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A Lieb-Thirring type inequality for δ-potentials supported on hyperplanes

MASTER'S THESIS

to achieve the university degree of

Diplom-Ingenieur

Mathematics

submitted to

Graz University of Technology

Supervisor

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Institute of Applied Mathematics

Graz, January 2018

STATUTORY DECLARATION

I declare that I have authored this thesis independently, that I have not used other than the declared sources / resources, and that I have explicitely marked all material which has been quoted either literally or by content from the used sources.

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Acknowledgements

First of all, I would like to express my sincere gratitude and appreciation to my supervisor Prof. Dr. Jussi Behrndt. In particular, his attitude to create a welcome atmosphere in every discussion or whenever I had questions was exceptional. For sure also Markus Holzmann has to be mentioned at this point, as he mostly was my first contact person and always managed to help me out of my troubles. I also want to thank Prof. Behrndt as well as Prof. Pavel Exner from the Nuclear Physics institute in Řež (Prague) to invite me there for a very pleasant working visit.

Moreover, I am grateful for my whole family which supported me during all my years at university. In particular, I want to mention my parents who helped me out with all my non-university problems and provided a parental home I could always visit. Not less important was the help of my grandparents and my uncle Klaus for supporting me financially.

Finally, the special thank go to my close friends Alice Soranidis and Simone Kaiser for investing more time than I expected in proofreading this thesis.

Contents

0. Introduction

A Schrödinger operator with a δ-potential supported on $\Sigma \subseteq \mathbb{R}^d$ can formally be written as

$$-\Delta + \alpha\delta_\Sigma \ , \tag{0.1}$$

where $\alpha : \Sigma \to \mathbb{R}$ is a real-valued measurable function describing the strength of the potential. Singular potentials of the form $\alpha\delta_\Sigma$ are used as an approximation of classical potentials V, which have large values in a small neighborhood of Σ and small values elsewhere. For Σ being a C^2-hypersurface, the justification that the spectral properties of Schrödinger operators with δ-potentials are close to the ones with approximating classical potentials can for example be found in [11]. If Σ consists of discrete points, one can think of (0.1) being an idealized model of impurities. The set Σ may also be a system of (at most countably many) one-dimensional line segments, in which case this system is called a leaky quantum graph and was investigated by P. Exner and co-authors, see e.g. [31, 36]. In this thesis we study the case that Σ is the $(d-1)$-dimensional hyperplane

$$\Sigma = \left\{ \, x \in \mathbb{R}^d \;\middle|\; x_d = 0 \, \right\} \ . \tag{0.2}$$

We are using the method of quasi boundary triples in order to assign a selfadjoint operator with the differential expression (0.1). In this context we also prove a new abstract theorem about self-adjoint extensions, which enlarges the class of allowed potentials to $\alpha \in L^p(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ for $p > \frac{4}{3}(d-1)$. Although one is already able to construct self-adjoint extensions for all $\alpha \in L^{d-1}(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ with the standard method of sesquilinear forms, this is only partly comparable, since we assume a higher Sobolev regularity for the operator domain. What we are mainly interested in, is to derive a Lieb-Thirring inequality for this operator, which is an upper bound for the sum of the negative discrete eigenvalues to some power $\gamma > 0$, consisting only of the integral over the negative part $\alpha_- := -\min\{\alpha, 0\}$ of the potential strength. For potentials with $\alpha \in L^{d-1+2\gamma}(\Sigma)$ R. Frank and A. Laptev [39] already proved the Lieb-Thirring inequality

$$\operatorname{tr}(-\Delta + \alpha\delta_\Sigma)_-^\gamma \le L_\gamma \int_\Sigma \alpha_-^{d-1+2\gamma} d\sigma \ . \tag{0.3}$$

The purpose of this thesis is now to derive a similar inequality for potentials $(\alpha+\alpha_0)\delta_\Sigma$ containing some constant negative shift α_0. In this setting immediately two problems occur. First of all, the right hand side becomes infinite if one simply replaces α by

$\alpha + \alpha_0$. Secondly, the bottom of the essential spectrum of $A_{\alpha+\alpha_0}$ shifts from 0 to $-\frac{\alpha_0^2}{4}$ and consequently one has to sum up powers of the distances between the discrete eigenvalues and $-\frac{\alpha_0^2}{4}$ on the left hand side.

The physical investigation of δ-interactions started in 1931, when R. de L. Kronig and W. G. Penney tried to describe the behaviour of electrons in a one dimensional crystal [52]. They used a periodic lattice $\Sigma = \mathbb{Z} \subseteq \mathbb{R}$ with constant potential strength α and confirmed various physical properties, as e.g. specific heat, electric and magnetic conductivity. Generalisations to higher dimensions and periodic lattices were done by [19, 43, 68]. In 1961, F. Berezin and L. Faddeev started to give the expression (0.1) a rigorous mathematical meaning in their work [18]. For $\Sigma = \{0\} \subseteq \mathbb{R}^3$ they considered at (0.1) as a self-adjoint extension of the Laplace operator $-\Delta$ with domain $\mathcal{C}_0^\infty(\mathbb{R}^3\backslash\{0\})$. Other approaches to this problem are methods like non-standard analysis [3, 4] or Dirichlet forms for a probabilistic interpretation [8, 9]. Again generalisations to other dimensons and periodic lattices of the mathematical exact definition were done in the following years in [7, 49, 44, 45, 5]. A standard reference treating point interactions is the book [6], where one can find further historic references as well as an overview over the mathematical methods describing these operators. For a more recent review paper we refer to [51].

In the case that Σ does not consist of discrete points, but some manifold with co-dimension 1, we refer to [21] for the basic theory of the mathematical realisation of (0.1) as a sesquilinear form. The spectral analysis in this case becomes more complicated, since the spectral properties additionally depend on the geometry of Σ. The fact that the geometry indeed induces bound states is proved in [32, 33, 34, 35]. Some further related publications investigating such models are [14, 37, 38, 50, 59].

Another approach beside the form method to find the self-adjoint realisation of (0.1) is the extension theory connected to quasi boundary triples. The method of ordinary boundary triples with their corresponding Weyl functions is the abstract counterpart to the trace maps for Sturm-Liouville operators and the Titchmarsh-Weyl function. They were introduced by V. M. Bruk and A. N. Kochubei [22, 48] and studied e.g. by V. A. Derkach and M. M. Malamud [25, 26, 60] and by M. L. Gorbachuk and V. I. Gorbachuk [42]. However, for the application on partial differential operators, like the Schrödinger operator in this thesis, it is more convenient to use the more general method of quasi boundary triples, initially defined by J. Behrndt and M. Langer in [12]. For its definition, basic properties as well as applications on partial differential operators we refer to [13, 15, 17].

The starting point of a quasi boundary triple is a densely defined, symmetric and closed operator S and a closable operator T with $\overline{T} = S^*$ in some Hilbert space V. A quasi boundary triple (W, Γ_0, Γ_1) then consists of another Hilbert space W, the boundary space, and a linear mapping $\Gamma = (\Gamma_0, \Gamma_1)^T : \mathrm{dom}(T) \to W \times W$, the boundary mapping, which has dense range and satisfies the abstract Green's identity

$$\langle Tv, \tilde{v}\rangle_V - \langle v, T\tilde{v}\rangle_V = \langle \Gamma_1 v, \Gamma_0 \tilde{v}\rangle_W - \langle \Gamma_0 v, \Gamma_1 \tilde{v}\rangle_W \ , \quad \forall v, \tilde{v} \in \operatorname{dom}(T) \ .$$

Symmetric extensions A_B of S can then be realised by

$$A_B v = Tv \quad \text{with} \quad \operatorname{dom}(A_B) = \{\, v \in \operatorname{dom}(T) \mid \Gamma_0 v = B\Gamma_1 v \,\} \ , \tag{0.4}$$

where B can be any symmetric operator in the boundary space W. If we claim certain additional properties of the operator B, we can also ensure the self-adjointness of the extension A_B. In Theorem 3.10 we consider a splitting $B \subseteq B_1 B_2$ and give sufficient conditions on B_1 and B_2 to ensure self-adjointness of A_B. This result is a generalisation of [16, Theorem 2.6], where similar conditions directly for the operator B are given. The advantage of the decomposition into B_1 and B_2 is that in general the operators B_1 and B_2 have better mapping properties than the initial operator B, for example in terms of boundedness between Sobolev spaces. In our application, Theorem 5.8, we use B as the multiplication with the potential strength α and B_1, B_2 as the multiplications with the powers $\operatorname{sign}(\alpha)|\alpha|^s$ and $|\alpha|^{1-s}$ for some $s \in [0, 1]$. According to the higher L^p-regularity of these powers we are able to allow a larger class of potentials to be treated with the method of quasi boundary triples.

Beside these self-adjoint extensions, there is another important object arising from the theory of quasi boundary triples, namely the operator-valued Weyl function M. Up to the multiplication with the potential strength, the values of the Weyl function equal the Birman-Schwinger operators $K_\lambda = |\alpha|^{\frac{1}{2}} M(\lambda)|\alpha|^{\frac{1}{2}}$. The importance of the Weyl function in our particular application is that there exists a one-to-one correspondence between the eigenvalues of the compact Birman-Schwinger operator and the ones from the unbounded Schrödinger operator. We will use this equivalence in order to count the number of eigenvalues in the Birman-Schwinger principle, Theorem 6.6, which will be an essential ingredient of the proof of the Lieb-Thirring inequality.

A Lieb-Thirring inequality or Cwikel-Lieb-Rozenblum-bound (CLR-bound) is an estimate of the form

$$\operatorname{tr}(-\Delta + V)_-^\gamma \le L_\gamma \int_{\mathbb{R}^d} V_-(x)^{\frac{d}{2}+\gamma} dx \ , \tag{0.5}$$

where $V : \mathbb{R}^d \to \mathbb{R}$ is some real-valued, measurable potential and $V_- := -\min\{V, 0\}$ is its negative part. The left hand side is ment to be the sum over the absolute values of all negative eigenvalues of $-\Delta + V$ to the power γ. The inequality (0.5) was first derived for $\gamma > \frac{1}{2}$ in the space dimension $d = 1$ and for $\gamma > 0$ in $d \ge 2$ dimensions by E. H. Lieb and W. Thirring in 1975-1976 [56, 57]. The generalisations to the cases $\gamma = \frac{1}{2}$ in $d = 1$ and $\gamma = 0$ in $d \ge 3$ were independently proven by G. Rozenblum in 1972 [65], M. Cwikel in 1977 [24] and E. H. Lieb in 1980 [54]. We would like to mention that $\gamma = 0$ is of special interest, because in this case the sum on the left hand side becomes the total number of negative eigenvalues of the quantum mechanical system. An overview of these basic Lieb-Thirring inequalities (including sharp constants and counterexamples) can be found in [53]. Since then several generalisations were proven, e.g. replacing $\mathbb{R}^d$ by some manifold [47, 63], including magnetic fields [29, 55, 62], considering complex valued potentials [40] or treating fractional Laplace operators [27, 41]. In 2000, A. Laptev and T. Weidl derived an inequality similar to (0.5), where they allowed V to be a family $(V(x))_{x\in\mathbb{R}^d}$ of self-adjoint, non-positive operators [53]. Using this operator-valued Lieb-Thirring inequality, R. Frank and A. Laptev [39] were able to derive the Lieb-Thirring inequality (0.3) mentioned above, for a singular potential $\alpha\delta_\Sigma$ supported on the hyperplane (0.2).

What we do in this thesis is to extend (0.3) to potentials $(\alpha + \alpha_0)\delta_\Sigma$, for which α is integrable and α_0 is some constant negative shift. In order to do this we have to restrict ourselves to powers $\gamma > \frac{d-1}{2}$ and are allowed to choose some arbitrary $0 < \eta < \frac{1}{2}(\gamma - \frac{d-1}{2})$. Then for every $\alpha_0 < 0$ and $\alpha \in L^{d-1+\gamma+\eta}(\Sigma) + L^{\frac{d-1}{2}+\gamma}(\Sigma)$ we prove the Lieb-Thirring type inequality

$$\operatorname{tr}\left(-\Delta + (\alpha+\alpha_0)\delta_\Sigma + \frac{\alpha_0^2}{4}\right)_-^\gamma \leq L_{\|\alpha\|,\alpha_0}\left(\int_\Sigma \alpha_-^{d-1+\gamma+\eta} d\sigma + \int_\Sigma \alpha_-^{\frac{d-1}{2}+\gamma} d\sigma\right). \qquad (0.6)$$

In comparison to (0.3) we do not obtain this single integral over $\alpha_-^{d-1+2\gamma}$ on the right hand side, but two integrals instead. The first one is over $\alpha_-^{d-1+\gamma+\eta}$ and corresponds to the one in (0.3). The second one is over $\alpha_-^{\frac{d-1}{2}+\gamma}$, where the exponent is only half the one of (0.3). Moreover, the prefactor $L_{\|\alpha\|,\alpha_0}$ is not completely constant, but still depends on the norm $\|\alpha\|_{L^{d-1+\gamma+\eta}}$. It may be that the mismatch of the exponent $d-1+\gamma+\eta$ and $d-1+2\gamma$ as well as the non-constant prefactor are due to our methods and could be improved using other techniques. However, the appearance of the second integral with half the exponent seems to be an intrinsic property of the shift $\alpha_0 < 0$.

After this introduction into the topics, methods and results let us now describe the structure of this thesis.

In Chapter 1 we briefly define the function spaces we will need throughout the chapters. In particular we mention Lebesgue, Sobolev and Distribution spaces. In Section 1.5 we prove the existence of Dirichlet and Neumann traces on the halfspace for all functions in the maximal domain $H^0_\Delta(\mathbb{R}^d_+)$. Once we have defined these traces we can also extend the first and second Greens identity, see Theorem 1.38 & 1.39, to those domains.

Chapter 2 gives an introduction into the topic of symmetric, semibounded and closed sesquilinear forms. After some basic properties, the first main result of this chapter is Proposition 2.10 about the stability of closed forms under relatively bounded perturbations. The second even more important result is the first representation theorem,

Theorem 2.11, which states the existence of a unique self-adjoint, semibounded operator representing the form.

The last preparatory part is Chapter 3 about quasi boundary triples. We start by collecting main properties of the related γ-field and Weyl function in Lemma 3.4 & 3.5. Above in (0.4) we already mentioned the possibility of defining symmetric extensions A_B associated with symmetric operators B in the boundary space. Sufficient conditions for the extension to be self-adjoint are proven in Theorem 3.10.

In Chapter 4 we first give a rigorous mathematical definition of the the expression (0.1) using the sesquilinear form a_α from Definition 4.1 in the case that Σ is a hyperplane and $\alpha \in L^{d-1}(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$. A corresponding semibounded and self-adjoint operator A_α is then given in Definition 4.3, using the abstract results from Chapter 2. Moreover, according to the special shape of the form a_α, we are also able to derive an explicit representation of A_α in Proposition 4.4 and calculate $\sigma_{\text{ess}}(A_{\alpha+\alpha_0})$ in Theorem 4.5 in the case that α fulfils some decay property at infinity and α_0 is some constant shift.

In Chapter 5 we apply the abstract theory of quasi boundary triples from Chapter 3 to the Laplace operator on $\mathbb{R}^d \setminus \Sigma$ to get a second definition of A_α. Suitable operators S and T, as well as boundary mappings Γ_0, Γ_1 are constructed in Theorem 5.1. For the γ-field, its adjoint and the Weyl function we derive explicit integral representations in Theorem 5.3. These integral representations will in particular be very useful in the proof of the Lieb-Thirring inequality in Chapter 7. The main result of this chapter is Theorem 5.8, an application of the abstract Theorem 3.10 on self-adjoint extensions. According to the more regular $\alpha \in L^p(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ for some $p > \frac{4}{3}(d-1)$ we obtain a $H^{3/2}_\Delta(\mathbb{R}^d_\pm)$-regularity of the functions in $\operatorname{dom}(A_\alpha)$, in comparison to the lower $H^1_\Delta(\mathbb{R}^d_\pm)$-regularity from Proposition 4.4 in the case $\alpha \in L^{d-1}(\Sigma) + L^\infty(\Sigma)$. In order to handle the constant shift $\alpha_0 < 0$ in the Lieb-Thirring inequality (0.6), we generalise the boundary triple to one for the shifted Laplace operator

$$-\Delta_{\alpha_0} := -\Delta + \alpha_0 \delta_\Sigma \ , \tag{0.7}$$

in the second part of this chapter. Also here we get a corresponding γ-field and Weyl function, with properties which follow immediately from the results of the quasi boundary triple associated with $-\Delta$.

Chapter 6 defines the Birman-Schwinger operator K_λ and proves its main properties and connections to the Schrödinger operator $A_{\alpha+\alpha_0}$. In Theorem 6.3 we show an equivalence of the eigenvalues of $A_{\alpha+\alpha_0}$ to the ones of $\overline{K_\lambda}$. This equivalence is then used to obtain the important Birman-Schwinger principle, Theorem 6.6, which will be essential in the proof of the Lieb-Thirring inequality.

The final Chapter 7 basically proves a version of (0.3), in the case that some constant shift $\alpha_0 < 0$ is added to the potential strength α. In order to separate α_0 from the rest of the potential we replace the Laplace operator $-\Delta$ by the shifted Laplace operator $-\Delta_{\alpha_0} = -\Delta + \alpha_0 \delta_\Sigma$ from (0.7) and only treat the integrable potential $\alpha\delta_\Sigma$

as a perturbation. For the shifted Laplace operator $-\Delta_{\alpha_0}$ we already derived a corresponding Birman-Schwinger principle in Chapter 6, with whose help the Lieb-Thirring inequality is proved in Theorem 7.3 using similar methods as Lieb and Thirring used in their original proof.

Eventually, this thesis has two appendices. In Appendix A we investigate boundedness and convergence properties of the multiplication operator with some function α. The first main result is on boundedness properties in Lemma A.1. The second main result is Theorem A.4, which states that a H^s-weak convergent sequence converges in the L^2-norm if one multiplies with a function α satisfying some decay property at infinity. On any occasion where compactness is being proved in this thesis, this theorem is crucial. In Appendix B we derive various properties of the integral operators which represent the γ-field and the Weyl function of the quasi boundary triple from Chapter 5.

1. Function spaces

In this chapter we introduce Banach and Hilbert spaces in which the operators in the later chapters will be defined. In Section 1.1 we define the concept of integrable functions and the corresponding Lebesgue spaces $L^p(X)$. In Section 1.2 weak derivatives of functions on an open subset $\Omega \subseteq \mathbb{R}^d$ are defined by the integral condition in Definition 1.2. As the resulting function spaces, the Sobolev spaces $W^{s,p}(\Omega)$ will appear in Definition 1.3 & 1.5. Moreover, an approximation result by smooth functions is given in Theorem 1.6 and for $\Omega = \mathbb{R}^d_\pm$ being the the halfspace, we also derive extensions and embedding properties of the Sobolev spaces. In Section 1.3 a short excursion into the theory of tempered distributions is made. Furthermore, the Fourier transformation of $L^p(\mathbb{R}^d)$-functions for $p \in [1, 2]$ as well as for tempered distributions is defined there. Section 1.4 gives an equivalent definition of weak derivatives and Sobolev spaces via distributions. Section 1.5 treats traces of Sobolev functions defined on the halfspaces $\mathbb{R}^d_\pm$. For instance, the usual definition of Dirichlet and Neumann traces on $H^s(\mathbb{R}^d_\pm)$ is extended to the maximal domain $H^0_\Delta(\mathbb{R}^d_\pm)$ in Theorem 1.37 and the corresponding Green's identities are stated in Theorem 1.38 & 1.39.

