\int_{\tau_\ell(t)} \frac{\partial}{\partial n_y} U^*(x,y)\varphi_{\ell_1}^{1,2d}(\tilde{y})ds_y ds_x,$$
$$K_{i,\ell}^{2d}[2](t) = \int_{\tau_i(t)}\int_{\tau_\ell(t)} \frac{\partial}{\partial n_y} U^*(x,y)\varphi_{\ell_2}^{1,2d}(\tilde{y})ds_y ds_x.$$

3 Numerical results

We consider a space-time cylinder $Q = (0, 0.5)^3$ with different kind of grids but we do not want to use a tensor-product grid. Instead we consider a structured mesh, see Fig. 3.1, for a coarse grid triangulation of the boundary and an unstructured mesh generated by NETGEN [16], see Fig. 3.2. But we compare the results with a sort of tensor product mesh, see Fig. 3.3.

In the numerical examples, we consider other transmission conditions with jump terms in our model problem (0.1)-(0.3), as mentioned in Remark 2.5, where we have small adjustments on the right-hand side of the system of linear equations. Furthermore, we consider the numerical examples without any stabilization term. As a conclusion we cannot completely compare our results with the theoretical optimal error estimate given in Cor. 2.12. For all examples, we consider very similar functions for the exterior domain, namely the fundamental solution regarding the two dimensional space setting with a singularity in the interior domain but for the time part we consider different situations. Of course this is a solution for an exterior Laplace equation and satisfies the radiation condition (0.3). The first example is very similar to the one of [6, Sect. 6.1], where we consider a smooth function in the interior domain. The order of convergence is here given by the eoc (experimental order of convergence), which is calculated by:

$$\text{eoc} = \frac{\frac{\ln(error_{\text{coarse}})}{\ln(error_{\text{fine}})}}{\ln(2)}.$$

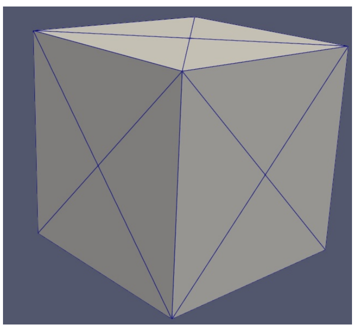

Figure 3.1: general structured grid: boundary of the space time cylinder.

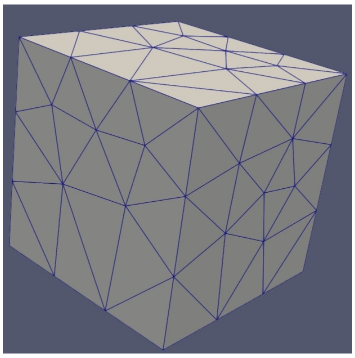

Figure 3.2: general unstructured grid produced by NETGEN: boundary of the space time cylinder.

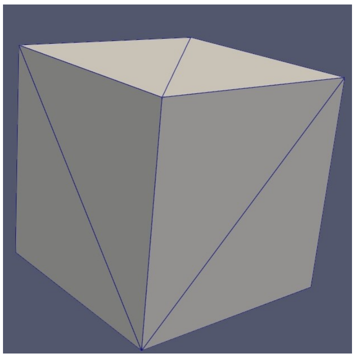

Figure 3.3: triangular subdivision of a tensor product grid: boundary of the space time cylinder.

Example 3.1. We consider the exact solution as

$$u_i(x_1, x_2, t) = \sin(2\pi t)(1 - 10x_1^2 - 10x_2^2)e^{-25(x_1^2+x_2^2)}$$
$$u_e(x_1, x_2, t) = -t \cdot \log(|x - x_0|) \quad \text{with the singularity in } x_0 = (0.2, 0.2).$$

In Tab. 3.1 we firstly consider Example 3.1 for the general structured grid of Fig. 3.1 and for the unstructured one of Fig. 3.2 in Tab. 3.2. Here we observe, that the eoc of the $L^2(Q)$ error seems to approach 2 for unstructured grid, while the one for the structured mesh is higher than 1.5 but far lower than 2 at this point of mesh refinement. For the $L^2(0, T, H^1(\Omega))$ norm, which we denote as $\|\cdot\|_Y$ in all tables, we observe an eoc higher than 1 for the general structured grid and for the unstructured mesh the expected eoc from the theory. From the error estimate of Corollary 2.12, we would expect an eoc of 1. For the $L^2(\Sigma)$ error regarding the conormal derivative we observe a typical order of convergence for the unstructured grid and slightly higher one for the structured one, since we would expect a convergence rate of 1 for smooth functions from the theory. If we compare the errors now with the sort of tensor product grid of