1.1. Lebesgue spaces

A triple $(X, \mathcal{A}, \mu)$ is called a *measure space*, if X is an arbitrary set, $\mathcal{A}$ is a σ-algebra of subsets of X, and $\mu : \mathcal{A} \to [0, \infty]$ is a measure. For a more detailed definition see for example [28, Kapitel II]. One example, and also the one we will use in this thesis, is the triple $(\Omega, \mathcal{B}(\Omega), \lambda)$, where $\Omega \subseteq \mathbb{R}^d$ is an arbitrary open set, $\mathcal{B}(\Omega)$ is the σ-algebra of Borel sets and λ is the Lebesgue measure, which gives d-dimensional rectangles the usual volume

$$\lambda([a_1, b_1] \times \cdots \times [a_d, b_d]) = \prod_{i=1}^{d} (b_i - a_i) \ .$$

A function $f : X \to \mathbb{C}$ is called *measurable*, if the pre-image $f^{-1}(O)$ of every open subset $O \subseteq \mathbb{C}$ is contained in the σ-algebra $\mathcal{A}$. If a measurable function f is additionally non-negative, it is possible to define the integral

$$\int_X f d\mu \in [0, \infty] \ . \tag{1.1}$$

Consequently, for $p \in [1, \infty]$ one can define the Lebesgue norm

$$\|f\|_{L^p(X)} := \begin{cases} \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} & \text{if } p < \infty \ , \\ \inf\limits_{\mu(N)=0} \sup\limits_{x \in N^c} |f(x)| & \text{if } p = \infty \ , \end{cases} \tag{1.2}$$

for every measurable function f and the corresponding Lebesgue spaces by

$$L^p(X) := \{ \, f : X \to \mathbb{C} \text{ measurable} \mid \|f\|_{L^p(X)} < \infty \, \} \ .$$

These Lebesgue spaces are complete and halfnormed, see [28, Kapitel IV 2.4], and by the usual construction of equivalence classes

$$[f] := \left\{ g \in L^p(X) \;\middle|\; \int_X |f - g| d\mu = 0 \right\} = \{ \, g \in L^p(X) \mid \mu(\{f \neq g\}) = 0 \, \} \ ,$$

$L^p(X)$ becomes a Banach space. Note, that although the space $L^p(X)$ does not consist of functions anymore, the norm is well-defined, because its value is the same for every function in the equivalence class.

In (1.1) we only defined the integral of non-negative functions. But if one separates the real and imaginary part as well as the positive and negative part $r^{\pm} := \max\{\pm r, 0\}$ of a complex-valued function $f \in L^1(X)$, we can also define the integral

$$\int_X f d\mu := \int_X (\mathrm{Re} f)^+ d\mu - \int_X (\mathrm{Re} f)^- d\mu + i \int_X (\mathrm{Im} f)^+ d\mu - i \int_X (\mathrm{Im} f)^- d\mu \ .$$

In the special case $p = 2$ we also define the inner product

$$\langle f, g \rangle_{L^2(X)} := \int_X f \, \bar{g} \, d\mu \ ,$$

which makes the space $L^2(X)$ a Hilbert space.

Lemma 1.1 (Hölder inequality)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for all measurable functions $f, g : X \to \mathbb{C}$, the inequality

$$\int_X |fg| d\mu \leq \|f\|_{L^p(X)} \|g\|_{L^q(X)} \tag{1.3}$$

holds true, in the sense, that one, or even both sides of the inequality are allowed to be infinite [28, Kapitel IV 1.5].

1.2. Sobolev spaces

The theory of Sobolev spaces will generalise the concept of differentiability of a function $f : \Omega \to \mathbb{C}$, defined on an open subset $\Omega \subseteq \mathbb{R}^d$. For multi-indices $\alpha \in \mathbb{N}_0^d$ and vectors $x \in \mathbb{R}^d$ we introduce the compact notations

$$|\alpha| := \sum_{i=1}^{d} \alpha_i \ , \quad x^\alpha := \prod_{i=1}^{d} x_i^{\alpha_i} \quad \text{and} \quad D^\alpha f := \frac{d^{|\alpha|}}{dx_1^{\alpha_1} \dots dx_d^{\alpha_d}} f \ . \tag{1.4}$$

The natural class of functions, for which a weak derivative can be defined, is

$$L^1_{\text{loc}}(\Omega) := \left\{ f : \Omega \to \mathbb{C} \text{ measurable} \ \middle| \ \int_K |f| dx < \infty \ \text{ for every compact } K \subseteq \Omega \right\} ,$$

which is especially a superset of $L^p(\Omega)$, for every $p \in [1, \infty]$.

Definition 1.2. Let $f \in L^1_{\text{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^d$. Then f is said to be *weakly differentiable of order* α, if there exists a function $g \in L^1_{\text{loc}}(\Omega)$, such that

$$\int_\Omega f \ D^\alpha \varphi \ dx = (-1)^{|\alpha|} \int_\Omega g \ \varphi \ dx \ , \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega) \ . \tag{1.5}$$

By the Fundamental Lemma of Calculus of Variations, the function g, if it exists, is unique and one can define the *weak derivative*

$$D^\alpha f := g \ .$$

Obviously, if f is sufficiently often continuously differentiable, one can partially integrate (1.5) and notice that the weak derivative coincides with the classical one, which justifies the notation $D^\alpha f$.

Definition 1.3. Let $\Omega \subseteq \mathbb{R}^d$ be open, $m \in \mathbb{N}_0$ and $p \in [1, \infty)$. Then define the *Sobolev space* by

$$W^{m,p}(\Omega) := \{ \ f \in L^p(\Omega) \mid D^\alpha f \text{ exists in } L^p(\Omega) \text{ for every } |\alpha| \leq m \ \} \ . \tag{1.6}$$

Together with the norm

$$\|f\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

$W^{m,p}(\Omega)$ is a Banach space [20, Theorem 2.15.4]. In the special case $p = 2$, the inner product

$$\langle f, g\rangle_{W^{m,2}(\Omega)} := \sum_{|\alpha|\le m} \langle D^\alpha f, D^\alpha g\rangle_{L^2(\Omega)}$$

makes $W^{m,2}(\Omega)$ a Hilbert space.

In order to extend the definition of Sobolev spaces to $W^{s,p}(\Omega)$ with non-integer $s \in \mathbb{R}_0^+ \setminus \mathbb{N}_0$, we define Sobolev Slobodeckij seminorm.

Definition 1.4. Let $\Omega \subseteq \mathbb{R}^d$ be open, $\sigma \in (0,1)$ and $p \in [1,\infty)$. Then define the *Sobolev-Slobodeckij seminorm* by

$$|f|_{W^{\sigma,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+\sigma p}} dydx\right)^{\frac{1}{p}} .$$

In the special case $p = 2$ we also define the semi inner product

$$\langle f, g\rangle_{W^{\sigma,2}(\Omega)} := \int_\Omega \int_\Omega \frac{(f(x)-f(y))\overline{(g(x)-g(y))}}{|x-y|^{d+2\sigma}} dydx .$$

Definition 1.5. Let $\Omega \subseteq \mathbb{R}^d$ be open, $m \in \mathbb{N}_0$, $\sigma \in (0,1)$ and $p \in [1,\infty)$. Then define the *Sobolev space of non-integer order* by

$$W^{m+\sigma,p}(\Omega) := \{\, f \in W^{m,p}(\Omega) \mid |D^\alpha f|_{W^{\sigma,p}(\Omega)} < \infty \text{ for every } |\alpha| = m \,\} . \tag{1.7}$$

The norm

$$\|f\|_{W^{m+\sigma}(\Omega)} := \left(\|f\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} |D^\alpha f|^p_{W^{\sigma,p}(\Omega)}\right)^{\frac{1}{p}}$$

makes $W^{m+\sigma,p}(\Omega)$ a Banach space [20, Theorem 8.10.10]. In the special case $p = 2$, the inner product

$$\langle f, g\rangle_{W^{m+\sigma,2}(\Omega)} := \langle f, g\rangle_{W^{m,2}(\Omega)} + \sum_{|\alpha|=m} \langle D^\alpha f, D^\alpha g\rangle_{W^{\sigma,2}(\Omega)} ,$$

makes $W^{m+\sigma,2}(\Omega)$ a Hilbert space.

Once we have defined Sobolev spaces, we are able to derive properties like approximation (Theorem 1.6), extension (Theorem 1.7) and embeddings (Theorem 1.8). Unfortunately, the extension and embedding property will no longer be true for all open domains Ω and we will need additional assumptions on the smoothness of the boundary. In order to avoid technical complications we will restrict ourselves to the case that Ω is either the full space $\mathbb{R}^d$ or the halfspace

$$\mathbb{R}^d_\pm = \left\{\, x \in \mathbb{R}^d \;\middle|\; \pm x_d > 0 \,\right\} .$$

The first theorem we want to state is one about approximating Sobolev functions on Ω by test functions on the whole space $\mathbb{R}^d$. This means that the approximating functions are not only smooth inside the domain Ω but also up to its boundary. A proof can for example be found in [2, Theorem 3.22].

Theorem 1.6. Let $\Omega \in \{\mathbb{R}^d, \mathbb{R}^d_\pm\}$, $s \in \mathbb{R}^+_0$ and $p \in [1, \infty)$. Then for every $f \in W^{s,p}(\Omega)$ there exist functions $(f_n)_{n\in\mathbb{N}} \in \mathcal{C}^\infty_0(\mathbb{R}^d)$, such that the restrictions $f_n|_\Omega$ converge in the Sobolev-norm

$$\lim_{n\to\infty} \|f - f_n|_\Omega\|_{W^{s,p}(\Omega)} = 0 \ .$$

The next theorem states the existence of an extension operator, where one possible construction can for example be found in the book [67, Chapter IV].

Theorem 1.7. Let $s \in \mathbb{R}^+_0$ and $p \in [1, \infty)$. Then there exists a bounded operator $E : W^{s,p}(\mathbb{R}^d_\pm) \to W^{s,p}(\mathbb{R}^d)$, such that $(Ef)|_{\mathbb{R}^d_\pm} = f$.

The last theorem according to properties of Sobolev spaces we want to state is the Sobolev embedding theorem, which proves certain inclusion of Sobolev spaces [20, Theorem 8.12.6].

Theorem 1.8. Let $\Omega \in \{\mathbb{R}^d, \mathbb{R}^d_\pm\}$, $s, r \in \mathbb{R}^+_0$, $p, q \in (1, \infty)$ with $p \le q$ and $r - \frac{d}{q} \le s - \frac{d}{p}$. Then there exists a constant $c_{s,p,q,r} > 0$, such that

$$W^{s,p}(\Omega) \subseteq W^{r,q}(\Omega) \quad \text{and} \quad \|f\|_{W^{r,q}(\Omega)} \le c_{s,p,r,q} \|f\|_{W^{s,p}(\Omega)} \ , \quad \forall f \in W^{s,p}(\Omega) \ .$$

The following corollary is the special case of Theorem 1.8 which will be mainly used in this thesis.

Corollary 1.9. Let $\Omega \in \{\mathbb{R}^d, \mathbb{R}^d_\pm\}$, $s \in [0, \frac{d}{2})$ and $q \in [2, \frac{2d}{d-2s}]$. Then there exists a constant $c_{s,q} > 0$, such that

$$W^{s,2}(\Omega) \subseteq L^q(\Omega) \quad \text{and} \quad \|f\|_{L^q(\Omega)} \le c_{s,q} \|f\|_{W^{s,2}(\Omega)} \ , \quad \forall f \in W^{s,2}(\Omega) \ .$$

1.3. Distributions, Fourier transformation and convolution

1.3.1. Space of tempered distributions

In order to define Sobolev spaces $W^{s,2}(\mathbb{R}^d)$ with negative exponents s in Section 1.4, we have to generalise the concept of functions to the concept of distributions, which have two nice properties. First of all, the weak derivative (1.5) can be generalised to the distributional derivative in Definition 1.10, such that every distribution becomes infinitely often differentiable. Secondly, the usual Fourier transformation of $L^p(\mathbb{R}^d)$-functions can be extended to the whole space of distributions. We will start with the space of infinitely often differentiable functions with rapid decay, the Schwartz space,

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) \;\middle|\; \sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta f(x)| < \infty \quad \text{for all } \alpha, \beta \in \mathbb{N}_0^d \right\} . \tag{1.8}$$

The power x^α and the derivative $D^\beta f$ in (1.8) have to be understood in the multi-index notation (1.4). Moreover, the Schwartz space can be equipped with a topology which makes it a metric space. Since the explicit form of the metric is of no interest in this thesis, we will not construct it in this thesis, but refer to [20, Definition 7.2.5]. With respect to the continuity of this topology, one can now define the *space of tempered distributions* as the dual space

$$\mathcal{S}'(\mathbb{R}^d) := \left\{ T : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C} \text{ linear, continuous} \right\} .$$

An example of a tempered distribution is for $p \in [1, \infty]$ and $f \in L^p(\mathbb{R}^d)$, the functional

$$T_f(\varphi) := \int_{\mathbb{R}^d} f(x)\varphi(x)dx , \tag{1.9}$$

which allows to identify the function f with an corresponding distribution.

Definition 1.10. For a distribution $T \in \mathcal{S}'(\mathbb{R}^d)$ and any $\beta \in \mathbb{N}_0^d$, its *distributional derivative* $D^\beta T$ is defined by

$$D^\beta T(\varphi) = (-1)^{|\beta|} T(D^\beta \varphi) , \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d) .$$

Remark 1.11. The distributional derivative $D^\beta T$ is again a tempered distribution (continuous with respect to the topology of the Schwartz space), which means that every distribution is infinitely often differentiable. For every function $f \in W^{m,p}(\mathbb{R}^d)$ and $|\alpha| \leq m$, the derivative of distributions is equivalent to the weak derivative of functions, in the sense that

$$D^\alpha T_f = T_{D^\alpha f} ,$$

if one uses the definition of a distribution of a function in (1.9).

Although distributions and functions are different objects, one can define a multiplication between them, being careful that the function and its derivatives does not grow too fast at infinity.

Definition 1.12. Let $T \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ with the property that for every $\beta \in \mathbb{N}_0^d$ there exists a constant $c_\beta > 0$ and a multi-index $\alpha_\beta \in \mathbb{N}_0^d$, such that

$$|D^\beta f(x)| \leq c_\beta \, |x^{\alpha_\beta}| , \quad \forall x \in \mathbb{R}^d .$$

In this case, the *multiplication of f and T* is defined by

$$(fT)(\varphi) := T(f\varphi) \ , \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d) \ . \tag{1.10}$$

1.3.2. Fourier transformation and convolution

We will define the Fourier transformation as a bounded and injective operator between the spaces $L^p(\mathbb{R}^d)$, for $p \in [1,2]$ and $L^{\frac{p}{p-1}}(\mathbb{R}^d)$ (unitary in the case $p = 2$), with the very useful properties that it turns derivatives, see Theorem 1.19, as well as convolutions, see Theorem 1.22, into a multiplication. In the end of this subsection we will even extend the Fourier transformation to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. Actually, there is not one single Fourier transformation, but different ones depending on the space it transforms. However, they all coincide on the common part of their domains and all of them will be denoted by $\mathcal{F}$ whenever it is clear from the context which space it transforms. Otherwise we will use an upper index, like $\mathcal{F}^{(p)}$, if we want to explicitly denote the specific Fourier transformation.

We will start to define the Fourier transformation on the space $L^1(\mathbb{R}^d)$, where it can be defined explicitly as an integral.

Definition 1.13. For any function $f \in L^1(\mathbb{R}^d)$ define its *Fourier transformation* $\mathcal{F}f : \mathbb{R}^d \to \mathbb{C}$ by the integral

$$\mathcal{F}f(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle k,x\rangle} f(x)dx \ , \quad \forall k \in \mathbb{R}^d \ . \tag{1.11}$$

Here, the brackets $\langle k, x\rangle := \sum_{j=1}^{d} k_j x_j$ denote the standard inner product in $\mathbb{R}^d$.

The following lemma gives some elementary properties of the L^1-Fourier transformation. They can be simply proven by hand, or found for example in [70, Satz V.2.2].

Lemma 1.14. Let $f \in L^1(\mathbb{R}^d)$. Then its Fourier transformation $\mathcal{F}f$ has the following properties:

a) $\mathcal{F}f$ is continuous,

b) $\|\mathcal{F}f\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \|f\|_{L^1(\mathbb{R}^d)}$,

c) $\lim\limits_{k\to\infty} \mathcal{F}f(k) = 0$.

The Fourier transformation $\mathcal{F}$ is, in particular, an everywhere defined bounded operator from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$.

However, the main space in which we will work later on is the space $L^2(\mathbb{R}^d)$, in which the integral definition (1.11) makes no sense, since the integral does not converge in general. To circumvent this problem, the following Theorem 1.15 proves the boundedness of $\mathcal{F}$ with respect to the L^2-norm on the dense Schwartz space $\mathcal{S}(\mathbb{R}^d)$. More specifically, this means that we are allowed to define the L^2-Fourier transformation in Definition 1.16 by the extension of this bounded operator to all of $L^2(\mathbb{R}^d)$. A proof of Theorem 1.15 can be found for instance in [20, Theorem 7.7.2 & Theorem 7.7.3].

Theorem 1.15. The L^1-Fourier transformations of Definition 1.11 restricted to the Schwartz space, $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is bijective. Furthermore, it is isometric according to the $L^2(\mathbb{R}^d)$-inner product

$$\langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2(\mathbb{R}^d)} = \langle f, g \rangle_{L^2(\mathbb{R}^d)} , \quad \forall f, g \in \mathcal{S}(\mathbb{R}^d) .$$

Definition 1.16. Let $f \in L^2(\mathbb{R}^d)$. Then define its Fourier transformation $\mathcal{F}f$ by the limit

$$\mathcal{F}f := \lim_{n\to\infty} \mathcal{F}f_n \quad \text{in } L^2(\mathbb{R}^d) , \tag{1.12}$$

with $(f_n)_{n\in\mathbb{N}} \in \mathcal{S}(\mathbb{R}^d)$, such that $\lim_{n\to\infty} \|f - f_n\|_{L^2(\mathbb{R}^d)} = 0$.

Corollary 1.17. Theorem 1.15 shows that the L^2-Fourier transformation $\mathcal{F}$ is a unitary operator in $L^2(\mathbb{R}^d)$.

By Lemma 1.14 and Corollary 1.17 we have defined the Fourier transformation as a bounded operator $\mathcal{F}^{(1)} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ and $\mathcal{F}^{(2)} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. Using the Interpolation Theorem of Riesz-Thorin [69, Satz 2.65], we can extend it also to every space $L^p(\mathbb{R}^d)$ for every $p \in [1, 2]$.

Theorem 1.18. Let $p \in [1, 2]$ and $q \in [2, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a unique bounded Fourier transformation

$$\mathcal{F} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d) \quad \text{with} \quad \|\mathcal{F}\| \leq \frac{1}{(2\pi)^{d(\frac{1}{p}-\frac{1}{2})}} , \tag{1.13}$$

which coincides with (1.11) and (1.12) on the common domain of definition.

The first main property of the Fourier transformation is, that it reduces the weak derivative (and therefore also the classical one) into a simple multiplication.

Theorem 1.19. Let $m \in \mathbb{N}_0$ and $f \in W^{m,2}(\mathbb{R}^d)$. Then for every multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$ the Fourier transformation of the weak derivative $D^\alpha f$ is given by

$$\mathcal{F}[D^\alpha f](k) = (ik)^\alpha \mathcal{F}f(k), \quad \forall k \in \mathbb{R}^d .$$

The second main property of the Fourier transformation is related to the convolution, which will be defined in the following.

Definition 1.20. Let $p, q \in [1, \infty]$, such that $\frac{1}{p} + \frac{1}{q} \geq 1$. Then for every $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ define its *convolution* $f * g$ by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y)dy , \quad \forall x \in \mathbb{R}^d .$$

It is not obvious that this integral exists, but choosing $r \in [1, \infty]$, such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and using the Hölder inequality (1.3) as well as the Interpolation Theorem of Riesz-Thorin, one can prove [20, Theorem 6.2.1] that the convolution is a well-defined bounded operator in $L^r(\mathbb{R}^d)$, such that

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^q(\mathbb{R}^d)} . \tag{1.14}$$

One of the main applications of the convolution is the approximation by regular functions. Also the solutions of partial differential equations can be expressed via the convolution with some integral kernel. The following theorem about differentiation of the convolution shows its smoothening properties.

Theorem 1.21. Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} \geq 1$, $m \in \mathbb{N}_0$, $f \in W^{m,p}(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Then $f * g \in W^{m,r}(\mathbb{R}^d)$, for $\frac{1}{r} := 1 - \frac{1}{p} - \frac{1}{q}$, and for every multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$ the weak derivative D^α of the convolution $f * g$ is given by

$$D^\alpha(f * g) = (D^\alpha f) * g . \tag{1.15}$$

If additionally $f \in \mathcal{C}^m(\mathbb{R}^d)$ is m-times continuously differentiable, then also $f * g \in \mathcal{C}^m(\mathbb{R}^d)$ is m-times continuously differentiable and (1.15) holds in the sense of classical derivatives.

The main property of the convolution is the following Theorem about its Fourier transformation.

Theorem 1.22. Let $p, q \in [1, 2]$ with $\frac{1}{p} + \frac{1}{q} \geq \frac{3}{2}$. Then $\frac{1}{r} := \frac{1}{p} + \frac{1}{q} - 1 \in [\frac{1}{2}, 1]$ and for $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ the Fourier transformation of $f * g$ is well-defined and has the form

$$\mathcal{F}^{(r)}[f * g] = (2\pi)^{\frac{d}{2}} \mathcal{F}^{(p)}[f] \mathcal{F}^{(q)}[g] . \tag{1.16}$$

Note that the three Fourier transformations are different ones, namely the ones in $L^r(\mathbb{R}^d)$, $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$, indicated by the upper indices.

The last result of this section will be the extension of the Fourier transformation to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

Definition 1.23. For every tempered distribution $T \in S'(\mathbb{R}^d)$ define its *Fourier transformation* $\mathcal{F}T$ by

$$\mathcal{F}T(\varphi) = T(\mathcal{F}\varphi) , \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d) .$$

Then the Fourier transformation $\mathcal{F}T \in \mathcal{S}'(\mathbb{R}^d)$ is again a tempered distribution (continuous with respect to the topology of the Schwartz space) and for any function $f \in L^p(\mathbb{R}^d)$, $p \in [1,2]$, the Fourier transformation of T_f from (1.9) is equivalent to the Fourier transformation of f, from Theorem 1.18, in the sense, that $\mathcal{F}T_f = T_{\mathcal{F}f}$.

1.4. Sobolev spaces of distributions

The Sobolev spaces $W^{s,p}(\mathbb{R}^d)$ were defined via weak derivatives in Definition 1.3 & 1.5 and in Theorem 1.19 we obtained a close relation between weak derivatives and Fourier transformation. This motivates the definition of the Sobolev spaces $H^s(\mathbb{R}^d)$ in (1.17) at least in the special case $p = 2$, $s \geq 0$ and $\Omega = \mathbb{R}^d$. However, using the concept of distributions from Section 1.3.1, we can also define the space $H^s(\mathbb{R}^d)$ for negative indices s.

Definition 1.24. For $s \in \mathbb{R}$ define the *distributional Sobolev space* by

$$H^s(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \,\middle|\, (1+|k|^2)^{\frac{s}{2}} \mathcal{F}f \in L^2(\mathbb{R}^d) \right\} . \tag{1.17}$$

The product of the function $(1+|k|^2)^{\frac{s}{2}}$ and the distribution $\mathcal{F}f$ is defined in (1.10) and the property that the distribution $(1+|k|^2)^{\frac{s}{2}}\mathcal{F}f$ is in the space $L^2(\mathbb{R}^d)$, has to be understood in the sense of (1.9). The corresponding inner product is defined as

$$\langle f, g\rangle_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1+|k|^2)^s \, \mathcal{F}f(k)\, \overline{\mathcal{F}g(k)}\, dk , \tag{1.18}$$

where the distribution $(1+|k|^2)^{\frac{s}{2}}\mathcal{F}f$ is identified with the corresponding function, according to (1.9).

Theorem 1.25. For every $s \in \mathbb{R}$ the distributional Sobolev space $(H^s(\mathbb{R}^d), \langle\cdot,\cdot\rangle_{H^s(\mathbb{R}^d)})$ is a Hilbert space. Moreover, if $s \geq 0$, then $H^s(\mathbb{R}^d) \cong W^{s,2}(\mathbb{R}^d)$ and the norms $\|\cdot\|_{H^s(\mathbb{R}^d)}$, $\|\cdot\|_{W^{s,2}(\mathbb{R}^d)}$ are equivalent.

Remark 1.26. For $s \geq 0$ the space $H^{-s}(\mathbb{R}^d)$ can be interpreted as the dual space of $H^s(\mathbb{R}^d)$. This means that there is a one-to-one correspondence between all bounded, antilinear functionals $F : H^s(\mathbb{R}^d) \to \mathbb{C}$ and tempered distributions $f \in H^{-s}(\mathbb{R}^d)$, see e.g. [20, Theorem 8.9.7]. Moreover, such a pair (F, f) satisfies

$$\langle f, g\rangle_{H^{-s},H^s} := F(g) = \int_{\mathbb{R}^d} \left((1+|k|^2)^{-\frac{s}{2}}\mathcal{F}f\right) \left(\overline{(1+|k|^2)^{\frac{s}{2}}\mathcal{F}g}\right) dx , \quad \forall g \in H^s(\mathbb{R}^d) .$$

The next lemma is a useful property of Sobolev spaces, because it allows to get an arbitrary small prefactor of the Sobolev norm if one slightly increases the order of the Sobolev space. The proof would have been quite technical using the original definition

$W^{s,2}(\mathbb{R}^d)$, but in the equivalent Fourier definition $H^s(\mathbb{R}^d)$ it basically reduces to one simple estimate.

Theorem 1.27. Let $s_1 < s < s_2 \in \mathbb{R}$ and $\varepsilon > 0$. Then every $f \in H^{s_2}(\mathbb{R}^d)$, the Sobolev norm $\|f\|_{H^s(\mathbb{R}^d)}$ can be estimated by

$$\|f\|_{H^s(\mathbb{R}^d)} \leq \varepsilon \, \|f\|_{H^{s_2}(\mathbb{R}^d)} + \frac{c_\theta}{\varepsilon^\theta} \, \|f\|_{H^{s_1}(\mathbb{R}^d)} \ ,$$

where the constant is given by $c_\theta = \left(\frac{\theta^\theta}{(1+\theta)^{1+\theta}}\right)^{\frac{1}{2}}$ with $\theta = \frac{s-s_1}{s_2-s}$.

Proof. For every $a \geq 0$ and $r > 1$ one can easily find the minimum of the function $a \mapsto a^r - a$ (for example by finding the zero of its derivative) and obtains the inequality

$$a^r - a \geq -\frac{r-1}{r^{\frac{r}{r-1}}} \ . \tag{1.19}$$

For every $k \in \mathbb{R}^d$ we can now use (1.19) with the values $a = \varepsilon^{2\theta}(1+|k|^2)^{s-s_1}$ and $r = \frac{1+\theta}{\theta}$ (where $\theta = \frac{s-s_1}{s_2-s}$) to derive

$$(1+|k|^2)^s \leq \varepsilon^2(1+|k|^2)^{s_2} + \frac{1}{\varepsilon^{2\theta}}\frac{\theta^\theta}{(1+\theta)^{1+\theta}}(1+|k|^2)^{s_1} \ .$$

Using Definition 1.24 of the Sobolev norm, we immediately get

$$\|f\|^2_{H^s(\mathbb{R}^d)} \leq \varepsilon^2\|f\|^2_{H^{s_2}(\mathbb{R}^d)} + \frac{1}{\varepsilon^{2\theta}}\frac{\theta^\theta}{(1+\theta)^{1+\theta}}\|f\|^2_{H^{s_1}(\mathbb{R}^d)} \ .$$

After taking the square root this inequality becomes the final result

$$\|f\|_{H^s(\mathbb{R}^d)} \leq \varepsilon\|f\|_{H^{s_2}(\mathbb{R}^d)} + \frac{1}{\varepsilon^{\theta}}\frac{\theta^{\frac{\theta}{2}}}{(1+\theta)^{\frac{1+\theta}{2}}}\|f\|_{H^{s_1}(\mathbb{R}^d)} \ .$$

□

In the last part of this section we want to introduce another variation of Sobolev spaces, which will be of great importance in the context of quasi boundary triples in Chapter 5.

Definition 1.28. For every $s \in \mathbb{R}_0^+$ define the Sobolev space

$$H^s_\Delta(\mathbb{R}^d_\pm) := \left\{ \, f \in H^s(\mathbb{R}^d_\pm) \,\middle|\, \Delta f \text{ exists in } L^2(\mathbb{R}^d_\pm) \, \right\} \ .$$

With the inner product

$$\langle f, g\rangle_{H^s_\Delta(\mathbb{R}^d_\pm)} := \langle f, g\rangle_{H^s(\mathbb{R}^d_\pm)} + \langle \Delta f, \Delta g\rangle_{L^2(\mathbb{R}^d_\pm)} \ ,$$

$H^s_\Delta(\mathbb{R}^d_\pm)$ is a Hilbert space.

Remark 1.29. In the case that $s \geq 2$ we obtain that $H^s_\Delta(\mathbb{R}^d_\pm) = H^s(\mathbb{R}^d_\pm)$ with equivalent norms $\|\cdot\|_{H^s_\Delta(\mathbb{R}^d_\pm)}$ and $\|\cdot\|_{H^s(\mathbb{R}^d_\pm)}$.

1.5. Traces on the hyperplane

In Section 1.2 we defined Sobolev spaces for open domains $\Omega \subseteq \mathbb{R}^d$. But to make sense of the potential $\alpha\delta_\Sigma$ later on in Definition 4.1, we also need to restrict these functions $f \in H^s(\mathbb{R}^d)$ to the hyperplane $\Sigma \cong \mathbb{R}^{d-1}$. In order to introduce quasi boundary triples in Chapter 5 and get the representation of $\operatorname{dom}(A_\alpha)$ via boundary values in Theorem 5.8, we will also need Dirichlet and Neumann traces of functions $f \in H^s(\mathbb{R}^d_\pm)$ defined on the halfspace. Furthermore we will derive additional properties like surjectivity or Green's first and second formula for these traces. Without loss of generality, most results will only be proven for the upper halfplane $\mathbb{R}^d_+$, but then trivially also hold for the lower one.

Lemma 1.30. For every $s > \frac{1}{2}$, there exists a unique, everywhere defined, bounded operator $\tau_D : H^s(\mathbb{R}^d) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$, such that

$$\tau_D f(x') = f(x', 0) , \quad \forall f \in \mathcal{S}(\mathbb{R}^d), x' \in \mathbb{R}^{d-1} . \tag{1.20}$$

Proof. First, define the operator $\tau_D : H^s(\mathbb{R}^d) \to L^2(\mathbb{R}^{d-1})$ by (1.20) on the subspace $\operatorname{dom}(\tau_D) = \mathcal{S}(\mathbb{R}^d)$, where it is obviously well-defined. Knowing that by Theorem 1.15 the Fourier transformation is a bijective mapping of the Schwartz space into itself, we can look at the action of τ_D in Fourier space.

$$\begin{aligned}
\tau_D f(x') = \mathcal{F}_d^{-1}\mathcal{F}_d f(x', 0) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\langle k,(x',0)\rangle} \mathcal{F}_d f(k) dk \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^{d-1}} e^{i\langle k',x'\rangle} \int_{\mathbb{R}} \mathcal{F}_d f(k', k_d) dk_d dk' \\
&= \frac{1}{\sqrt{2\pi}} \mathcal{F}_{d-1}^{-1} \left[\int_{\mathbb{R}} \mathcal{F}_d f(\,\cdot\,, k_d) dk_d \right] (x')
\end{aligned}$$

Applying the Fourier transformation on both sides of this equation, one obtains the action of the Dirichlet trace in Fourier space as the integration over the d-th variable

$$\mathcal{F}_{d-1}\tau_D f(k') = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}_d f(k', k_d) dk_d . \tag{1.21}$$

Inserting $\frac{(1+|k|^2)^{\frac{s}{2}}}{(1+|k|^2)^{\frac{s}{2}}}$ and applying the Hölder inequality (1.3) gives

$$|\mathcal{F}_{d-1}\tau_D f(k')|^2 \leq \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{(1+|k|^2)^s} dk_d \int_{\mathbb{R}} (1+|k|^2)^s |\mathcal{F}_d f(k)|^2 dk_d \;, \tag{1.22}$$

where the first integral converges because of $s > \frac{1}{2}$ and has the value

$$\int_{\mathbb{R}} \frac{1}{(1+|k|^2)^s} dk_d = \frac{c_s}{(1+|k'|^2)^{s-\frac{1}{2}}} \;,$$

with some constant $c_s > 0$. Note that in this integral we can see the reduction of the power of the Sobolev space by $\frac{1}{2}$ when reducing the dimension by 1. Using this explicit integral in (1.22) as well as the definition of the $H^s(\mathbb{R}^d)$- and $H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$-norm from (1.18), we immediately get the boundedness of the Dirichlet trace operator

$$\begin{aligned} \|\tau_D f\|^2_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})} &= \int_{\mathbb{R}^{d-1}} (1+|k'|^2)^{s-\frac{1}{2}} |\mathcal{F}_{d-1}\tau_0 f(k')|^2 dk' \\ &\leq \frac{c_s}{2\pi} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} (1+|k|^2)^s |\mathcal{F}_d f(k)|^2 dk_d dk' \\ &= \frac{c_s}{2\pi} \|f\|^2_{H^s(\mathbb{R}^d)} \;. \end{aligned}$$

Since the domain $\mathrm{dom}(\tau_D) = \mathcal{S}(\mathbb{R}^d)$ was dense in $H^s(\mathbb{R}^d)$, τ_D has a bounded continuation to the whole space. □

The Dirichlet and the Neumann trace can also be defined for functions $f \in H^s(\mathbb{R}^d_+)$, defined only on the halfspace, see [46, Theorem 9.2].

Theorem 1.31. For every $s > \frac{1}{2}$, there exists a unique, everywhere defined, bounded operator $\tau_D : H^s(\mathbb{R}^d_+) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$, called the *Dirichlet trace*, such that

$$\tau_D f(x') = f(x', 0) \;, \quad \forall x' \in \mathbb{R}^{d-1}$$

for all continuous functions $f \in \mathcal{C}(\overline{\mathbb{R}^d_+}) \cap H^s(\mathbb{R}^d_+)$.

Furthermore, for every $s > \frac{3}{2}$, there exists a unique, everywhere defined, bounded operator $\tau_N : H^s(\mathbb{R}^d_+) \to H^{s-\frac{3}{2}}(\mathbb{R}^{d-1})$, called the *Neumann trace*, such that

$$\tau_N f(x') = -\partial_{x_d} f(x', 0) \;, \quad \forall x' \in \mathbb{R}^{d-1}$$

for all continuously differentiable functions $f \in \mathcal{C}^1(\overline{\mathbb{R}^d_+}) \cap H^s(\mathbb{R}^d_+)$.

Remark 1.32.

- The minus sign in the definition of the Neumann trace is caused by the normal derivative, which is $-e_d$ for the halfspace $\mathbb{R}^d_+$.

- Despite the fact that for every s the Dirichlet (Neumann) trace is a different operator, we will nevertheless name all of them τ_D (τ_N), because they are extensions from one another. We can also look at this in the way that the Dirichlet trace is a mapping (not an operator) $\tau_D : \bigcup_{s>\frac{1}{2}} H^s(\mathbb{R}^d_+) \to \bigcup_{s>\frac{1}{2}} H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$, such that

$$\tau_D\restriction_{H^s(\mathbb{R}^d_+)} : H^s(\mathbb{R}^d_+) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1}) \text{ is a bounded operator.}$$

  Also for the Neumann trace $\tau_N : \bigcup_{s>\frac{3}{2}} H^s(\mathbb{R}^d_+) \to \bigcup_{s>\frac{3}{2}} H^{s-\frac{3}{2}}(\mathbb{R}^{d-1})$, the restriction

$$\tau_N\restriction_{H^s(\mathbb{R}^d_+)} : H^s(\mathbb{R}^d_+) \to H^{s-\frac{3}{2}}(\mathbb{R}^{d-1}) \text{ is a bounded operator.}$$

From [61, Theorem 2] we get the following surjectivity of the combined mapping of the Dirichlet and Neumann trace.

Theorem 1.33. For every $s > \frac{3}{2}$ the combined mapping

$$\begin{pmatrix} \tau_D \\ \tau_N \end{pmatrix} : H^s(\mathbb{R}^d_+) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1}) \times H^{s-\frac{3}{2}}(\mathbb{R}^{d-1}) \text{ is surjective.}$$

Furthermore, if $s - \frac{1}{2} \notin \mathbb{N}$, there exists an everywhere defined bounded right inverse

$$R : H^{s-\frac{1}{2}}(\mathbb{R}^{d-1}) \times H^{s-\frac{3}{2}}(\mathbb{R}^{d-1}) \to H^s(\mathbb{R}^d_+)\ .$$

The same construction for the Dirichlet and the Neumann trace can also be done for the lower halfplane $\mathbb{R}^d_-$ and we get another pair of traces. Without causing confusions we will also call them τ_D and τ_N, and they will be identified by the function they act at.

In the rest of this section (and also of the thesis), we will often restrict functions $f \in L^2(\mathbb{R}^d)$ to the halfspaces $\mathbb{R}^d_\pm$. Therefore, we will use the notation

$$f_+ := f|_{\mathbb{R}^d_+} \in L^2(\mathbb{R}^d_+) \quad \text{and} \quad f_- := f|_{\mathbb{R}^d_-} \in L^2(\mathbb{R}^d_-)\ . \tag{1.23}$$

Theorem 1.34. For $s > \frac{1}{2}$ and $f \in H^s(\mathbb{R}^d)$, the Dirichlet trace is continuous in the sense that

$$\tau_D f_+ = \tau_D f_-\ .$$

If $s > \frac{3}{2}$, also the Neumann trace is continuous in the sense that

$$\tau_N f_+ = -\tau_N f_-\ .$$

The different signs are caused by the different unit normal vectors of $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$.

Proof. We will only prove the continuity of the Dirichlet trace, since the continuity of the Neumann trace follows the same steps.

Let $f \in H^s(\mathbb{R}^d)$. Then by Theorem 1.6 there exists functions $(f_n)_{n\in\mathbb{N}} \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, such that

$$\lim_{n\to\infty} \|f - f_n\|_{H^s(\mathbb{R}^d)} = 0 \,. \tag{1.24}$$

The difference of the two Dirichlet traces can then be estimated as

$$\left\|\tau_D f_+ - \tau_D f_-\right\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})} \leq \|\tau_D\| \, \|f_+ - (f_n)_-\|_{H^s(\mathbb{R}^d_+)} + \|\tau_D\| \, \|(f_n)_- - f_-\|_{H^s(\mathbb{R}^d_-)} \,, \tag{1.25}$$

where the mixed term $\left\|\tau_D(f_n)_+ - \tau_D(f_n)_-\right\|_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})}$ vanishes, because of $\tau_D(f_n)_+ = f_n|_\Sigma = \tau_D(f_n)_-$.

Equation (1.24) shows that the right hand side of (1.25) converges to zero, which proves the continuity of the Dirichlet trace. □

We will now derive the two Green's identities which are an important connection between the Sobolev spaces on the domain and on the boundary. The first one is a generalisation of partial integration and the second one follows immediately from the first one.

Theorem 1.35 (First Green's identity)**.** The *first Green's identity*

$$\langle \Delta f, g\rangle_{L^2(\mathbb{R}^d_+)} = -\langle \nabla f, \nabla g\rangle_{L^2(\mathbb{R}^d_+)} + \langle \tau_N f, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} \tag{1.26}$$

holds true for every $f \in H^2(\mathbb{R}^d_+)$ and $g \in H^1(\mathbb{R}^d_+)$.

Proof. From Theorem 1.6 we know that we can approximate f and g by functions $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, such that

$$\lim_{n\to\infty} \|f - (f_n)_+\|_{H^2(\mathbb{R}^d_+)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|g - (g_n)_+\|_{H^1(\mathbb{R}^d_+)} = 0 \,.$$

These approximations especially give the convergences of the three inner products

$$\begin{aligned}
&\lim_{n\to\infty} \langle \Delta f_n, g_n\rangle_{L^2(\mathbb{R}^d_+)} = \langle \Delta f, g\rangle_{L^2(\mathbb{R}^d_+)} \,, \\
&\lim_{n\to\infty} \langle \nabla f_n, \nabla g_n\rangle_{L^2(\mathbb{R}^d_+)} = \langle \nabla f, \nabla g\rangle_{L^2(\mathbb{R}^d_+)} \quad \text{and} \\
&\lim_{n\to\infty} \langle \tau_N f_n, \tau_D g_n\rangle_{L^2(\mathbb{R}^{d-1})} = \langle \tau_N f, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} \,.
\end{aligned}$$

It is well-known, for example using Gauss theorem, that (1.26) holds true for all functions $f_n, g_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and by the above limits it can be extended to f and g. □

Theorem 1.36 (Second Green's identity)**.** The *second Green's identity*

$$\langle \Delta f, g\rangle_{L^2(\mathbb{R}^d_+)} - \langle f, \Delta g\rangle_{L^2(\mathbb{R}^d_+)} = \langle \tau_N f, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} - \langle \tau_D f, \tau_N g\rangle_{L^2(\mathbb{R}^{d-1})} \tag{1.27}$$

holds true for every $f, g \in H^2(\mathbb{R}^d_+)$.