L	FE vertices	BE triangles	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|\gamma_1^{ext}u-w_h\|_{L^2(\Sigma)}$	eoc	$\|u-u_h\|_Y$	eoc
0	15	16	0.0419438		0.406976		0.26365	
1	69	64	0.0196611	1.09	0.410358	-0.01	0.133842	0.98
2	409	256	0.0088893	1.15	0.210929	0.96	0.0765904	0.81
3	2801	1024	0.00357079	1.32	0.0959766	1.14	0.0351169	1.12
4	20705	4096	0.00127178	1.49	0.0394574	1.28	0.0141728	1.31
5	159169	16384	0.000382579	1.59	0.0164315	1.26	0.00513567	1.46

Table 3.1: Numerical results of example 3.1 regarding the structured general grid

L	FE vertices	BE triangles	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|\gamma_1^{ext}u-w_h\|_{L^2(\Sigma)}$	eoc	$\|u-u_h\|_Y$	eoc
0	75	84	0.0127889		0.224007		0.0783363	
1	411	336	0.00435066	1.56	0.129453	0.79	0.0387175	1.02
2	2669	1344	0.00145886	1.58	0.0665595	0.96	0.0176406	1.13
3	19129	5376	0.0004567	1.68	0.0305334	1.13	0.00861917	1.03
4	144625	21504	0.000131289	1.80	0.0140137	1.12	0.00427956	1.01

Table 3.2: Numerical results of example 3.1 regarding the general unstructured grid generated by NETGEN

L	FE vertices	BE triangles	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|\gamma_1^{ext}u-w_h\|_{L^2(\Sigma)}$	eoc	$\|u-u_h\|_Y$	eoc
0	8	8	0.0443614		0.564902		0.24911	
1	27	32	0.0277419	0.68	0.48902	0.21	0.144901	0.78
2	125	128	0.0181042	0.62	0.344827	0.51	0.0989709	0.55
3	729	512	0.00772285	1.23	0.186144	0.89	0.0470548	1.07
4	4913	2048	0.00316265	1.29	0.0813145	1.19	0.010826	2.12
5	35937	8192	0.00107451	1.56	0.0337524	1.27	0.00925362	0.23

Table 3.3: Numerical results of example 3.1 regarding a sort of tensor product grid

Fig. 3.3 in Tab. 3.3, we observe that the $L^2(Q)$ error seems to have a lower eoc and the $\|\cdot\|_Y$ error is highly irregular. If we compare the $\|\cdot\|_Y$ error on level 3 with level 5 the eoc seems good, but both steps alone are quite unusual.

If we compare the errors from the different grids we observe that for similar numbers of vertices and triangles the errors for the general grids are better than the ones for the tensor product grid. Furthermore the general unstructured grid has better results for the $L^2(Q)$ error and the convergence from the theory is nearly fulfilled in comparison to the structured one.

In a second example we consider a slightly different situation for the interior part, where we consider a smooth function.

Example 3.2. We consider the exact solution as

$$u_i(x_1, x_2, t) = \sin(2\pi t)\sin(\pi x_1)\sin(\pi x_2)$$
$$u_e(x_1, x_2, t) = -t \cdot \log(|x - x_0|) \quad \text{with the singularity in } x_0 = (0.2, 0.2).$$

L	FE vertices	BE triangles	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|\gamma_1^{ext}u-w_h\|_{L^2(\Sigma)}$	eoc	$\|u-u_h\|_Y$	eoc
0	15	16	0.0878445		0.612943		0.398156	
1	69	64	0.0525024	0.74	0.420497	0.54	0.300484	0.41
2	409	256	0.0241155	1.12	0.210411	1.00	0.211987	0.50
3	2801	1024	0.00872143	1.47	0.0912639	1.21	0.128132	0.73
4	20705	4096	0.00273507	1.67	0.0374126	1.29	0.0695695	0.88
5	159169	16384	0.000796917	1.78	0.0158499	1.24	0.0358354	0.96

Table 3.4: Numerical results of example 3.2 regarding the general structured grid