Proof. The first Green's identity, Theorem 1.35, holds true in the form (1.26) and also if one exchanges f and g:

$$\begin{aligned}\langle \Delta f, g\rangle_{L^2(\mathbb{R}^d_+)} &= -\langle \nabla f, \nabla g\rangle_{L^2(\mathbb{R}^d_+)} + \langle \tau_N f, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})}\\ \langle \Delta g, f\rangle_{L^2(\mathbb{R}^d_+)} &= -\langle \nabla g, \nabla f\rangle_{L^2(\mathbb{R}^d_+)} + \langle \tau_N g, \tau_D f\rangle_{L^2(\mathbb{R}^{d-1})} .\end{aligned}$$

Subtracting the second from the first equation immediately gives the second Green's identity (1.27). □

In Theorem 1.31 we defined the Dirichlet (Neumann) trace only for spaces $H^s(\mathbb{R}^d_+)$ with $s > \frac{1}{2}$ ($s > \frac{3}{2}$). Unfortunately, for the case $s \leq \frac{1}{2}$ ($s \leq \frac{3}{2}$) this is no longer possible. However, it is still possible to define a Dirichlet (Neumann) traces on the space $H^s_\Delta(\mathbb{R}^d_+)$, see Definition 1.28, for every $s \in [0, 2]$.

Theorem 1.37. For every $s \in [0, 2]$ there exists a unique, everywhere defined and bounded *Dirichlet trace*

$$\tau_D : H^s_\Delta(\mathbb{R}^d_+) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$$

and a unique, everywhere defined and bounded *Neumann trace*

$$\tau_N : H^s_\Delta(\mathbb{R}^d_+) \to H^{s-\frac{3}{2}}(\mathbb{R}^{d-1}) ,$$

which are extensions of the Dirichlet and Neumann traces from Theorem 1.31.

Proof. We will only prove the existence of the Dirichlet trace. The same proof with obvious changes then also works for the Neumann trace.

Let $R : H^{\frac{3}{2}}(\mathbb{R}^{d-1}) \times H^{\frac{1}{2}}(\mathbb{R}^{d-1}) \to H^2(\mathbb{R}^d_+)$ be the bounded right inverse of the pair (τ_D, τ_N) from Theorem 1.33 in the case $s = 2$. Then for every $f \in H^2(\mathbb{R}^d_+)$ we can define the linear functional $\Gamma_D f : H^{\frac{1}{2}}(\mathbb{R}^{d-1}) \to \mathbb{C}$ by

$$\Gamma_D f(\phi) := \langle f, \Delta R(0, \phi)\rangle_{L^2(\mathbb{R}^d_+)} - \langle \Delta f, R(0, \phi)\rangle_{L^2(\mathbb{R}^d_+)} , \quad \forall \phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}) .$$

This functional is bounded by the estimate

$$\begin{aligned}|\Gamma_D f(\phi)| &\leq \|\Delta f\|_{L^2(\mathbb{R}^d_+)} \|R(0, \phi)\|_{L^2(\mathbb{R}^d_+)} + \|f\|_{L^2(\mathbb{R}^d_+)} \|\Delta R(0, \phi)\|_{L^2(\mathbb{R}^d_+)}\\ &\leq \sqrt{2}\, \|f\|_{H^0_\Delta(\mathbb{R}^d_+)} \|R(0, \phi)\|_{H^2(\mathbb{R}^d_\pm)}\\ &\leq \sqrt{2}\, \|R\|\, \|f\|_{H^0_\Delta(\mathbb{R}^d_+)} \|\phi\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})} \end{aligned} \tag{1.28}$$

and therefore an element in the dual space $H^{\frac{1}{2}}(\mathbb{R}^{d-1})'$ which is equivalent to $H^{-\frac{1}{2}}(\mathbb{R}^{d-1})$ by Remark 1.26. Furthermore, by the second Green's identity (1.27) we have

$$\Gamma_D f(\phi) = \langle \tau_D f, \tau_N R(0, \phi)\rangle_{L^2(\mathbb{R}^{d-1})} - \langle \tau_N f, \tau_D R(0, \phi)\rangle_{L^2(\mathbb{R}^{d-1})} = \langle \tau_D f, \phi\rangle_{L^2(\mathbb{R}^{d-1})}$$

and $\Gamma_D f \cong \tau_D f$ coincide. By (1.28) we then get the boundedness

$$\|\tau_D f\|_{H^{-\frac{1}{2}}(\mathbb{R}^{d-1})} \leq \sqrt{2}\,\|R\|\,\|f\|_{H^0_\Delta(\mathbb{R}^d_+)}\ , \qquad \forall f \in H^2(\mathbb{R}^d_+)$$

of τ_D, which consequently has a bounded extension to the whole space $H^0_\Delta(\mathbb{R}^d_+)$.

Moreover, from Theorem 1.31 and Remark 1.29 we know that $\tau_D : H_{\Delta^2}(\mathbb{R}^d_+) \to H^{\frac{3}{2}}(\mathbb{R}^{d-1})$ is a bounded operator. By the interpolation property of the Sobolev spaces [58], we can conclude that the restriction of τ_D to $H^s_\Delta(\mathbb{R}^d_+)$ is then also bounded as an operator into the space $H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$, for every $s \in [0,2]$. □

With this expansion of the traces we can now also prove Green's first and second identity, Theorem 1.35 & 1.36, for a larger class of functions. One has to only exchange the inner products $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^{d-1})}$ on the right hand side by the evaluation of the functional $\langle\cdot,\cdot\rangle_{H^{-t},H^t}$ defined in Remark 1.26.

Theorem 1.38 (First Green's identity)**.** For every $s \in [1,2]$, the *first Green's identity*

$$\langle \Delta f, g\rangle_{L^2(\mathbb{R}^d_+)} = -\langle \nabla f, \nabla g\rangle_{L^2(\mathbb{R}^d_+)} + \langle \tau_N f, \tau_D g\rangle_{H^{s-\frac{3}{2}},H^{\frac{3}{2}-s}} \tag{1.29}$$

holds true for every $f \in H^s_\Delta(\mathbb{R}^d_+)$ and $g \in H^1(\mathbb{R}^d_+)$.

Proof. Since $H^s_\Delta(\mathbb{R}^d_+)$ is dense in $H^s(\mathbb{R}^d_+)$, there exists functions $(f_n)_{n\in\mathbb{N}} \in H^2(\mathbb{R}^d_+)$, such that

$$\lim_{n\to\infty} \|f - f_n\|_{H^s_\Delta(\mathbb{R}^d_+)} = 0\ .$$

In particular, by the continuity of the traces, this implies the convergences

$$\begin{aligned}
&\lim_{n\to\infty} \langle \Delta f_n, g\rangle_{L^2(\mathbb{R}^d_+)} = \langle \Delta f, g\rangle_{L^2(\mathbb{R}^d_+)}\ ,\\
&\lim_{n\to\infty} \langle \nabla f_n, \nabla g\rangle_{L^2(\mathbb{R}^d_+)} = \langle \nabla f, \nabla g\rangle_{L^2(\mathbb{R}^d_+)} \quad \text{and,}\\
&\lim_{n\to\infty} \langle \tau_N f_n, \tau_D g\rangle_{H^{s-\frac{3}{2}},H^{\frac{3}{2}-s}} = \langle \tau_N f, \tau_D g\rangle_{H^{s-\frac{3}{2}},H^{\frac{3}{2}-s}}\ .
\end{aligned}$$

Since by Theorem 1.35 Green's first identity already holds for $f_n \in H^2(\mathbb{R}^d_+)$ and $g \in H^1(\mathbb{R}^d_+)$, it follows immediately that it also holds for $f \in H^s_\Delta(\mathbb{R}^d_+)$ by these limits. □

Theorem 1.39 (Second Green's identity)**.** For every $s \in [0,2]$, the *second Green's identity*

$$\langle \Delta f, g\rangle_{L^2(\mathbb{R}^d_+)} - \langle f, \Delta g\rangle_{L^2(\mathbb{R}^d_+)} = \langle \tau_N f, \tau_D g\rangle_{H^{s-\frac{3}{2}},H^{\frac{3}{2}-s}} - \langle \tau_D f, \tau_N g\rangle_{H^{s-\frac{1}{2}},H^{\frac{1}{2}-s}} \tag{1.30}$$

holds for every $f \in H^s_\Delta(\mathbb{R}^d_+)$, $g \in H^{2-s}_\Delta(\mathbb{R}^d_+)$.

Proof. Since the spaces $H^s_\Delta(\mathbb{R}^d_+)$ and $H^{2-s}_\Delta(\mathbb{R}^d_+)$ are dense in $H^s(\mathbb{R}^d_+)$, there exists functions $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \in H^2(\mathbb{R}^d_+)$, such that

$$\lim_{n\to\infty} \|f - f_n\|_{H^s_\Delta(\mathbb{R}^d_+)} = \lim_{n\to\infty} \|g - g_n\|_{H^{2-s}_\Delta(\mathbb{R}^d_+)} = 0 \ .$$

In particular by the continuity of the traces, this implies the convergences

$$\begin{aligned}
&\lim_{n\to\infty} \langle \Delta f_n, g_n\rangle_{L^2(\mathbb{R}^d_+)} = \langle \Delta f, g\rangle_{L^2(\mathbb{R}^d_+)} \ , \\
&\lim_{n\to\infty} \langle f_n, \Delta g_n\rangle_{L^2(\mathbb{R}^d_+)} = \langle f, \Delta g\rangle_{L^2(\mathbb{R}^d_+)} \ , \\
&\lim_{n\to\infty} \langle \tau_N f_n, \tau_D g_n\rangle_{H^{s-\frac{3}{2}}, H^{\frac{3}{2}-s}} = \langle \tau_N f, \tau_D g\rangle_{H^{s-\frac{3}{2}}, H^{\frac{3}{2}-s}} \quad \text{and} \\
&\lim_{n\to\infty} \langle \tau_D f_n, \tau_N g_n\rangle_{H^{s-\frac{1}{2}}, H^{\frac{1}{2}-s}} = \langle \tau_D f, \tau_N g\rangle_{H^{s-\frac{1}{2}}, H^{\frac{1}{2}-s}} \ .
\end{aligned}$$

Since by Theorem 1.36 Green's second identity already holds for $f_n, g_n \in H^2(\mathbb{R}^d_+)$, it follows immediately that it also holds for $f \in H^s_\Delta(\mathbb{R}^d_+), g \in H^{2-s}_\Delta(\mathbb{R}^d_+)$ by these limits. □

Lemma 1.40. The Dirichlet as well as the Neumann trace have the same kernels

$$\ker(\tau_D\restriction_{H^s_\Delta(\mathbb{R}^d_+)}) = \ker(\tau_D\restriction_{H^2(\mathbb{R}^d_+)}) \quad \text{and} \quad \ker(\gamma_N\restriction_{H^s_\Delta(\mathbb{R}^d_+)}) = \ker(\gamma_N\restriction_{H^2(\mathbb{R}^d_+)}) \ ,$$

independent of $s \in [0, 2]$.

Proof. We will only prove the equality for the kernels of the Dirichlet trace. The same proof then also works for the kernels of the Neumann trace. Moreover, by the inclusions

$$H^2(\mathbb{R}^d_+) = H^2_\Delta(\mathbb{R}^d_+) \subseteq H^s_\Delta(\mathbb{R}^d_+) \subseteq H^0_\Delta(\mathbb{R}^d_+) \ ,$$

the prove further reduces to the inclusion $\ker(\tau_D\restriction_{H^0_\Delta(\mathbb{R}^d_+)}) \subseteq \ker(\tau_D\restriction_{H^2(\mathbb{R}^d_+)})$.

Define the self-adjoint Dirichlet Laplace operator

$$A_D f = -\Delta f \quad \text{with} \quad \operatorname{dom}(A_D) = \ker(\tau_D\restriction_{H^2(\mathbb{R}^d_+)}) \ ,$$

and its extension

$$T_{\max} f = -\Delta f \quad \text{with} \quad \operatorname{dom}(T_{\max}) = H^0_\Delta(\mathbb{R}^d_+) \ .$$

We now claim that for any $\lambda \in \rho(A_D)$ the Sobolev space $H^0_\Delta(\mathbb{R}^d_+)$ can be decomposed into the direct sum

$$H^0_\Delta(\mathbb{R}^d_+) = \ker(\tau_D\restriction_{H^2(\mathbb{R}^d)}) \dotplus \ker(T_{\max} - \lambda) \ . \tag{1.31}$$

The inclusion " $\supseteq$ " is clear. For the inverse inclusion " $\subseteq$ " let $f \in H^0_\Delta(\mathbb{R}^d_+)$. Since we chose $\lambda \in \rho(A_D)$, we can use $\operatorname{ran}(A_D - \lambda) = L^2(\mathbb{R}^d_+)$ to ensure the existence of an element $g \in \ker(\tau_D \restriction_{H^2(\mathbb{R}^d)})$ with

$$(A_D - \lambda)g = (T_{\max} - \lambda)f \ .$$

Since A_D is a restriction of $T_{\max}$, we clearly have $f - g \in \ker(T_{\max} - \lambda)$ and the decomposition $f = g + (f - g)$ satisfies (1.31).

For the actual proof let $f \in \ker(\tau_D \restriction_{H^0_\Delta(\mathbb{R}^d_+)})$. Then the above claim (1.31) states the decomposition

$$f = f_0 + f_\lambda \ , \quad \text{for some} \quad f_0 \in \ker(\tau_D \restriction_{H^2(\mathbb{R}^d_+)}),\ f_\lambda \in \ker(T_{\max} - \lambda) \ .$$

Moreover, for every $g \in L^2(\mathbb{R}^d_+)$ we can use Green's second identity (1.30) for the functions f_λ and $(A_D - \bar{\lambda})^{-1}g$ to get

$$\begin{aligned}\langle \Delta f_\lambda, (A_D - \bar{\lambda})^{-1}g\rangle_{L^2(\mathbb{R}^d_+)} - \langle f_\lambda, \Delta(A_D - \bar{\lambda})^{-1}g\rangle_{L^2(\mathbb{R}^d_+)} \\ = \langle \tau_N f_\lambda, \tau_D(A_D - \bar{\lambda})^{-1}g\rangle_{H^{-\frac{3}{2}},H^{\frac{3}{2}}} - \langle \tau_D f_\lambda, \tau_N(A_D - \bar{\lambda})^{-1}g\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}\end{aligned}$$

Since $(A_D - \bar{\lambda})^{-1}g \in \ker(\tau_D \restriction_{H^2(\mathbb{R}^d_+)})$ and $f_\lambda = f - f_0 \in \ker(\tau_D \restriction_{H^0_\Delta(\mathbb{R}^d_+)})$ by assumption, the complete right hand side of this formula vanishes. Also the left hand side can be simplified by using the fact that $-\Delta f_\lambda = \lambda f_\lambda$. Doing all this we end up with the equation

$$\langle f_\lambda, g\rangle_{L^2(\mathbb{R}^d_+)} = 0 \ .$$

Since this holds for every $g \in L^2(\mathbb{R}^d_+)$ we conclude $f_\lambda = 0$ and consequently $f = f_0 \in \ker(\tau_D \restriction_{H^2(\mathbb{R}^d)})$. ☐

Theorem 1.41. For every $s \in [0, 2]$, the Dirichlet trace

$$\tau_D : H^s_\Delta(\mathbb{R}^d_+) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$$

as well as the Neumann trace

$$\tau_N : H^s_\Delta(\mathbb{R}^d_+) \to H^{s-\frac{3}{2}}(\mathbb{R}^{d-1}) \text{ are surjective.}$$

Proof. We will only treat the Dirichlet trace here. The proof of the surjectivity of the Neumann trace then follows the same steps.

First note that by [46, Theorem 9.16], the Dirichlet trace $\tau_D : H^0_\Delta(\mathbb{R}^d_+) \to H^{-\frac{1}{2}}(\mathbb{R}^{d-1})$ is surjective. Moreover, we know that its kernel is given by Lemma 1.40. The decomposition (1.31) then shows that $\tau_D \restriction_{\ker(T_{\max}-\lambda)}$ is bijective and therefore has a bounded inverse

$$R_D : H^{-\frac{1}{2}}(\mathbb{R}^{d-1}) \to H^0_\Delta(\mathbb{R}^d_+) \ . \tag{1.32}$$

By Theorem 1.33 also the restriction $\tau_D : H_{\Delta^2}(\mathbb{R}^d) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$ is surjective and again by the decomposition (1.31), $\tau_D\restriction_{\ker(T_{\max}-\lambda)\cap H_{\Delta^2}(\mathbb{R}^d_+)}$ is bijective. Therefore, it also has a bounded inverse

$$R_D : H^{\frac{3}{2}}(\mathbb{R}^{d-1}) \to H^2_\Delta(\mathbb{R}^d_+) ,$$

which is obviously the restriction of (1.32).

In the same way as we interpolated τ_D at the end of the proof of Theorem 1.37, we can also interpolate R_D to get the boundedness of the restriction

$$R_D : H^{s-\frac{1}{2}}(\mathbb{R}^{d-1}) \to H^s_\Delta(\mathbb{R}^d_+) .$$

From Lemma 1.40 we know that $\tau_D\restriction_{\ker(T_{\max}-\lambda)\cap H^s_\Delta(\mathbb{R}^d_+)}$ is injective and it is obvious that $R_D\restriction_{H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})}$ is its inverse. This in particular shows that $\tau_D\restriction_{H^s_\Delta(\mathbb{R}^d_+)}$ is surjective. □

Theorem 1.42. If we use the notation (1.23) of the restriction of functions to the halfspaces $\mathbb{R}^d_\pm$. Then for every $s \in [0,2]$ the space $H^2(\mathbb{R}^d)$ can be characterized as

$$H^2(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) \,\middle|\, \begin{array}{l} f_\pm \in H^s_\Delta(\mathbb{R}^d_\pm) \\ \tau_D f_+ = \tau_D f_- \\ \tau_N f_+ = -\tau_N f_- \end{array} \right\} .$$

Proof. The inclusion " $\subseteq$ " is clear by Theorem 1.34. For the inverse inclusion " $\supseteq$ " let $f \in L^2(\mathbb{R}^d)$ with the properties

$$f_\pm \in H^s_\Delta(\mathbb{R}^d_\pm) , \quad \tau_D f_+ = \tau_D f_- \quad \text{and} \quad \tau_N f_+ = -\tau_N f_- .$$

We have to show, that Δf exists in $L^2(\mathbb{R}^d)$.

Let $\varphi \in \mathcal{C}^\infty_0(\mathbb{R}^d)$. Then by using Green's second identity, Theorem 1.39 on every halfplane $\mathbb{R}^d_\pm$, we get

$$\begin{aligned}
\langle f, \Delta\varphi\rangle_{L^2(\mathbb{R}^d)} &= \langle f_+, \Delta\varphi_+\rangle_{L^2(\mathbb{R}^d_+)} + \langle f_-, \Delta\varphi_-\rangle_{L^2(\mathbb{R}^d_-)} \\
&= \sum_{i=\pm} \Big(\langle \Delta f_i, \varphi_i\rangle_{L^2(\mathbb{R}^d_i)} + \langle \tau_N f_i, \tau_D\varphi_i\rangle_{H^{s-\frac{3}{2}},H^{\frac{3}{2}-s}} - \langle \tau_D f_i, \tau_N\varphi_i\rangle_{H^{s-\frac{1}{2}},H^{\frac{1}{2}-s}} \Big) \\
&= \sum_{i=\pm} \langle \Delta f_i, \varphi_i\rangle_{L^2(\mathbb{R}^d_i)} + \langle (\tau_N f_+ + \tau_N f_-), \varphi|_\Sigma\rangle_{H^{s-\frac{3}{2}},H^{\frac{3}{2}-s}} \\
&\qquad + \langle (\tau_D f_+ - \tau_D f_-), \partial_{x_d}\varphi|_\Sigma\rangle_{H^{s-\frac{1}{2}},H^{\frac{1}{2}-s}} \\
&= \langle \Delta f_+, \varphi_+\rangle_{L^2(\mathbb{R}^d_+)} + \langle \Delta f_-, \varphi_-\rangle_{L^2(\mathbb{R}^d_-)}
\end{aligned}$$

By the definition of the weak derivative this shows, that

$$\Delta f(x) = \begin{cases} \Delta f_+(x) & \text{if } x \in \mathbb{R}^d_+ \ , \\ \Delta f_-(x) & \text{if } x \in \mathbb{R}^d_- \ , \end{cases} \in L^2(\mathbb{R}^d) \ , \tag{1.33}$$

which finishes the proof. □

2. Semibounded forms

This chapter will give a short introduction into sesquilinear forms. This introduction will be by no means complete, but provide all the tools we will need to work with the Schrödinger form in Chapter 4. In the first section of this chapter we will define symmetric, semibounded and closed forms and its form-induced inner product. Since the closedness of a form is a very important property in our applications, we will derive sufficient conditions for the closedness of the sum of two forms in Section 2.2. In the third section of this chapter, the Theorem 2.11 proves a connection between symetric, semibounded and closed forms and self-adjoint, semibounded operators.

2.1. Symmetric, semibounded and closed forms

We will start this section by the definition of a sesquilinear form.

Definition 2.1. Let V be a Hilbert space and $D_a \subseteq V$ a subspace. A mapping $a : D_a \times D_a \to \mathbb{C}$ is called *form*, if

i) $a(\,\cdot\,, w)$ is linear, $\forall w \in D_a$ and

ii) $a(v, \,\cdot\,)$ is antilinear, $\forall v \in D_a$.

In order to prove the important results of this chapter, these sesquilinear forms have to have certain additional properties. One of which is symmetry.

Definition 2.2. A form $a : D_a \times D_a \to \mathbb{C}$ is called *symmetric*, if

$$a(v, w) = \overline{a(w, v)}\ , \quad \forall v, w \in D_a\ .$$

A symmetric form can furthermore have the property of being semibounded.

Definition 2.3. A symmetric form a is called *semibounded*, if there exists some $\gamma \in \mathbb{R}$, such that

$$a(v, v) \geq \gamma \|v\|^2\ , \quad \forall v \in D_a\ . \tag{2.1}$$

Furthermore, define the *lower bound* $\gamma_a := \sup\{\, \gamma \in \mathbb{R} \mid \gamma \text{ satisfies } (2.1)\,\}$.

Corollary 2.4. For a symmetric, semibounded form a, also the lower bound γ_a satisfies the inequality

$$a(v, v) \geq \gamma_a \|v\|^2\ , \quad \forall v \in D_a\ .$$

For these semibounded forms one can define an inner products, which especially generate a topology on D_a in order to define closed forms in Definition 2.7.

Definition 2.5. For a symmetric, semibounded form a and $\gamma < \gamma_a$, we define the *induced inner product*

$$\langle v, w\rangle_{a,\gamma} \coloneqq a(v, w) - \gamma\langle v, w\rangle \ , \quad \forall v, w \in D_a \ .$$

In the following lemma we will prove that $\langle\cdot,\cdot\rangle_{a,\gamma}$ is indeed an inner product and that all the corresponding norms with different values of γ are equivalent. However, these norms are not equivalent to the original norm of the Hilbert space. Whenever the special value of the norm is not of importance (e.g. in convergences or closures), we will not specify a certain γ, but write $\|\cdot\|_a$ instead.

Lemma 2.6. Let a be a symmetric, semibounded form with lower bound γ_a. Then $\langle\cdot,\cdot\rangle_{a,\gamma}$ is an inner product for every $\gamma < \gamma_a$. Furthermore, for every $v \in D_a$ the induced norm $\|\cdot\|_{a,\gamma}$ satisfies the norm inequality

$$\|v\| \leq \frac{1}{\sqrt{\gamma_a - \gamma}}\|v\|_{a,\gamma} \ , \tag{2.2}$$

as well as for every $\gamma_1, \gamma_2 < \gamma_a$ the norm equivalence

$$\min\left\{1, \sqrt{\frac{\gamma_a - \gamma_2}{\gamma_a - \gamma_1}}\right\}\|v\|_{a,\gamma_1} \leq \|v\|_{a,\gamma_2} \leq \max\left\{1, \sqrt{\frac{\gamma_a - \gamma_2}{\gamma_a - \gamma_1}}\right\}\|v\|_{a,\gamma_1} \ . \tag{2.3}$$

Proof. Linearity in the first and antilinearity in the second argument, as well as the symmetry of $\langle\cdot,\cdot\rangle_\gamma$ follows immediately from the respective properties of $a(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$. Corollary 2.4 gives the inequality

$$\langle v, v\rangle_{a,\gamma} = a(v, v) - \gamma\|v\|^2 \geq (\gamma_a - \gamma)\|v\|^2 \ ,$$

which proves (2.2) and immediately implies the positivity as well as the definiteness of $\langle\cdot,\cdot\rangle_{a,\gamma}$ and makes it an inner product.

In order to prove the equivalence property (2.3), it is enough to show the first inequality. The second one then follows immediately by interchanging $\gamma_1 \leftrightarrow \gamma_2$. Distinguish two different cases.

- $\gamma_1 \geq \gamma_2$: In this case we simply estimate

$$\|v\|^2_{a,\gamma_1} = a(v, v) - \gamma_1\|v\|^2 \leq a(v, v) - \gamma_2\|v\|^2 = \|v\|^2_{a,\gamma_2} \ . \tag{2.4}$$

- $\gamma_1 < \gamma_2$: Choosing $c = \frac{\gamma_2 - \gamma_1}{\gamma_a - \gamma_1} \in (0, 1)$ and using Corollary 2.4 yields the stated inequality

$$\begin{aligned}\|v\|_{a,\gamma_2}^2 &= (1-c)a(v,v) + ca(v,v) - \gamma_2\|v\|^2 \\ &\geq (1-c)a(v,v) + (c\gamma_a - \gamma_2)\|v\|^2 \\ &= \frac{\gamma_a - \gamma_2}{\gamma_a - \gamma_1}\|v\|_{a,\gamma_1}^2 \ . \end{aligned} \tag{2.5}$$

Combining (2.4) & (2.5) then gives the left inequality of (2.3). □

Definition 2.7. A symmetric, semibounded form a is called *closed*, if $(D_a, \langle\cdot,\cdot\rangle_{a,\gamma})$ is a Hilbert space for some $\gamma < \gamma_a$.

Equation (2.3) shows that this property is independent of the choice of $\gamma < \gamma_a$.

2.2. Perturbation of closed forms

While the symmetry and the semiboundedness of a form are quite easy to prove, the verification of the closedness is quite challenging in general. With Proposition 2.8 and especially Proposition 2.10 we will now derive two useful criteria of the closedness of the sum of two forms.

Proposition 2.8. Let a, b be symmetric, semibounded and closed forms. Then also its sum $a + b$ is a symmetric, semibounded and closed form.

Note that the domain of the sum $D_{a+b} = D_a \cap D_b$ is given by the intersection of the single domains.

Proof. It follows directly from the Definitions 2.2 & 2.3 that $a + b$ is symmetric and semibounded with lower bound $\gamma_{a+b} \geq \gamma_a + \gamma_b$. In order to prove the closedness, we fix some $\gamma < \gamma_a + \gamma_b$ and have to show that the domain D_{a+b}, equipped with the inner product $\langle\cdot,\cdot\rangle_{a+b,\gamma}$, is a Hilbert space. To do so, choose

$$\gamma_1 := \frac{\gamma_a - \gamma_b + \gamma}{2} < \gamma_a \quad \text{and} \quad \gamma_2 := \frac{\gamma_b - \gamma_a + \gamma}{2} < \gamma_b \ ,$$

as lower bounds of a and b to use the inner products $\langle\cdot,\cdot\rangle_{a,\gamma_1}$ and $\langle\cdot,\cdot\rangle_{b,\gamma_2}$. By Definition 2.5, we get the equality

$$\|v\|_{a+b,\gamma}^2 = \|v\|_{a,\gamma_1}^2 + \|v\|_{b,\gamma_2}^2 \ , \quad \forall v \in D_{a+b} \ . \tag{2.6}$$

Now let $(v_n)_{n\in\mathbb{N}} \in D_{a+b}$ be a $\|\cdot\|_{a+b,\gamma}$-Cauchy-sequence. Then by (2.6) it is also a $\|\cdot\|_{a,\gamma_1}$- as well as a $\|\cdot\|_{b,\gamma_2}$-Cauchy-sequence and since a and b are assumed to be closed, there exist elements $v_a \in D_a$ and $v_b \in D_b$, such that

$$\lim_{n\to\infty} \|v_n - v_a\|_{a,\gamma_1} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|v_n - v_b\|_{b,\gamma_2} = 0 \ .$$

By the norm inequality (2.2), these two limits also converge in the Hilbert space norm $\|\cdot\|$ and hence the limits v_a and v_b have to coincide and will be denoted by $v_0 \coloneqq v_a = v_b \in D_{a+b}$. Using (2.6) again, gives also the convergence in the $\|\cdot\|_{a+b,\gamma}$-norm

$$\lim_{n\to\infty} \|v_n - v_0\|^2_{a+b,\gamma} = 0 ,$$

which proves the completeness of $(D_{a+b}, \langle\cdot,\cdot\rangle_{a+b,\gamma})$. □

In order to state the second criteria, Proposition 2.10, related to the closedness of forms, we have to introduce the concept of relative boundedness first.

Definition 2.9. Let a, b be symmetric forms with domains $D_a \subseteq D_b$. Then b is called *a-bounded*, if there exist constants $\alpha, \beta \geq 0$, such that

$$|b(v,v)| \leq \alpha\, a(v,v) + \beta\, \|v\|^2 , \quad \forall v \in D_a . \tag{2.7}$$

Furthermore, define the *a-bound of b* by $\gamma_{a,b} = \inf\{\, \alpha \geq 0 \mid \exists \beta \geq 0 \text{ satisfying (2.7)} \,\}$.

Proposition 2.10. Let a, b be symmetric forms with $D_a \subseteq D_b$. Furthermore, let a be semibounded and b shall be a-bounded with bound $\gamma_{a,b} < 1$. Then the sum $a + b$ is symmetric, semibounded and

$$a \text{ is closed if and only if } a + b \text{ is closed.}$$

Proof. First of all, the symmetry of $a + b$ is clear by the symmetry of a and b. By the assumption $D_a \subseteq D_b$ we get $D_{a+b} = D_a$ for the domain of the sum. Furthermore, by the a-boundedness of b with bound $\gamma_{a,b} < 1$, there exist constants $\alpha \in [0, 1), \beta \geq 0$, such that

$$|b(v,v)| \leq \alpha\, a(v,v) + \beta\, \|v\|^2 , \quad \forall v \in D_a .$$

From this inequality we immediately conclude the semiboundedness of $a + b$, with lower bound $\gamma_{a+b} \geq (1-\alpha)\gamma_a - \beta$, by the estimate

$$\begin{aligned}(a+b)(v,v) &\geq a(v,v) - |b(v,v)| \geq \\ &\geq a(v,v) - \alpha\, a(v,v) - \beta\, \|v\|^2 \geq \\ &\geq ((1-\alpha)\gamma_a - \beta)\|v\|^2 .\end{aligned}$$

For the stated equivalence of the closedness of a and $a+b$, it is enough to show that for some $\mu < (1-\alpha)\gamma_a - \beta$ and $\gamma < \gamma_a$ the norms $\|\cdot\|_{a+b,\mu}$ and $\|\cdot\|_{a,\gamma}$ are equivalent. Because in this case $(D_a, \langle\cdot,\cdot\rangle_{a,\gamma})$ is a Hilbert space if and only if $(D_{a+b}, \langle\cdot,\cdot\rangle_{a+b,\mu})$ is a Hilbert space.

For the first estimate of the norm equivalence we use the inequality

$$\begin{aligned}\|v\|^2_{a+b,\mu} &= (a+b)(v,v) - \mu\|v\|^2 \\ &\leq (1+\alpha)a(v,v) + (\beta-\mu)\|v\|^2 \\ &= (1+\alpha)\|v\|^2_{a,\frac{\mu-\beta}{1+\alpha}} \ ,\end{aligned}$$

which is true for every $v \in D_a$. From $\mu < (1-\alpha)\gamma_a - \beta \leq (1+\alpha)\gamma_a + \beta$ we get $\frac{\mu-\beta}{1+\alpha} < \gamma_a$ and together with (2.3), this leads to the inequality

$$\|\cdot\|_{a+b,\mu} \leq (1+\alpha)\max\left\{1, \sqrt{\frac{\gamma_a - \frac{\mu-\beta}{1+\alpha}}{\gamma_a - \gamma}}\right\}\|\cdot\|_{a,\gamma} \ . \tag{2.8}$$

For the inverse inequality we use

$$\begin{aligned}\|v\|^2_{a+b,\mu} &= (a+b)(v,v) - \mu\|v\|^2 \\ &\geq (1-\alpha)a(v,v) - (\beta+\mu)\|v\|^2 \\ &= (1-\alpha)\|v\|^2_{a,\frac{\beta+\mu}{1-\alpha}}\end{aligned}$$

for every $v \in D_a$. From $\mu < (1-\alpha)\gamma_a - \beta$ we get $\frac{\beta+\mu}{1-\alpha} < \gamma_a$ and together with the equivalence (2.3), this leads to the inequality

$$\|\cdot\|_{a+b,\mu} \geq (1-\alpha)\min\left\{1, \sqrt{\frac{\gamma_a - \frac{\beta+\mu}{1-\alpha}}{\gamma_a - \gamma}}\right\}\|\cdot\|_{a,\gamma} \ . \tag{2.9}$$

Equation (2.8) & (2.9) finally ensure the equivalence of the norms $\|\cdot\|_{a,\gamma}$ and $\|\cdot\|_{a+b,\mu}$, which finishes the proof. □

2.3. First representation theorem

Now we come to the main theorem of this chapter, the first representation theorem. It proves a connection between symmetric, semibounded, closed forms and self-adjoint, semibounded operators.

Theorem 2.11. Let a be a symmetric, semibounded and closed form, with a dense domain D_a. Then there exists a unique self-adjoint operator A, satisfying the representation

$$a(w,v) = \langle Aw, v\rangle \ , \quad \forall w \in \mathrm{dom}(A), v \in D_a \ . \tag{2.10}$$

Furthermore the operator A has the following properties:

a) The domain of A has the abstract representation

$$\operatorname{dom}(A) = \{\, w \in D_a \mid \exists v \in V, \text{ such that } a(u,w) = \langle u, v\rangle \text{ for all } u \in D_a \,\} . \tag{2.11}$$

b) A is semibounded from below with the same lower bound as a.

c) The operator A is maximal in the sense that $\overline{\operatorname{dom}(A)}^{\|\cdot\|_a} = D_a$ and every subspace $D \subseteq D_a$ with $\overline{D}^{\|\cdot\|_a} = D_a$ has for every $w \in D_a, u \in V$ the property:

$$\text{If } a(w,v) = \langle u, v\rangle \text{ for every } v \in D \text{ then } w \in \operatorname{dom}(A) \text{ and } Aw = u \,. \tag{2.12}$$

Proof. For the proof fix some $\gamma < \gamma_a$ as a representative of the induced inner product $\langle\cdot,\cdot\rangle_{a,\gamma}$. In the first step we will construct an everywhere defined operator $T : V \to D_a$, with the property

$$\langle u, Tv\rangle_{a,\gamma} = \langle u, v\rangle \ , \qquad \forall u \in D_a, v \in V \ . \tag{2.13}$$

For every $v \in V$ define the functional

$$F_v : D_a \to \mathbb{C} \quad \text{ with } \quad F_v(u) = \langle u, v\rangle \ ,$$

which is obviously linear and also bounded by (2.2). The Riesz representation theorem [70, Theorem V.3.6] states the existence of a unique element $u_v \in D_a$, such that

$$F_v = \langle\, \cdot\, , u_v\rangle_\gamma \ . \tag{2.14}$$

Equation (2.14) is now true for every $v \in V$ and one finds $Tv \coloneqq u_v$ as the claimed operator in (2.13).

Up to a constant shift, the inverse of T shall now be the stated operator A. The fact that T is injective, and we are allowed to invert it, follows directly from its definition (2.13) and the assumption, that D_a is dense in V. The inverse operator operator T^{-1} satisfies

$$\langle u, T^{-1}w\rangle = \langle u, TT^{-1}w\rangle_\gamma = \langle u, w\rangle_\gamma \ , \qquad \forall w \in \operatorname{ran}(T), u \in D_a$$

by definition. If one defines

$$A \coloneqq T^{-1} + \gamma \quad \text{ with } \quad \operatorname{dom}(A) = \operatorname{ran}(T) \ ,$$

this proves the stated representation (2.10)

$$\langle Aw, u\rangle = \langle T^{-1}w + \gamma w, u\rangle = \langle w, u\rangle_\gamma + \gamma\langle w, u\rangle = a(w,u) \ ,$$

for every $w \in \operatorname{dom}(A), u \in D_a$.

We will continue in proving the properties a) - c) of the theorem. Also the self-adjointness will be proven in b).

a) From (2.13) we know that

$$\operatorname{ran}(T) = \{\, w \in D_a \mid \exists v \in V, \text{ such that } \langle u, w\rangle_\gamma = \langle u, v\rangle \text{ for all } u \in D_a \,\} \,.$$

Using that $\operatorname{dom}(A) = \operatorname{ran}(T)$ and that $v \in V$ if and only if $v + \gamma u \in V$ ensures the form (2.11) of $\operatorname{dom}(A)$.

c1) We will only prove the property $\overline{\operatorname{dom}(A)}^{\|\cdot\|_{a,\gamma}} = D_a$ in this first part of point c). It is obvious that $\operatorname{dom}(A) \subseteq D_a$. So, if we denote $^{\perp_{a,\gamma}}$ as the orthogonal complement in the Hilbert space $(D_a, \langle\cdot,\cdot\rangle_{a,\gamma})$, it is left to show that $\operatorname{dom}(A)^{\perp_{a,\gamma}} = \{0\}$. Let $u \in \operatorname{dom}(A)^{\perp_{a,\gamma}}$. Then by $\operatorname{dom}(A) = \operatorname{ran}(T)$ this means that

$$\langle u, Tv\rangle_{a,\gamma} = 0 \;, \quad \forall v \in V$$

and (2.13) then implies $v = 0$.

b) The semiboundedness of A is clear by the inequality

$$\langle Aw, w\rangle = a(w, w) \geq \gamma_a \|w\|^2 \;, \quad \forall w \in \operatorname{dom}(A) \;.$$

In order to prove the equality of the lower bounds, let $\mu \in \mathbb{R}$ be any lower bound of A. Then the inequality

$$a(w, w) = \langle Aw, w\rangle \geq \mu \|w\|^2$$

holds for all $w \in \operatorname{dom}(A) \subseteq D_a$. However, since $\overline{\operatorname{dom}(A)}^{\|\cdot\|_a} = D_a$ is dense by c1), this inequality holds on the whole form domain D_a, which shows that μ is a lower bound of a as well.

The symmetry of A follows trivially from the symmetry of a and (2.10). Once we know the semiboundedness of A, we also know that the chosen $\gamma < \gamma_a$ is in the resolvent set of A. Hence the self-adjointness follows from the surjectivity of $(A - \gamma)$, which is obviously satisfied by

$$\operatorname{ran}(A - \gamma) = \operatorname{ran}(T^{-1} + \gamma - \gamma) = \operatorname{ran}(T^{-1}) = \operatorname{dom}(T) = V \;.$$

c2) In the second part of c) we prove the maximality condition (2.12). Let $D \subseteq D_a$ be a subspace with $\overline{D}^{\|\cdot\|_\gamma} = D_a$. Furthermore let $w \in D_a, u \in V$ elements with

$$a(w, v) = \langle u, v\rangle \;, \quad \forall v \in D \;. \tag{2.15}$$

Because of the density $\overline{D}^{\|\cdot\|_\gamma} = D_a$ and the norm inequality (2.2), the equation (2.15) can be extended to the whole form domain D_a:

$$a(w, v) = \langle u, v\rangle \;, \quad \forall v \in D_a \;. \tag{2.16}$$

This means that (2.16) is especially true on $\operatorname{dom}(A)$ and together with (2.10) one obtains

$$\langle w, Av \rangle = \langle u, v \rangle \ , \quad \forall v \in \operatorname{dom}(A) \ .$$

This shows that

$$w \in \operatorname{dom}(A^*) \quad \text{and} \quad A^* w = u$$

and (2.12) follows immediately by the self-adjointness of A.

We will finish the proof by verifying the uniqueness of A. Let A_1, A_2 be two self-adjoint operators with the stated properties. Because of the self-adjointness it is enough to check just one inclusion $A_2 \subseteq A_1$.

Let $w \in \operatorname{dom}(A_2)$. Then the property (2.10) gives

$$a(w, v) = \langle A_2 w, v \rangle \ , \quad \forall v \in \operatorname{dom}(A_1) \ .$$

Property (2.12), together with the density $\overline{\operatorname{dom}(A_1)}^{\|\cdot\|_\gamma} = D_a$ of the domain, then shows the uniqueness

$$w \in \operatorname{dom}(A_1) \quad \text{and} \quad A_2 w = A_1 w \ .$$

□

3. Quasi boundary triples

Quasi boundary triples are an abstract tool for finding self-adjoint extensions of symmetric operators. The names of all the objects like boundary mapping, Weyl function, abstract Green's identity, come from its counterparts in the application on partial differential equations, where also the origin of this method lies.

We will introduce quasi boundary triples in Definition 3.1 and the corresponding γ-field and Weyl function in Definition 3.3. The main result of this chapter will then be Theorem 3.10, which gives a sufficient condition for an extension to be self-adjoint and satisfying a Krein type resolvent formula in Theorem 3.10.