L	FE vertices	BE triangles	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|\gamma_1^{ext}u-w_h\|_{L^2(\Sigma)}$	eoc	$\|u-u_h\|_Y$	eoc
0	75	84	0.0192695		0.220868		0.223629	
1	411	336	0.00932787	1.04	0.135986	0.70	0.137416	0.70
2	2669	1344	0.00410737	1.18	0.0741023	0.88	0.0790118	0.80
3	19129	5376	0.00147328	1.48	0.0342711	1.11	0.043258	0.87
4	144625	21504	0.000438204	1.75	0.0152373	1.17	0.0226249	0.94

Table 3.5: Numerical results of example 3.2 regarding the general unstructured grid generated by NETGEN

L	FE vertices	BE triangles	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|\gamma_1^{ext}u-w_h\|_{L^2(\Sigma)}$	eoc	$\|u-u_h\|_Y$	eoc
0	8	8	0.107724		0.61386		0.50418	
1	27	32	0.050361	1.10	0.516271	0.25	0.371913	0.44
2	125	128	0.0192381	1.39	0.257499	1.00	0.232875	0.68
3	729	512	0.00662282	1.54	0.1225566	1.07	0.132288	0.82
4	4913	2048	0.00247143	1.43	0.0573803	1.09	0.0704436	0.91
5	35937	8192	0.000832177	1.57	0.0267837	1.10	0.0364303	0.95

Table 3.6: Numerical results of example 3.2 regarding a sort of tensor product grid

In Tab. 3.4–3.6 we observe, that the eoc of the $L^2(Q)$ error is larger than 1.5 and seems to tend to 2. For the Y error, we clearly see that it is going to 1 for all three meshes, this completely fits the theory mentioned in Corollary 2.12. For the $L^2(\Sigma)$ error of the conormal derivative, we observe a slightly higher order of convergence than the expected eoc of 1 in Tab. 3.4–3.6. If we compare the errors from the different grids we observe that for similar number of vertices and triangles the $L^2(Q)$ and $L^2(0,T;H^1(\Omega))$ errors for the tensor product grid are now slightly better than both general grids, which are quite the same for the structured and unstructured one. Furthermore the $L^2(\Sigma)$ errors of the conormal derivative are also quite the same for all three types of grids.

In a third example, we consider a quite similar smooth function in the interior part as before. For the exterior domain we consider a quite arbitrary choice of function in the time setting.

L	FE vertices	BE triangles	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|\gamma_1^{ext}u-w_h\|_{L^2(\Sigma)}$	eoc	$\|u-u_h\|_Y$	eoc
0	15	16	0.0295122		0.554768		0.295023	
1	69	64	0.0180166	0.71	0.374879	0.55	0.184217	0.68
2	409	256	0.00799915	1.17	0.185948	1.03	0.1067008	0.79
3	2801	1024	0.0026917	1.57	0.0904405	1.04	0.0592529	0.85
4	20705	4096	0.000770101	1.81	0.0438996	1.04	0.0309319	0.94
5	159169	16384	0.000210936	1.87	0.0215444	1.03	0.0156908	0.98
6	1248129	65536	$5.59846 \cdot 10^{-5}$	1.91	0.0106655	1.01	0.00788025	0.99

Table 3.7: Numerical results of example 3.3 regarding the general structured grid

L	FE vertices	BE triangles	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|\gamma_1^{ext}u-w_h\|_{L^2(\Sigma)}$	eoc	$\|u-u_h\|_Y$	eoc
0	75	84	0.00710913		0.292917		0.147786	
1	411	336	0.00301395	1.24	0.155526	0.91	0.0835133	0.82
2	2669	1344	0.0012133	1.31	0.0794485	0.97	0.0450244	0.89
3	19129	5376	0.0004114	1.56	0.0390808	1.02	0.0235071	0.94
4	144625	21504	0.000117792	1.80	0.0192149	1.02	0.0119871	0.97

Table 3.8: Numerical results of example 3.3 regarding the general unstructured grid generated by NETGEN

L	FE vertices	BE triangles	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|\gamma_1^{ext}u-w_h\|_{L^2(\Sigma)}$	eoc	$\|u-u_h\|_Y$	eoc
0	8	8	0.0478604		0.971068		0.414511	
1	27	32	0.0268021	0.62	0.629937	0.88	0.282359	0.56
2	125	128	0.0102023	1.39	0.328655	0.94	0.165584	0.77
3	729	512	0.00395813	1.37	0.165382	0.99	0.0918858	0.85
4	4913	2048	0.001337755	1.57	0.0819573	1.01	0.0487253	0.91
5	35937	8192	0.000388757	1.78	0.0405738	1.01	0.02502	0.92
6	274625	32768	$9.9454 \cdot 10^{-5}$	1.97	0.0201773	1.01	0.0126321	0.99