3.1. Definition and basic properties

Definition 3.1. Let S be a densely defined, symmetric and closed operator in a Hilbert space V and T a closable operator with closure $\overline{T} = S^*$. Furthermore, let $\Gamma_0, \Gamma_1 : \mathrm{dom}(T) \to W$ be linear mappings into another Hilbert space W. Then the triple (W, Γ_0, Γ_1) is called a *quasi boundary triple* for S^*, if it satisfies the following properties:

a) $\overline{\mathrm{ran}} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = W \times W$.

b) $A_0 := T\restriction_{\ker(\Gamma_0)}$ is self-adjoint.

c) The *abstract Green's identity*

$$\langle Tv, \tilde{v}\rangle_V - \langle v, T\tilde{v}\rangle_V = \langle \Gamma_1 v, \Gamma_0 \tilde{v}\rangle_W - \langle \Gamma_0 v, \Gamma_1 \tilde{v}\rangle_W \tag{3.1}$$

holds for every $v, \tilde{v} \in \mathrm{dom}(T)$.

It is not stated explicitly, but it comes up as a direct consequence that also the operator S is a restriction of T.

Lemma 3.2. Let (W, Γ_0, Γ_1) be a quasi boundary triple for S^*. Then $S = T\restriction_{\ker(\Gamma)}$.

Proof. By the Definition 3.1 it is assumed that $\overline{T} = S^*$ or equivalently $S = T^*$. Therefore it is enough to validate $T^* = T\restriction_{\ker(\Gamma)}$.

Let $v \in \operatorname{dom}(T^*)$. Then by using $T \subseteq A_0 = A_0^* \subseteq T^*$ and the abstract Green's identity (3.1) we obtain for every $\tilde{v} \in \operatorname{dom}(T)$ the equation

$$0 = \langle Tv, \tilde{v} \rangle_V - \langle v, T\tilde{v} \rangle_V = \langle \Gamma_1 v, \Gamma_0 \tilde{v} \rangle_W - \langle \Gamma_0 v, \Gamma_1 \tilde{v} \rangle_W \ .$$

From the dense range $\overline{\operatorname{ran}}(\Gamma) = W \times W$, the boundary mappings $\Gamma_0 v = 0$ and $\Gamma_1 v = 0$ have to vanish in the strong sense, which shows that $v \in \ker(\Gamma)$.

For the inverse inclusion, let $v \in \ker(\Gamma)$. Then the abstract Green's identity (3.1) reduces to

$$\langle Tv, \tilde{v} \rangle_V - \langle v, T\tilde{v} \rangle_V = 0 \ , \qquad \forall \tilde{v} \in \operatorname{dom}(T) \ .$$

This in particular shows that $v \in \operatorname{dom}(T^*)$. □

By the definition of $A_0 = T\restriction_{\ker(\Gamma_0)}$ and the fact that $\ker(A_0 - \lambda) = \{0\}$ for every $\lambda \in \rho(A_0)$, the restriction $\Gamma_0\restriction_{\ker(T-\lambda)}$ is injective. This allows us to define the following two main operators corresponding to a quasi boundary triple.

Definition 3.3. Let (W, Γ_0, Γ_1) be a quasi boundary triple for S^*. For every $\lambda \in \rho(A_0)$ define the *γ-field*

$$\gamma(\lambda) \coloneqq (\Gamma_0\restriction_{\ker(T-\lambda)})^{-1} \tag{3.2}$$

and the *Weyl function*

$$M(\lambda) \coloneqq \Gamma_1 \gamma(\lambda) \ . \tag{3.3}$$

As a first step, we will now collect simple functional analytic properties of the γ-field in Lemma 3.4 and of the Weyl function in Lemma 3.5.

Lemma 3.4. Let (W, Γ_0, Γ_1) be a quasi boundary triple for S^* and $\lambda, \mu \in \rho(A_0)$. Then the γ-field (3.2) has the following properties:

a) $\operatorname{dom}(\gamma(\lambda)) = \operatorname{ran}(\Gamma_0)$

b) $\operatorname{ran}(\gamma(\lambda)) = \ker(T - \lambda)$

c) $\ker(\gamma(\lambda)) = \{0\}$

d) $\operatorname{dom}(\gamma(\lambda)^*) = V$ and $\gamma(\lambda)^* = \Gamma_1 (A_0 - \bar{\lambda})^{-1}$

e) $\gamma(\lambda), \gamma(\lambda)^*$ are bounded.

f) $\gamma(\mu) = (1 + (\mu - \lambda)(A_0 - \mu)^{-1})\gamma(\lambda)$

g) $\frac{d}{d\lambda}\gamma(\lambda) w = (A_0 - \lambda)^{-1} w \ , \quad \forall w \in \operatorname{ran}(\Gamma_0)$

Proof. We will prove this lemma by verifying every point a) - g) seperately.

a) Let $v \in \operatorname{dom}(\Gamma_0)$. Then, because of $\lambda \in \rho(A_0)$, we know that $\operatorname{ran}(A_0 - \lambda) = V$ and the representation

$$(T - \lambda) v = (A_0 - \lambda) \tilde{v} \tag{3.4}$$

holds true for some $\tilde{v} \in \operatorname{dom}(A_0)$. Equation (3.4) and $\operatorname{dom}(A_0) = \ker(\Gamma_0)$ then give

$$\Gamma_0 v = \Gamma_0(v - \tilde{v}) \quad \text{and} \quad v - \tilde{v} \in \ker(T - \lambda) . \tag{3.5}$$

Since (3.5) is true for every $v \in \operatorname{dom}(\Gamma_0)$ this implies

$$\operatorname{dom}(\gamma(\lambda)) = \operatorname{ran}(\Gamma_0 \restriction_{\ker(T-\lambda)}) = \operatorname{ran}(\Gamma_0) .$$

b) It follows immediately from the definition of the γ-field (3.2) that

$$\operatorname{ran}(\gamma(\lambda)) = \operatorname{dom}(\Gamma_0 \restriction_{\ker(T-\lambda)}) = \ker(T - \lambda) .$$

c) Since $\gamma(\lambda)$ is defined as the inverse of an injective operator, it has to be injective itself.

d) Let $v \in V$ and choose $\tilde{v} := (A_0 - \bar{\lambda})^{-1} v \in \operatorname{dom}(A_0) = \ker(\Gamma_0)$. Using the abstract Green's identity (3.1), property b) of this lemma and the definition of the γ-field (3.2), we observe for every $w \in \operatorname{dom}(\gamma(\lambda))$ the identity

$$\begin{aligned}
\langle v, \gamma(\lambda) w \rangle_V &= \langle (T - \bar{\lambda})\tilde{v}, \gamma(\lambda) w \rangle_V \\
&= \langle \tilde{v}, T\gamma(\lambda) w \rangle_V + \langle \Gamma_1 \tilde{v}, \Gamma_0 \gamma(\lambda) w \rangle_W - \langle \Gamma_0 \tilde{v}, \Gamma_1 \gamma(\lambda) w \rangle_W - \bar{\lambda} \langle \tilde{v}, \gamma(\lambda) w \rangle_V \\
&= \langle \tilde{v}, \lambda \gamma(\lambda) w \rangle_V + \langle \Gamma_1 \tilde{v}, w \rangle_W - \bar{\lambda} \langle \tilde{v}, \gamma(\lambda) w \rangle_V \\
&= \langle \Gamma_1 \tilde{v}, w \rangle_W .
\end{aligned}$$

This shows that $v \in \operatorname{dom}(\gamma(\lambda)^*)$ and $\gamma(\lambda)^* v = \Gamma_1 \tilde{v} = \Gamma_1 (A_0 - \bar{\lambda})^{-1} v$.

e) From d) it follows that $\gamma(\lambda)^*$ is an everywhere defined operator and the closed graph theorem then shows that $\gamma(\lambda)^*$ is also bounded.

If the adjoint operator $\gamma(\lambda)^*$ is bounded and everywhere defined, it follows from basic functional analysis that the operator $\gamma(\lambda)$ itself has to be bounded as well.

f) First note that by property a) of this lemma, the left and right hand side of the assertion f) have the same domains $\operatorname{dom}(\gamma(\lambda)) = \operatorname{dom}(\gamma(\mu)) = \operatorname{ran}(\Gamma_0)$. In order to show that also the actions of both sides are the same, let $w \in \operatorname{ran}(\Gamma_0)$. Then from property b) we know that

$$\gamma(\lambda) w \in \ker(T - \lambda) \quad \text{and} \quad \gamma(\mu) w \in \ker(T - \mu) . \tag{3.6}$$

Additionally, by the definition of the γ-field (3.2), it follows that

$$\gamma(\lambda) w - \gamma(\mu) w \in \ker(\Gamma_0) = \operatorname{dom}(A_0) . \tag{3.7}$$

Using (3.6) & (3.7) gives the stated formula

$$(A_0 - \mu)(\gamma(\lambda) w - \gamma(\mu) w) = (T - \mu)(\gamma(\lambda) w - \gamma(\mu) w) = (\lambda - \mu)\gamma(\lambda) w .$$

g) The derivative $\frac{d}{d\lambda}\gamma(\lambda)w = (A_0 - \lambda)^{-1}w$ follows directly from the representation f) and the convergence of the resolvent $(A_0 - \lambda)^{-1} = \lim_{\mu\to\lambda}(A_0 - \mu)^{-1}$.

□

Lemma 3.5. Let (W, Γ_0, Γ_1) be a quasi boundary triple for S^* and $\lambda, \mu \in \rho(A_0)$. Then the Weyl function (3.3) has the following properties:

a) $\mathrm{dom}(M(\lambda)) = \mathrm{ran}(\Gamma_0)$

b) $\mathrm{ran}(\Gamma_0) \subseteq \mathrm{dom}(M(\bar{\lambda})^*)$ and $M(\bar{\lambda})^*\restriction_{\mathrm{ran}(\Gamma_0)} = M(\mu) + (\lambda - \mu)\gamma(\bar{\lambda})^*\gamma(\mu)$

c) $\frac{d}{d\lambda}M(\lambda)w = \gamma(\bar{\lambda})^*\gamma(\lambda)w\ , \quad \forall w \in \mathrm{ran}(\Gamma_0)$

Proof.

a) By the definition of the Weyl function (3.3), its domain is given by $\mathrm{dom}(M(\lambda)) = \mathrm{dom}(\gamma(\lambda))$, which equals $\mathrm{ran}(\Gamma_0)$ by Lemma 3.4 a).

b) For every $w, \tilde{w} \in \mathrm{ran}(\Gamma_0)$, Lemma 3.4 b) implies

$$\gamma(\bar{\lambda})\tilde{w} \in \ker(T - \bar{\lambda}) \quad \text{and} \quad \gamma(\mu)w \in \ker(T - \mu)\ .$$

By the definition of the γ-field, the Weyl function and the abstract Green's identity (3.1) we obtain

$$\begin{aligned}\langle M(\mu)w, \tilde{w}\rangle_W - \langle w, M(\bar{\lambda})\tilde{w}\rangle_W &= \langle \Gamma_1\gamma(\mu)w, \Gamma_0\gamma(\bar{\lambda})\tilde{w}\rangle_W - \langle \Gamma_0\gamma(\mu)w, \Gamma_1\gamma(\bar{\lambda})\tilde{w}\rangle_W = \\ &= \langle T\gamma(\mu)w, \gamma(\bar{\lambda})\tilde{w}\rangle_V - \langle \gamma(\mu)w, T\gamma(\bar{\lambda})\tilde{w}\rangle_V = \\ &= \langle \mu\gamma(\mu)w, \gamma(\bar{\lambda})\tilde{w}\rangle_V - \langle \gamma(\mu)w, \bar{\lambda}\gamma(\bar{\lambda})\tilde{w}\rangle_V = \\ &= (\mu - \lambda)\langle \gamma(\bar{\lambda})^*\gamma(\mu)w, \tilde{w}\rangle_V\ .\end{aligned}$$

Since this is true for every $\tilde{w} \in \mathrm{ran}(\Gamma_0)$, this proves that

$$w \in \mathrm{dom}(M(\bar{\lambda})^*) \quad \text{and} \quad M(\bar{\lambda})^*w = M(\mu)w + (\lambda - \mu)\gamma(\bar{\lambda})^*\gamma(\lambda)w\ .$$

c) For every $\mu \in \rho(A_0) \setminus \{\lambda\}$, the identity

$$\frac{M(\mu)w - M(\lambda)w}{\mu - \lambda} = \frac{M(\mu)w - M(\bar{\lambda})^*w}{\mu - \lambda} = \gamma(\bar{\lambda})^*\gamma(\mu)w \tag{3.8}$$

holds true by point b) for every $w \in \mathrm{ran}(\Gamma_0)$. Moreover, Lemma 3.4 g) yields the differentiability and hence the continuity of $\gamma(\lambda)$. Using also the boundedness of $\gamma(\bar{\lambda})^*$, the right hand side of (3.8) converges and the derivative

$$\frac{d}{d\lambda}M(\lambda)w = \gamma(\bar{\lambda})^*\gamma(\lambda)w \quad \text{exists.}$$

□

In the next lemma we will derive more properties of the Weyl function in the special case, where λ is real-valued.

Lemma 3.6. Let (W, Γ_0, Γ_1) be a quasi boundary triple for S^*. Then for every $\lambda \in \rho(A_0) \cap \mathbb{R}$ the Weyl function $M(\lambda)$ is a symmetric and the function $\lambda \mapsto M(\lambda)$ is monotone increasing in the sense that for every $\lambda_1 \leq \lambda_2 \in \rho(A_0) \cap \mathbb{R}$ the quadratic form fulfils

$$\langle M(\lambda_1)w, w\rangle_W \leq \langle M(\lambda_2)w, w\rangle_W \ , \quad \forall w \in \mathrm{ran}(\Gamma_0) \ .$$

Proof. The symmetry of $M(\lambda)$ follows from Lemma 3.5 b) with $\mu = \lambda = \bar{\lambda}$ and the monotonicity from its non-negative derivative

$$\frac{d}{d\lambda}\langle M(\lambda)w, w\rangle_W = \langle \gamma_\lambda^* \gamma(\lambda)w, w\rangle = \|\gamma(\lambda)w\|^2 \geq 0 \ , \quad \forall w \in \mathrm{dom}(M(\lambda)) \ ,$$

which was calculated in Lemma 3.5 c). □

Usually the conditions a) - c) of Definition 3.1 are easy to verify, but the assumption that one is in the setting of two operators S and T, satisfying $\overline{T} = S^*$ may be difficult to check. For this reason, the following theorem constructs to a given operator T and boundary mappings Γ_0, Γ_1 a densely defined, symmetric and closed operator S, which satisfies $\overline{T} = S^*$ and makes (W, Γ_0, Γ_1) a quasi boundary triple of S^*.

Theorem 3.7. Let T be an operator in V and $\Gamma = (\Gamma_0, \Gamma_1)^T : \mathrm{dom}(T) \to W \times W$ a linear mapping with the following properties:

a) There exists a self-adjoint $A_0 \subseteq T\restriction_{\ker(\Gamma_0)}$.

b) $\overline{\mathrm{ran}}(\Gamma) = W \times W$

c) $\overline{\ker}(\Gamma) = V$

d) Γ_0, Γ_1 satisfy the abstract Green's identity (3.1).

Then $S := T\restriction_{\ker(\Gamma)}$ is a densely defined, closed, symmetric operator with $\overline{T} = S^*$ and (W, Γ_0, Γ_1) is a quasi boundary triple for S^*.

Proof. The proof mainly consists two statements, from which one can then conclude all the assertions.

Statement 1: $A_0 = T\restriction_{\ker(\Gamma_0)}$

The abstract Green's identity (3.1) gives

$$\langle Tv_1, v_2\rangle_V - \langle v_1, Tv_2\rangle_V = 0 \ , \quad \forall v_1, v_2 \in \ker(\Gamma_0) \ ,$$

which shows that $T\restriction_{\ker(\Gamma_0)}$ is a symmetric extension of A_0 and so they have to coincide.

Statement 2: $T\upharpoonright_{\ker(\Gamma)} = T^*$

Let $v \in \ker(\Gamma)$. Then the abstract Green's identity (3.1) reduces to

$$\langle Tv, \tilde{v}\rangle_V - \langle v, T\tilde{v}\rangle_V = 0 \;, \quad \forall \tilde{v} \in \mathrm{dom}(T) \;.$$

This shows that $v \in \mathrm{dom}(T^*)$ and $T^*v = Tv$. Since this is true for every $v \in \ker(\Gamma)$, this verifies the operator inclusion $T\upharpoonright_{\ker(\Gamma)} \subseteq T^*$.

For the inverse inclusion, notice first that, because of $A_0 \subseteq T$ and a), we obtain the operator inclusion

$$T^* \subseteq A_0^* = A_0 = T\upharpoonright_{\ker(\Gamma_0)} \;.$$

Now, for every $v \in \mathrm{dom}(T^*)$, we can use this in the abstract Green's identity (3.1) to conclude

$$0 = \langle Tv, \tilde{v}\rangle_V - \langle v, T\tilde{v}\rangle_V = \langle \Gamma_1 v, \Gamma_0 \tilde{v}\rangle_W \;, \quad \forall \tilde{v} \in \mathrm{dom}(T) \;.$$

By the density $\overline{\mathrm{ran}}(\Gamma_0) = W$, assumed in b), also $\Gamma_1 v = 0$ has to vanish and it follows that $v \in \ker(\Gamma)$. This proves the second inclusion $T^* \subseteq T\upharpoonright_{\ker(\Gamma_0)}$ and therefore the whole Statement 2.

From Statement 2 it follows that $T\upharpoonright_{\ker(\Gamma)}$ is a closed operator and also symmetric because of

$$T\upharpoonright_{\ker(\Gamma)} = T^* \subseteq (T\upharpoonright_{\ker(\Gamma)})^* \;.$$

Furthermore,

$$\overline{T} = (T^*)^* = (T|_{\ker(\Gamma)})^* = S^*$$

follows immediately. Statement 1 then finally shows that $T\upharpoonright_{\ker(\Gamma_0)}$ is a self-adjoint operator and hence (W, Γ_0, Γ_1) a quasi boundary triple for S^*. □

3.2. Self-adjoint extensions

Until this point, A_0 was the only self-adjoint operator we have seen. But with the help of the boundary mappings Γ_0, Γ_1 and operators B in the boundary space W, we are able to construct extensions A_B, which, under additional assumptions on the operator B, will turn out to be self-adjoint.

Definition 3.8. Let (W, Γ_0, Γ_1) be a quasi boundary triple for S^*. Then for any operator B in W we define the operator

$$A_B := T\upharpoonright_{\ker(\Gamma_0 - B\Gamma_1)} \;. \tag{3.9}$$

Lemma 3.9. Let (W, Γ_0, Γ_1) be a quasi boundary triple for S^*. For any operator B

in W the corresponding operator A_B satisfies the operator inclusion

$$S \subseteq A_B \subseteq S^* \,. \tag{3.10}$$

Moreover, for a symmetric operator B also A_B is symmetric.

Proof. It follows from Lemma 3.2 and Definition 3.8 that $S = T\restriction_{\ker(\Gamma)} \subseteq A_B$. The second inclusion follows trivially from $A_B \subseteq T \subseteq S^*$.

For the validation of the symmetry of A_B, let $v_1, v_2 \in \operatorname{dom}(A_B)$, which by (3.9) particularly means that

$$\Gamma_1 v_i \in \operatorname{dom}(B) \quad \text{and} \quad \Gamma_0 v_i = B\Gamma_1 v_i \,, \qquad \forall i \in \{1,2\} \,.$$

Using this in the abstract Green's identity (3.1) gives

$$\begin{aligned}\langle A_B v_1, v_2\rangle_V - \langle v_1, A_B v_2\rangle_V &= \langle \Gamma_1 v_1, \Gamma_0 v_2\rangle_W - \langle \Gamma_0 v_1, \Gamma_1 v_2\rangle_W = \\ &= \langle \Gamma_1 v_1, B\Gamma_1 v_1\rangle_W - \langle B\Gamma_1 v_1, \Gamma_1 v_2\rangle_W = 0 \,,\end{aligned}$$

where the right hand side vanishes, because of the symmetry of B. Because this is true for every $v_1, v_2 \in \operatorname{dom}(A_B)$, this proves the symmetry of A_B. □

The previous Lemma 3.9 is not completely satisfying, in the sense that one wants the operator A_B to be self-adjoint and not only symmetric. However, the simple conclusion that self-adjointenss of B implies self-adjointness of A_B is not true in general. The following Theorem 3.10 shows that additional properties are necessary, and also gives a Krein type formula (3.11) for the resolvents of A_0 and A_B. A version of this theorem without splitting B into B_1 and B_2 can be found in [16, Theorem 2.6]. However, we will present here the more general case, in foresight that we want to separate our potential strength α later in Theorem 5.8 into the parts $\operatorname{sign}(\alpha)|\alpha|^{\frac{1}{3}}$ and $|\alpha|^{\frac{2}{3}}$ to get better integrability properties.

Theorem 3.10. Let (W, Γ_0, Γ_1) be a quasi boundary triple for S^*. Furthermore, let $\lambda_0 \in \rho(A_0) \cap \mathbb{R}$ and B, B_1, B_2 be operators in W with B symmetric, $B \subseteq B_1 B_2$ and the following properties:

a) $1 \in \rho\left(B_2\overline{M(\lambda_0)B_1}\right)$

b) $\operatorname{ran}\left(B_2\overline{M(\lambda_0)B_1}\right) \subseteq \operatorname{ran}(\Gamma_0) \cap \operatorname{dom}(B_1)$

c) $\operatorname{ran}\left(B_1\restriction_{\operatorname{ran}(\Gamma_0)}\right) \subseteq \operatorname{ran}(\Gamma_0)$

d) $\operatorname{ran}\left(B_2\restriction_{\operatorname{ran}(\Gamma_1)}\right) \subseteq \operatorname{ran}(\Gamma_0) \cap \operatorname{dom}(B_1)$

e) $\operatorname{ran}(\Gamma_1) \subseteq \operatorname{dom}(B)$

Then the boundary operator A_B is self-adjoint and for every $\lambda \in \rho(A_0) \cap \rho(A_B)$ the Krein type resolvent formula

$$(A_B - \lambda)^{-1} - (A_0 - \lambda)^{-1} = \gamma(\lambda) B_1 (1 - B_2 M(\lambda) B_1)^{-1} B_2 \gamma(\bar{\lambda})^* \tag{3.11}$$

holds true.

Remark 3.11. If one already knows that A_B is self-adjoint and only wants to derive (3.11), one only needs the assumptions c) - e).

If we choose $B_1 = I$ being the identity operator and $B_2 = B$ we regain **??**Theorem 2.6]BeLaLoRo2017, which shows that Theorem 3.10 is indeed a generalisation.

Corollary 3.12. Let (W, Γ_0, Γ_1) be a quasi boundary triple for S^*. Furthermore, let $\lambda_0 \in \rho(A_0) \cap \mathbb{R}$ and B be a symmetric operator in B with the following properties:

a) $1 \in \rho\left(B\overline{M(\lambda_0)}\right)$
b) $\operatorname{ran}\left(B\overline{M(\lambda_0)}\right) \subseteq \operatorname{ran}(\Gamma_0)$
c) $\operatorname{ran}\left(B\restriction_{\operatorname{ran}(\Gamma_1)}\right) \subseteq \operatorname{ran}(\Gamma_0)$
d) $\operatorname{ran}(\Gamma_1) \subseteq \operatorname{dom}(B)$

Then the boundary operator A_B is self-adjoint and for every $\lambda \in \rho(A_0) \cap \rho(A_B)$ the Krein type resolvent formula

$$(A_B - \lambda)^{-1} - (A_0 - \lambda)^{-1} = \gamma(\lambda)(1 - BM(\lambda))^{-1} B \gamma(\bar{\lambda})^*$$

holds true.

Proof of Theorem 3.10. The proof is splitted into four steps.

Step 1: In the first step we claim that

$$\operatorname{ran}(B_2 \gamma(\lambda_0)^*) \subseteq \operatorname{ran}(1 - B_2 M(\lambda_0) B_1) , \tag{3.12}$$

where it is important that the inclusion holds without the closure of $M(\lambda_0)B_1$ on the right hand side.

Let $w \in \operatorname{ran}(B_2\gamma(\lambda_0)^*)$. Then by assumption d) we also get

$$w \in \operatorname{ran}(B_2\restriction_{\operatorname{ran}(\Gamma_1)}) \subseteq \operatorname{ran}(\Gamma_0) \cap \operatorname{dom}(B_1) . \tag{3.13}$$

By assumption a), the element $\tilde{w} := \left(1 - B_2\overline{M(\lambda_0)B_1}\right)^{-1} w$ is well-defined and admits the implicit representation

$$\tilde{w} - w = B_2 \overline{M(\lambda_0) B_1} \tilde{w} \in \operatorname{ran}(\Gamma_0) \cap \operatorname{dom}(B_1) , \tag{3.14}$$

by assumption b). Combining (3.13) & (3.14) implies that also $\tilde{w} \in \operatorname{ran}(\Gamma_0) \cap \operatorname{dom}(B_1)$, and by assumption c) that $B_1\tilde{w} \in \operatorname{ran}(\Gamma_0) = \operatorname{dom}(M(\lambda_0))$. Therefore (3.14) can be written as

$$\tilde{w} - w = B_2 M_{\lambda_0} B_1 \tilde{w} \ ,$$

which finally confirms the claim (3.12).

Step 2: In the second step we will prove the self-adjointness of A_B. The essential point is the claim

$$\operatorname{ran}(A_B - \lambda_0) = V \ . \tag{3.15}$$

Let $v \in V$. Then, by assumption e) and $B \subseteq B_1 B_2$, the element $B_2\gamma(\lambda_0)^* v$ is well-defined and Step 1 (3.12) gives the existence of an element $w \in \operatorname{dom}(B_2 M(\lambda_0) B_1)$, such that

$$B_2\gamma(\lambda_0)^* v = (1 - B_2 M(\lambda_0) B_1) w \ . \tag{3.16}$$

Defining now the element

$$\tilde{v} \coloneqq (A_0 - \lambda_0)^{-1} v + \gamma(\lambda_0) B_1 w \ , \tag{3.17}$$

one sees that $(T - \lambda_0)\tilde{v} = v$, since $\gamma(\lambda_0) B_1 w \in \ker(T - \lambda_0)$. In order to prove (3.15) it is enough to show that $\tilde{v} \in \operatorname{dom}(A_B)$. To do so, we apply the boundary mappings Γ_0, Γ_1 onto it and end up with the equations

$$\Gamma_0 \tilde{v} = B_1 w \quad \text{and} \quad \Gamma_1 \tilde{v} = \gamma(\lambda_0)^* v + M(\lambda_0) B_1 w \ , \tag{3.18}$$

where we used the definitions of γ-field and Weyl function as well as their mapping properties, collected in Lemma 3.4. Applying B to the second equation of (3.18), allowed by assumption e), and using the definition of w in (3.16) as well as $B \subseteq B_1 B_2$ gives

$$B\Gamma_1 \tilde{v} = B_1 B_2 (\gamma(\lambda_0)^* v + M(\lambda_0) B_1 w) = B_1 w = \Gamma_0 \tilde{v} \ ,$$

where the last equality follows from the first equation of (3.18). Therefore we obtain $\tilde{v} \in \operatorname{dom}(A_B)$, which finishes the claim (3.15)

Let now $\lambda \in \rho(A_0) \cap \rho(A_B)$ and it remains to prove that the Krein type formula (3.11) holds true.

Step 3: prepares the proof (3.11), as it claims that

$$(1 - B_2 M(\lambda) B_1) \text{ is injective.} \tag{3.19}$$

Let $w \in \ker(1 - B_2 M(\lambda) B_1)$. Then, by the assumptions c) & d), we get

$$B_1 w = B_1 B_2 M(\lambda) B_1 w \in \operatorname{ran}(B_1 \restriction_{\operatorname{ran}(\Gamma_0)}) \subseteq \operatorname{ran}(\Gamma_0) \ .$$

This means that by assumption e), $B\Gamma_1\gamma(\lambda)$ can be applied to $B_1 w$, which gives the identity

$$B\Gamma_1\gamma(\lambda)B_1 w = BM(\lambda)B_1 w = B_1 w = \Gamma_0\gamma(\lambda)B_1 w \ .$$

This shows that $\gamma(\lambda)B_1 w \in \mathrm{dom}(A_B)$ by definition and since $\mathrm{ran}(\gamma(\lambda)) = \ker(T-\lambda)$ we get

$$\gamma(\lambda)B_1 w \in \ker(A_B - \lambda) = \{0\} \ ,$$

because of $\lambda \in \rho(A_B)$. Hence $w = B_2\Gamma_1\gamma(\lambda)B_1 w = 0$ as well, which proves the staed injectivity (3.19).

Step 4: In the last step show that $\mathrm{ran}(B_2\gamma(\bar{\lambda})^*) \subseteq \mathrm{ran}(1 - B_2 M(\lambda)B_1)$ and (3.11) holds true.

For $v \in V$ define the elements

$$v_B := (A_B - \lambda)^{-1} v \quad \text{and} \quad v_0 := (A_0 - \lambda)^{-1} v \ .$$

Then especially $v_B - v_0 \in \ker(T-\lambda)$ and hence

$$\gamma(\lambda)\Gamma_0(v_B - v_0) = v_B - v_0 \ , \tag{3.20}$$

by the definition of the γ-field. Furthermore, Lemma 3.4 and $v_B \in \mathrm{dom}(A_B)$ gives

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} v_0 = \begin{pmatrix} 0 \\ \gamma(\bar{\lambda})^* v \end{pmatrix} \quad \text{and} \quad \Gamma_0 v_B = B\Gamma_1 v_B \ . \tag{3.21}$$

The element $B_2 M(\lambda)B_1 B_2\Gamma_1 v_B$ is well-defined by the assumptions c), d) & e) and with the equations (3.20) & (3.21) we obtain the identity

$$\begin{aligned}(1 - B_2 M(\lambda)B_1)B_2\Gamma_1 v_B &= B_2\Gamma_1 v_B - B_2 M(\lambda)\Gamma_0 v_B \\ &= B_2\Gamma_1 v_B - B_2 M(\lambda)\Gamma_0(v_B - v_0) \\ &= B_2\Gamma_1 v_B - B_2\Gamma_1(v_B - v_0) \\ &= B_2\gamma(\bar{\lambda})^* v \ .\end{aligned}$$

By the injectivity (3.19) we then obtain

$$B_2\Gamma_1 v_B = (1 - B_2 M(\lambda)B_1)^{-1} B_2\gamma(\bar{\lambda})^* v \ .$$

Using again $\Gamma_0(v_B - v_0) = B\Gamma_1 v_B$ from (3.21), finally leads to

$$v_B - v_0 = \gamma(\lambda)B_1(1 - B_2 M(\lambda)B_1)^{-1} B_2\gamma(\bar{\lambda})^* v \ ,$$

which is the stated Krein type formula (3.11). □

4. Schrödinger operator and corresponding form

In this chapter, we want to give the formal differential expression (0.1) from the introduction a rigorous mathematical meaning via the sesquilinear form a_α. We will restrict ourselves to the case where $\Sigma \subseteq \mathbb{R}^d$ is the hyperplane

$$\Sigma = \left\{ x \in \mathbb{R}^d \;\middle|\; x_d = 0 \right\} \cong \mathbb{R}^{d-1} . \tag{4.1}$$

In Lemma 4.2 we prove that the form a_α is symmetric, semibounded and closed. The abstract Theorem 2.11 will then give a corresponding Schrödinger operator A_α in Definition 4.3. Beyond this application of the abstract theory of sesquilinear forms we derive explicit forms of the domain, action and essential spectrum of A_α in Proposition 4.4 and Theorem 4.5.

4.1. Schrödinger operator defined by a form

Definition 4.1. Let $d \geq 3$ and $\alpha \in L^{d-1}(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ be real-valued. Then define the sesquilinear form a_α by

$$a_\alpha(f,g) = \langle \nabla f, \nabla g \rangle_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^{d-1}} \alpha \, \tau_D f \, \overline{\tau_D g} \, d\tilde{x} \quad \text{with} \quad D_{a_\alpha} = H^1(\mathbb{R}^d) . \tag{4.2}$$

Here τ_D denotes the Dirichlet trace operator from Lemma 1.30.

By the trace theorem we get $\tau_D f \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ and Lemma A.1 then shows that $|\alpha|^{\frac{1}{2}} \tau_D f \in L^2(\mathbb{R}^{d-1})$, which confirms the well-definedness of the integral.

The next lemma basically verifies the assumptions of the Theorem 2.11.

Lemma 4.2. Let $d \geq 3$ and $\alpha \in L^{d-1}(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ be real-valued. Then the form a_α is symmetric, semibounded and closed.

For the definition of symmetry, semiboundedness and closedness of a form, see the Definitions 2.2, 2.3 & 2.7 in Chapter 2.

Proof. In this proof we want to look at a_α as the sum of the two forms

$$a_0(f,g) := \langle \nabla f, \nabla g \rangle_{L^2(\mathbb{R}^d)} \qquad \text{with} \quad D_{a_0} = H^1(\mathbb{R}^d) \quad \text{and}$$
$$a_1(f,g) := \int_{\mathbb{R}^{d-1}} \alpha \, \tau_D f \, \overline{\tau_D g} \, d\tilde{x} \quad \text{with} \quad D_{a_1} = H^1(\mathbb{R}^d) \, .$$

From the real-valued potential strength α we immediately obtain the symmetry of a_0 and a_1. Also the semiboundedness of a_0 (with lower bound $\gamma_{a_0} = 0$) follows immediately from its definition. One possible induced inner product of a_0 is $\langle \cdot, \cdot \rangle_{H^1(\mathbb{R}^d)}$ and since $D_{a_0} = H^1(\mathbb{R}^d)$ equipped with this norm is a Hilbert space, a_0 is also closed.

To conclude now the symmetry, semiboundedness and closedness of a_α, we want to use Proposition 2.10. So all we have to check is the a_0-boundedness with bound $\gamma_{a_0,a_1} < 1$. First decompose the potential

$$\alpha = u + v \quad \text{for some} \quad u \in L^{d-1}(\mathbb{R}^{d-1}), \; v \in L^\infty(\mathbb{R}^{d-1}) \, .$$

The bounded part v of the potential can simply be estimated by the L^∞-norm later on. But to deal with the unbounded u, the strategy will be, to cut it off at some bound b, such that the bounded part can again be estimated by the L^∞-norm and the unbounded remainder becomes arbitrary small. For this purpose define the sets

$$\Sigma_b = \left\{ \tilde{x} \in \mathbb{R}^{d-1} \;\middle|\; |u(\tilde{x})| \leq b \right\} \, , \quad \forall b \geq 0 \, .$$

Because of the integrability $\|u\|_{L^{d-1}(\mathbb{R}^{d-1})} < \infty$, there exists a bound $b_\varepsilon \geq 0$ for every $\varepsilon > 0$, such that the integral over the unbounded part of u becomes

$$\|u\|_{L^{d-1}(\mathbb{R}^{d-1} \setminus \Sigma_{b_\varepsilon})} \leq \frac{\varepsilon}{2 c_E^2 c_{D,1}^2} \, , \tag{4.3}$$

where $c_E = c_{\frac{1}{2}, \frac{2(d-1)}{d-2}}$ is the constant of Corollary 1.9 and $c_{D,1} = \|\tau_D\restriction_{H^1(\mathbb{R}^d)}\|$ is the operator norm of the trace operator on $H^1(\mathbb{R}^d)$. This means that these constants appear in the inequalities

$$\|\tau_D f\|_{L^{\frac{2(d-1)}{d-2}}(\mathbb{R}^{d-1})} \leq c_E \|\tau_D f\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})} \quad \text{and} \quad \|\tau_D f\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})} \leq c_{D,1} \|f\|_{H^1(\mathbb{R}^d)} \tag{4.4}$$

for every $f \in H^1(\mathbb{R}^d)$. If one splits now $\alpha = (v + u\,\chi_{\Sigma_{b_\varepsilon}}) + u\,\chi_{\mathbb{R}^{d-1}\setminus\Sigma_{b_\varepsilon}}$ into a bounded part and an unbounded remainder, we can also split the form a_1 into

$$|a_1(f,f)| \leq \int_{\mathbb{R}^{d-1}} |v + u\,\chi_{\Sigma_{b_\varepsilon}}| \, |\tau_D f|^2 d\tilde{x} + \int_{\mathbb{R}^{d-1}\setminus\Sigma_{b_\varepsilon}} |u| \, |\tau_D f|^2 d\tilde{x} \, , \tag{4.5}$$

and can estimate both integrals separately.

- Since the potential in the first integral is bounded, it can simply be estimated by its upper bound

$$\int_{\mathbb{R}^{d-1}} |v + u\ \chi_{\Sigma_{b_\varepsilon}}|\ |\tau_D f|^2 d\tilde{x} \leq (\|v\|_\infty + b_\varepsilon)\|\tau_D f\|^2_{L^2(\mathbb{R}^{d-1})}\ . \tag{4.6}$$

  Using Lemma 1.30 for some arbitrary $s \in (\frac{1}{2}, 1)$ we can estimate

$$\|\tau_D f\|_{L^2(\mathbb{R}^{d-1})} \leq c_{D,s}\|f\|_{H^s(\mathbb{R}^d)}\ , \tag{4.7}$$

  where $c_{D,s} = \|\tau_D \restriction_{H^s(\mathbb{R}^d)}\|$ is the operator norm of the Dirichlet trace on $H^s(\mathbb{R}^d)$. With Theorem 1.27 we find a constant $c_\varepsilon > 0$, such that

$$\|f\|^2_{H^s(\mathbb{R}^d)} \leq \frac{1}{(\|v\|_\infty + b_\varepsilon)c^2_{D,s}} \left(\frac{\varepsilon}{2}\ \|f\|^2_{H^1(\mathbb{R}^d)} + c_\varepsilon\|f\|_{L^2(\mathbb{R}^d)}\right)\ . \tag{4.8}$$

  Using now (4.7) & (4.8) in (4.6), gives the final estimate of the first integral

$$\int_{\mathbb{R}^{d-1}} |v + u\ \chi_{\Sigma_{b_\varepsilon}}|\ |\tau_D f|^2 d\tilde{x} \leq \frac{\varepsilon}{2}\ \|f\|^2_{H^1(\mathbb{R}^d)} + c_\varepsilon\|f\|^2_{L^2(\mathbb{R}^d)}\ . \tag{4.9}$$

- For the estimate of the second integral in (4.5) we first use Hölder's inequality in combination with (4.3) to obtain

$$\int_{\mathbb{R}^{d-1}\setminus\Sigma_{b_\varepsilon}} |u|\ |\tau_D f|^2 d\tilde{x} \leq \frac{\varepsilon}{2c_E^2 c_{D,1}^2}\|\tau_D f\|^2_{L^{\frac{2(d-1)}{d-2}}(\mathbb{R}^{d-1})}\ .$$

  The inequalities (4.4) then give the final estimate of the second integral

$$\int_{\mathbb{R}^{d-1}\setminus\Sigma_{b_\varepsilon}} |u|\ |\tau_D f|^2 d\tilde{x} \leq \frac{\varepsilon}{2}\ \|f\|^2_{H^1(\mathbb{R}^d)}\ . \tag{4.10}$$

Combining (4.9) & (4.10) gives the estimate

$$|a_1(f,f)| \leq \varepsilon\ \|f\|^2_{H^1(\mathbb{R}^d)} + c_\varepsilon\|f\|^2_{L^2(\mathbb{R}^d)} = \varepsilon\ a_0(f,f) + (c_\varepsilon + \varepsilon)\|f\|^2_{L^2(\mathbb{R}^d)} \tag{4.11}$$

which confirms the a_0-boundedness of a_1. Since (4.11) is true for every $\varepsilon > 0$, the bound is given by $\gamma_{a_0,a_1} = 0$. □

The previous Lemma 4.2 showed that the form a_α fulfils the assumptions of Theorem 2.11 and there exists a unique self-adjoint and semibounded operator A_α which represents the form a_α.

Definition 4.3. Let $d \geq 3$ and $\alpha \in L^{d-1}(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ be real-valued. Then define the *Schrödinger operator* A_α by

$$\mathrm{dom}(A_\alpha) = \left\{ f \in H^1(\mathbb{R}^d) \;\middle|\; \exists h \in L^2(\mathbb{R}^d) : a_\alpha(f,g) = \langle h, g\rangle_{L^2(\mathbb{R}^d)},\ \forall g \in H^1(\mathbb{R}^d) \right\} \tag{4.12}$$

and

$$a_\alpha(f,g) = \langle A_\alpha f, g\rangle_{L^2(\mathbb{R}^d)}\ , \quad \forall f \in \mathrm{dom}(A_\alpha), g \in H^1(\mathbb{R}^d)\ . \tag{4.13}$$

4.2. Further properties of the Schrödinger operator

While in Section 4.1 the Schrödinger operator A_α was defined in an abstract way, its precise action, a more explicit representation of the domain via boundary values and the essential spectrum $\sigma_{\mathrm{ess}}(A_\alpha)$ will be calculated in the rest of this Section 4.2.