Table 3.9: Numerical results of example 3.3 regarding a sort of tensor product grid

Example 3.3. We consider the exact solution as

$$u_i(x_1, x_2, t) = \sin(\pi t)\sin(\pi x_1)\sin(\pi x_2)$$
$$u_e(x_1, x_2, t) = -\exp(t) \cdot t \cdot \log(|x - x_0|) \quad \text{with the singularity in } x_0 = (0.2, 0.2),$$

In all three Tables 3.7–3.9 we most clearly see that all experimental orders of convergence fit the theory for typical FEM and BEM approximations, see also the error estimate of Corollary 2.12. If we compare the errors from the different grids we observe that for similar numbers of vertices and triangles the errors are quite the same.

Regarding the numbers of iterations for the GMRES solver of the Schurr complement system of equation (2.36), we observe that the number of iterations approximately doubles in each iteration step and therefore we need for the highest levels of the mesh refinement in our case around 1000 to 4000 iterations steps depending on the typ of grid. This is a large number of iteration steps and a preconditioner or a multilevel method is mandatory if we want to compute numerical solutions on higher levels of mesh refinements.

4 Conclusions and Outlook

In this thesis the non-symmetric coupling of finite and boundary element methods for a parabolic-elliptic interface problem is firstly discussed for a space-time setting, where space and time are explicitly not considered separately. In our model problem we considered the heat equation regarding the parabolic partial differential equation in the interior domain and the Laplace equation in the exterior domain. A proof for the unique solvability is given in the analytic as well in the discrete setting. As a starting point to proof the injectivity of the bilinear form we took the idea of the derivation of the inf-sup condition from the space-time formulation for a boundary value problem of the heat equation and adapted it for our case. There we firstly discussed the related quasistatic problem and proved a boundedness and ellipticity estimate in order to derive unique solvability in this setting. The surjectivity followed then also quite similar to the one of the heat equation. Unfortunately in the discrete setting the stabilized variational formulation is not equivalent to the original one for our specific choice of test and ansatz functions and therefore, there is a lack of equivalence in the discrete setting, which needs to be fixed. First numerical tests are provided for the non modified formulation, which give very promising results. We also tested for an unstructured mesh. In some numerical examples the expected order of convergence completely fulfills the theory, but in some other cases it differs in the sense that we get better convergence rates than expected from the FEM and BEM approximation theory.

For an outlook regarding the topic of a non-symmetric coupling for a parabolic-elliptic interface equation, it remains to proof and test the results for more arbitrary choices of parabolic partial differential equations. There it would be advantageous to firstly discuss the regarding quasistatic problem and in some cases also the elliptic–elliptic interface equation firstly. From there on similar techniques could lead to a proof of unique solvability. In order to apply this concept in industrial application and simulate this for an electric motor we would also need the derivation and implementation of a preconditioner or a multilevel method in order to reduce the number of iterations solving the system of linear equations and the computational time. Furthermore a Fast BEM implementation could also be very useful in order to compute numerical solutions on higher levels of mesh refinement. While the theoretical results are equivalent for the three dimensional space setting, an implementation here keeps open for now, since in a general case for a space-time method it is quite challenging to implement it in a four dimensional space-time domain. To test this firstly on a tensor product mesh could be helpful. Also the discussion on the corresponding coupling for boundary value problems for similar situation keeps for now open.

Bibliography

[1] R. A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] H. W. Alt. *Lineare Funktionalanalysis*. Universitext. Springer-Verlag Berlin Heidelberg, 2011. Eine anwendungsorientierte Einführung.

[3] M. Aurada, M. Feischl, T. Führer, M. Karkulik, J. M. Melenk, and D. Praetorius. Classical FEM-BEM coupling methods: nonlinearities, well-posedness, and adaptivity. *Comput. Mech.*, 51(4):399–419, 2013. doi:10.1007/s00466-012-0779-6.

[4] M. Costabel, V. J. Ervin, and E. P. Stephan. Symmetric coupling of finite elements and boundary elements for a parabolic-elliptic interface problem. *Quart. Appl. Math.*, 48(2):265–279, 1990. doi:10.1090/qam/1052136.

[5] A. Eftekhari, D. Pasadakis, M. Bollhöfer, S. Scheidegger, and O. Schenk. Block-enhanced precision matrix estimation for large-scale datasets. *Journal of Computational Science*, 53:1877–7503, 2021. doi:10.1016/j.jocs.2021.101389.

[6] H. Egger, C. Erath, and R. Schorr. On the nonsymmetric coupling method for parabolic-elliptic interface problems. *SIAM J. Numer. Anal.*, 56(6):3510–3533, 2018. doi:10.1137/17M1158276.

[7] L. Gaedke-Merzhäuser, J. van Niekerk, O. Schenk, and H. Rue. Parallelized integrated nested Laplace approximations for fast Bayesian inference. *Statistics and Computing*, 33(1):25, 2023.

[8] C. Johnson and J.-C. Nédélec. On the coupling of boundary integral and finite element methods. *Math. Comp.*, 35(152):1063–1079, 1980. doi:10.2307/2006375.

[9] T. Kaltenbacher. A boundary element method for the Laplace equation with time-dependent Dirichlet data. Project Technomathematics, Institut für Numerische Mathematik, Technische Universität Graz, 2023.

[10] J. Lions, P. Kenneth, and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications: Vol. 1.* Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2012.

[11] R. C. MacCamy and M. Suri. A time-dependent interface problem for two-dimensional eddy currents. *Quarterly of Applied Mathematics*, 44(4):675–690, 1987.

[12] P. Mund and E. P. Stephan. Adaptive coupling and fast solution of FEM-BEM equations for parabolic-elliptic interface problems. *Math. Methods Appl. Sci.*, 20(5):403–423, 1997. `doi:10.1002/(SICI)1099-1476(19970325)20:5<403::AID-MMA816>3.0.CO;2-5`.

[13] G. Of and O. Steinbach. Is the one-equation coupling of finite and boundary element methods always stable? *ZAMM Z. Angew. Math. Mech.*, 93(6-7):476–484, 2013. `doi:10.1002/zamm.201100188`.

[14] D. Pasadakis, M. Bollhöfer, and O. Schenk. Sparse quadratic approximation for graph learning. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 45(9):11256–11269, 2023. `doi:10.1109/TPAMI.2023.3263969`.

[15] F.-J. Sayas. The validity of Johnson-Nédélec's BEM-FEM coupling on polygonal interfaces. *SIAM J. Numer. Anal.*, 47(5):3451–3463, 2009. `doi:10.1137/08072334X`.

[16] J. Schoeberl. NETGEN An advancing front 2D/3D-mesh generator based on abstract rules. *Computing and Visualization in Science*, 1:41–52, 07 1997. `doi:10.1007/s007910050004`.

[17] B. Schweizer. *Partielle Differentialgleichungen*. Springer-Verlag, Berlin, 2013. Eine anwendungsorientierte Einführung. `doi:10.1007/978-3-642-40638-6`.

[18] O. Steinbach. Galerkin-und Kollokationsdiskretisierungen für Randintegralgleichungen in 2d. Technical report, Universität Stuttgart, Mathematisches Institut A, 1996.

[19] O. Steinbach. *Numerical Approximation Methods for Elliptic Boundary Value Problems*. Springer, New York, 2008.

[20] O. Steinbach. A note on the stable one-equation coupling of finite and boundary elements. *SIAM J. Numer. Anal.*, 49(4):1521–1531, 2011. `doi:10.1137/090762701`.

[21] O. Steinbach. Space-time finite element methods for parabolic problems. *Comput. Methods Appl. Math.*, 15(4):551–566, 2015. `doi:10.1515/cmam-2015-0026`.

[22] O. Steinbach and M. Zank. Coercive space-time finite element methods for initial boundary value problems. *Electron. Trans. Numer. Anal.*, 52:154–194, 2020. `doi:10.1553/etna_vol52s154`.

[23] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis.* Studies in mathematics and its applications. North-Holland Amsterdam, 1984.

[24] J. Wloka. *Funktionalanalysis und Anwendungen.* Walter de Gruyter, Berlin, 1971.

[25] M. Zank. *Inf-Sup Stable Space-time Methods for Time-dependent Partial Differential Equations.* Computation in Engineering and Science. Verlag der Technischen Universität Graz, 2020.

[26] O. C. Zienkiewicz, D. W. Kelly, and P. Bettess. Marriage à la mode—the best of both worlds (finite elements and boundary integrals). In *Energy methods in finite element analysis*, Wiley Ser. Numer. Methods Engrg., pages 81–107. Wiley, Chichester, 1979.