Proposition 4.4. Let $d \geq 3$ and $\alpha \in L^{d-1}(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ be real-valued. Then the Schrödinger operator A_α has the domain

$$\mathrm{dom}(A_\alpha) = \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \begin{array}{l} f_\pm \in H^1_\Delta(\mathbb{R}^d_\pm) \\ \tau_D f_+ = \tau_D f_- \\ \tau_N f_+ + \tau_N f_- = -\alpha \tau_D f \end{array} \right\} \tag{4.14}$$

and the action

$$A_\alpha f = (-\Delta f_+) \oplus (-\Delta f_-)\ , \quad \forall f \in \mathrm{dom}(A_\alpha)\ . \tag{4.15}$$

Note, that we used the notation $f_\pm = f|_{\mathbb{R}^d_\pm}$ for the restriction of functions to the halfspace, from (1.23).

Proof. Let us start the proof with the inclusion " $\subseteq$ " of (4.14). Let $f \in \mathrm{dom}(A_\alpha)$. Then, by (4.12), f is in $H^1(\mathbb{R}^d)$ and there exists a function $h \in L^2(\mathbb{R}^d)$, such that

$$\langle \nabla f, \nabla g\rangle_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^{d-1}} \alpha\ \tau_D f\ \overline{\tau_D g}\ d\tilde{x} = \langle h, g\rangle_{L^2(\mathbb{R}^d)}\ , \quad \forall g \in H^1(\mathbb{R}^d)\ . \tag{4.16}$$

By Theorem 1.34 we immediately get $\tau_D f_+ = \tau_D f_-$. Equation (4.16) especially holds true for all $g \in \mathcal{C}_0^\infty(\mathbb{R}^d_+)$, for which it has the simple form

$$\int_{\mathbb{R}^d_+} \nabla f\ \overline{\nabla g}\ dx = \int_{\mathbb{R}^d_+} h\ \overline{g}\ dx\ .$$

Partial integration of the left hand side leads to

$$-\int_{\mathbb{R}^d_+} f\ \overline{\Delta g}\ dx = \int_{\mathbb{R}^d_+} h\ \overline{g}\ dx\ , \quad \forall g \in \mathcal{C}_0^\infty(\mathbb{R}^d_+)\ ,$$

which is the definition of the weak Laplacian $-\Delta f_+ = h_+ \in L^2(\mathbb{R}^d)$. Analogously we obtain $-\Delta f_- = h_- \in L^2(\mathbb{R}^d)$ on the negative halfplane. Consequently, it follows that

$$f_\pm \in H^1_\Delta(\mathbb{R}^d_\pm) \quad \text{and} \quad A_\alpha f = h = (-\Delta f_+) \oplus (-\Delta f_-) \ .$$

Knowing $h = (-\Delta f_+) \oplus (-\Delta f_-)$, we use again (4.16) to obtain for every $g \in H^1(\mathbb{R}^d)$ the equation

$$\langle \nabla f, \nabla g \rangle_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^{d-1}} \alpha\, \tau_D f\, \overline{\tau_D g}\, d\tilde{x} = -\langle \Delta f_+, g_+ \rangle_{L^2(\mathbb{R}^d_+)} - \langle \Delta f_-, g_- \rangle_{L^2(\mathbb{R}^d_-)} \ .$$

Partial integration on $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$ of the right hand side admits the identity

$$\int_{\mathbb{R}^{d-1}} \alpha\, \tau_D f\, \overline{\tau_D g}\, d\tilde{x} = -\int_{\mathbb{R}^{d-1}} \tau_N f_+\, \overline{\tau_D g_+}\, d\tilde{x} - \int_{\mathbb{R}^{d-1}} \tau_N f_-\, \overline{\tau_D g_-}\, d\tilde{x} \ , \quad \forall g \in H^1(\mathbb{R}^d)$$

consisting only of integrals on the boundary. Since $\tau_D g_+ = \tau_D g_-$, by Theorem 1.34 and $\operatorname{ran}(\tau_D \restriction_{H^1(\mathbb{R}^d)}) = H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ is dense in $L^2(\mathbb{R}^{d-1})$ we confirmed the boundary condition

$$-\alpha \tau_D f = \tau_N f_+ + \tau_N f_- \ .$$

For the inclusion " $\supseteq$ ", let $f \in L^2(\mathbb{R}^d)$ with the properties

$$f_\pm \in H^1_\Delta(\mathbb{R}^d_\pm) \ , \qquad \tau_D f_+ = \tau_D f_- \quad \text{and} \quad \tau_N f_+ + \tau_N f_- = -\alpha \tau_D f \ . \tag{4.17}$$

From the first two properties of (4.17) we conclude that $f \in H^1(\mathbb{R}^d)$. Furthermore, defining $h \coloneqq (-\Delta f_+) \oplus (-\Delta f_-)$ we obtain for every $g \in H^1(\mathbb{R}^d)$ the identity

$$\begin{aligned} \langle h, g \rangle_{L^2(\mathbb{R}^d)} &= -\langle \Delta f_+, g_+ \rangle_{L^2(\mathbb{R}^d_+)} - \langle \Delta f_-, g_- \rangle_{L^2(\mathbb{R}^d_-)} \\ &= \langle \nabla f, \nabla g \rangle_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^{d-1}} \tau_N f_+\, \overline{\tau_D g_+}\, d\tilde{x} - \int_{\mathbb{R}^{d-1}} \tau_N f_-\, \overline{\tau_D g_-}\, d\tilde{x} \\ &= \langle \nabla f, \nabla g \rangle_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^{d-1}} \alpha\, \tau_D f\, \overline{\tau_D g}\, d\tilde{x} \\ &= a_\alpha(f, g) \ . \end{aligned}$$

By the definition of the operator domain in (4.12), we validated $f \in \operatorname{dom}(A_\alpha)$. □

Theorem 4.5. Let $d \geq 3$, $\alpha_0 \in \mathbb{R}$ and $\alpha \in L^{d-1}(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ be real-valued with the decay property:

$$\left\{ \tilde{x} \in \mathbb{R}^{d-1} \;\middle|\; |\alpha(\tilde{x})| \geq \varepsilon \right\} \text{ has finite measure for all } \varepsilon > 0 \ . \tag{4.18}$$

Then the Schrödinger operator $A_{\alpha+\alpha_0}$ has the essential spectrum

$$\sigma_{\text{ess}}(A_{\alpha+\alpha_0}) = \begin{cases} [0,\infty) & \text{if } \alpha_0 \geq 0\ , \\ [-\frac{\alpha_0^2}{4},\infty) & \text{if } \alpha_0 \leq 0\ . \end{cases}$$

Remark 4.6. Since the decay property (4.18) is fulfilled for all functions in $L^p(\mathbb{R}^{d-1})$, if $p < \infty$, it is only a restriction to the L^∞-part of α. It forces this L^∞-part to become small at infinity, although maybe in an non-integrable way.

Proof. We know that the essential spectrum of A_{α_0}, only treating the constant potential α_0, is given by

$$\sigma_{\text{ess}}(A_{\alpha_0}) = \begin{cases} [0,\infty) & \text{if } \alpha_0 \geq 0\ , \\ [-\frac{\alpha_0^2}{4},\infty) & \text{if } \alpha_0 \leq 0\ . \end{cases}\ .$$

In order to prove that the essential spectrum of $A_{\alpha+\alpha_0}$ looks the same, it is sufficient to check that the resolvent difference $(A_{\alpha_0}-\lambda)^{-1} - (A_{\alpha+\alpha_0}-\lambda)^{-1}$ is compact for some $\lambda \in \rho(A_{\alpha_0}) \cap \rho(A_{\alpha+\alpha_0})$. Since $A_{\alpha+\alpha_0}$ as well as A_{α_0} are semibounded, we can choose λ to be real-valued. For a shorter notation denote the resolvents as

$$R_{\alpha_0} \coloneqq (A_{\alpha_0}-\lambda)^{-1} \quad \text{and} \quad R_{\alpha+\alpha_0} \coloneqq (A_{\alpha+\alpha_0}-\lambda)^{-1}$$

for the rest of the proof. For every $f, g \in L^2(\mathbb{R}^d)$ we obtain the representation

$$\begin{aligned} \langle (R_{\alpha_0} - R_{\alpha+\alpha_0})f, g\rangle_{L^2(\mathbb{R}^d)} &= \langle R_{\alpha_0} f, (A_{\alpha+\alpha_0}-\lambda)R_{\alpha+\alpha_0} g\rangle_{L^2(\mathbb{R}^d)} \\ &\quad - \langle (A_{\alpha_0}-\lambda)R_{\alpha_0} f, R_{\alpha+\alpha_0} g\rangle_{L^2(\mathbb{R}^d)} \\ &= \langle R_{\alpha_0} f, A_{\alpha+\alpha_0} R_{\alpha+\alpha_0} g\rangle_{L^2(\mathbb{R}^d)} - \langle A_{\alpha_0} R_{\alpha_0} f, R_{\alpha+\alpha_0} g\rangle_{L^2(\mathbb{R}^d)} \\ &= a_{\alpha+\alpha_0}(R_{\alpha_0} f, R_{\alpha+\alpha_0} g) - a_{\alpha_0}(R_{\alpha_0} f, R_{\alpha+\alpha_0} g) \\ &= \int_{\mathbb{R}^{d-1}} \alpha\ \tau_D R_{\alpha_0} f\ \overline{\tau_D R_{\alpha+\alpha_0} g}\ d\tilde{x}\ . \end{aligned} \tag{4.19}$$

of the resolvent difference by just an integral on the boundary. The estimate (4.11) with $\varepsilon = \frac{1}{2}$ and α replaced by $\alpha+\alpha_0$, yields the existence of some constant $c > 0$, such that

$$a_{\alpha+\alpha_0}(f,f) \geq \frac{1}{2}\|f\|^2_{H^1(\mathbb{R}^d)} - c\|f\|^2_{L^2(\mathbb{R}^d)}\ , \quad \forall f \in H^1(\mathbb{R}^d)\ .$$

This inequality admits the bound

$$\begin{aligned}\|(A_{\alpha+\alpha_0}-\lambda)f\|_{L^2(\mathbb{R}^d)} &= \sup_{h\in L^2(\mathbb{R}^d)} \frac{\langle (A_{\alpha+\alpha_0}-\lambda)f,h\rangle_{L^2(\mathbb{R}^d)}}{\|h\|_{L^2(\mathbb{R}^d)}}\\ &\geq \frac{a_{\alpha+\alpha_0}(f,f)-\lambda\|f\|^2_{L^2(\mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{R}^d)}}\\ &\geq \frac{1}{2}\|f\|_{H^1(\mathbb{R}^d)} - (\lambda+c)\,\|f\|_{L^2(\mathbb{R}^d)}\ .\end{aligned}$$

Since $\lambda \in \rho(A_{\alpha_0}) \cap \rho(A_{\alpha+\alpha_0}) \cap \mathbb{R}$ was arbitrary, we can especially choose $\lambda < -c$ to make $R_{\alpha+\alpha_0} = (A_{\alpha+\alpha_0}-\lambda)^{-1}$ bounded as an operator from $H^1(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Consequently $\operatorname{sign}(\alpha)|\alpha|^{\frac{1}{2}}\tau_D R_{\alpha+\alpha_0} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{d-1})$ is everywhere defined and bounded by Lemma A.1. Hence, also its adjoint is everywhere defined and bounded. Since (4.19) holds for every $f, g \in L^2(\mathbb{R}^d)$ we obtain the identity

$$R_{\alpha_0} - R_{\alpha+\alpha_0} = (\operatorname{sign}(\alpha)|\alpha|^{\frac{1}{2}}\tau_D R_{\alpha+\alpha_0})^*|\alpha|^{\frac{1}{2}}\tau_D R_{\alpha_0}$$

and it remains to prove that $|\alpha|^{\frac{1}{2}}\tau_D R_{\alpha_0}$ is compact. Therefore, let $(f_n)_{n\in\mathbb{N}} \in L^2(\mathbb{R}^d)$ a bounded sequence. Since $\tau_D R_{\alpha_0} : L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^{d-1})$ is, analogously to $\tau_D R_{\alpha+\alpha_0}$ above, a bounded operator, there exists a weak convergent subsequence

$$\phi_0 \coloneqq \underset{k\to\infty}{\operatorname{wlim}}_{H^1(\mathbb{R}^{d-1})}\, \tau_D R_{\alpha_0} f_{n_k}\ .$$

After the multiplication with $|\alpha|^{\frac{1}{2}}$, this sequence then also converges in norm

$$\lim_{k\to\infty} \||\alpha|^{\frac{1}{2}}\tau_D R_{\alpha_0} f_{n_k} - |\alpha|^{\frac{1}{2}}\phi_0\|_{L^2(\mathbb{R}^{d-1})} = 0$$

by Theorem A.4, which proves the compactness of $|\alpha|^{\frac{1}{2}}\tau_D R_{\alpha_0}$ and hence of the resolvent difference $R_{\alpha_0} - R_{\alpha+\alpha_0}$. □

5. Schrödinger operator and corresponding quasi boundary triples

In the last chapter we introduced the Schrödinger operator A_α in Definition 4.3 by the form a_α and the first representation theorem. In this chapter we define a quasi boundary triple, which in the end will turn out to be the right choice to also describe the operator A_α. Although we already calculated the domain, action and essential spectrum of A_α in Chapter 2 for very general $\alpha \in L^{d-1}(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$, the advantage of the definition via quasi boundary triples is, that if we use more regular $\alpha \in L^{\frac{4}{3}(d-1)}(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$, we also obtain the higher regularity $f \in H^{3/2}_\Delta(\mathbb{R}^d_\pm)$ for all $f \in \mathrm{dom}(A_\alpha)$, see Theorem 5.8. Another reason why it is advantageous to consider quasi boundary triples is, that we will need the corresponding Weyl function in order to define the Birman-Schwinger operator in Chapter 6 and prove the Lieb-Thirring inequality in Chapter 7. In particular the explicit integral representations of Theorem 5.3 will be essential.

This chapter will be splitted into two sections. In Section 5.1 we will define a quasi boundary triple $(L^2(\mathbb{R}^{d-1}), \Gamma_0, \Gamma_1)$ for the Laplacian $-\Delta$ on $\mathbb{R}^d \setminus \Sigma$, where Σ is the hyperplane (4.1). In Section 5.2 we will generalise this triple to a quasi boundary triple $(L^2(\mathbb{R}^{d-1}), \Gamma_0', \Gamma_1')$ for the shifted Laplacian $-\Delta + \alpha_0\delta_\Sigma$ for some $\alpha_0 \in \mathbb{R}$.

5.1. Quasi boundary triple for the Laplacian

We will start to construct a quasi boundary triple (W, Γ_0, Γ_1), in the sense of Definition 3.1, for $-\Delta$ on $\mathbb{R}^d \setminus \Sigma$. For the appearing Sobolev space $H^s_\Delta(\mathbb{R}^d_\pm)$ in (5.1) and the traces in (5.2) we recall Definition 1.28 and Theorem 1.37.

Theorem 5.1. Define the operator

$$\begin{aligned} Tf &:= (-\Delta f_+) \oplus (-\Delta f_-) \\ \mathrm{dom}(T) &:= \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; f_\pm \in H^{\frac{3}{2}}_\Delta(\mathbb{R}^d_\pm),\ \tau_D f_+ = \tau_D f_- \right\} \end{aligned} \tag{5.1}$$

and the boundary mappings

$$\Gamma_0 f := \tau_N f_+ + \tau_N f_- \quad \text{and} \quad \Gamma_1 f := \tau_D f_+ = \tau_D f_- \ , \quad \forall f \in \mathrm{dom}(T) \ . \tag{5.2}$$

Then the operator

$$S := T\restriction_{\ker(\Gamma)} \text{ is a densely defined, closed, symmetric operator} \tag{5.3}$$

and $(L^2(\mathbb{R}^{d-1}), \Gamma_0, \Gamma_1)$ is a quasi boundary triple for S^*. Furthermore the corresponding operator $A_0 = T\restriction_{\ker(\Gamma_0)}$ is given by

$$A_0 = -\Delta \quad \text{and} \quad \mathrm{dom}(A_0) = H^2(\mathbb{R}^d) \quad \text{and} \quad \rho(A_0) = \mathbb{C} \setminus [0, \infty). \tag{5.4}$$

Proof. All the assertions, except of (5.4), follow immediately from the abstract Theorem 3.7. So all we have to do is to verify its assumptions a) - d).

a) The free Laplace operator

$$A_{\text{free}} = -\Delta \quad \text{with} \quad \mathrm{dom}(A_{\text{free}}) = H^2(\mathbb{R}^d)$$

is well-known to be self-adjoint. Furthermore, for every $f \in H^2(\mathbb{R}^d)$, its Neumann trace from the upper and from the lower halfplane coincide (except of a minus sign) by Theorem 1.34, which shows that

$$\Gamma_0 f = \tau_N f_+ + \tau_N f_- = 0$$

and hence $f \in \ker(\Gamma_0)$. Also by the continuity of the Dirichlet and the Neumann trace it follows that

$$Tf = (-\Delta f_1) \oplus (-\Delta f_2) = -\Delta f = A_{\text{free}} f \ ,$$

where a calculation can for example be found in (1.33). This proves that $T\restriction_{\ker(\Gamma_0)}$ is an extension of the self-adjoint operator A_{free}.

b) Let $\phi \in H^{\frac{3}{2}}(\mathbb{R}^{d-1})$ and $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$. Then by the surjectivity of the combined mapping of Dirichlet and Neumann trace in Theorem 1.33, there exist functions $f_+ \in H^2(\mathbb{R}^d_+)$ and $f_- \in H^2(\mathbb{R}^d_-)$, such that

$$\begin{pmatrix} \tau_D \\ \tau_N \end{pmatrix} f_+ = \begin{pmatrix} \phi \\ \varphi \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tau_D \\ \tau_N \end{pmatrix} f_- = \begin{pmatrix} \phi \\ 0 \end{pmatrix} .$$

If we define $f := f_+ \oplus f_-$, we obtain

$$f \in \mathrm{dom}(T) \quad \text{and} \quad \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} f = \begin{pmatrix} \varphi \\ \phi \end{pmatrix} .$$

This verifies $H^{\frac{3}{2}}(\mathbb{R}^{d-1}) \times H^{\frac{1}{2}}(\mathbb{R}^{d-1}) \subseteq \operatorname{ran}(\Gamma)$ and therefore the density of the range $\overline{\operatorname{ran}}(\Gamma) = L^2(\mathbb{R}^{d-1}) \times L^2(\mathbb{R}^{d-1})$.

c) We know that the space $\mathcal{C}_0^\infty(\mathbb{R}^d_\pm)$ of testfunctions is dense in $L^2(\mathbb{R}^d_\pm)$ and furthermore, it is obvious that $\mathcal{C}_0^\infty(\mathbb{R}^d_+) \oplus \mathcal{C}_0^\infty(\mathbb{R}^d_-) \subseteq \ker(\Gamma)$. This leads to the density of the kernel

$$\overline{\ker}(\Gamma) = \overline{\mathcal{C}_0^\infty(\mathbb{R}^d_+)} \oplus \overline{\mathcal{C}_0^\infty(\mathbb{R}^d_-)} = L^2(\mathbb{R}^d_+) \oplus L^2(\mathbb{R}^d_-) = L^2(\mathbb{R}^d) \, .$$

d) Let $f, g \in \operatorname{dom}(T)$. Then especially $f_\pm, g_\pm \in H^s_\Delta(\mathbb{R}^d_\pm)$ and Green's second formula (1.30) holds on $\mathbb{R}^d_+$ and $\mathbb{R}^d_-$ respectively. Together with the definition of the boundary mappings (5.2), this gives the abstract Green's identity

$$\begin{aligned}
\langle Tf, g\rangle_{L^2(\mathbb{R}^d)} - \langle f, Tg\rangle_{L^2(\mathbb{R}^d)} &= -\langle \Delta f_+, g_+\rangle_{L^2(\mathbb{R}^d_+)} - \langle \Delta f_-, g_-\rangle_{L^2(\mathbb{R}^d_-)} \\
&\quad + \langle f_+, \Delta g_+\rangle_{L^2(\mathbb{R}^d_+)} + \langle f_-, \Delta g_-\rangle_{L^2(\mathbb{R}^d_-)} \\
&= \langle \tau_D f_+, \tau_N g_+\rangle_{L^2(\mathbb{R}^{d-1})} - \langle \tau_N f_+, \tau_D g_+\rangle_{L^2(\mathbb{R}^{d-1})} \\
&\quad + \langle \tau_D f_-, \tau_N g_-\rangle_{L^2(\mathbb{R}^{d-1})} - \langle \tau_N f_-, \tau_D g_-\rangle_{L^2(\mathbb{R}^{d-1})} \\
&= \langle \Gamma_1 f, \Gamma_0 g\rangle_{L^2(\mathbb{R}^{d-1})} - \langle \Gamma_0 f, \Gamma_1 g\rangle_{L^2(\mathbb{R}^{d-1})} \, .
\end{aligned}$$

These points a) - d) verify the assumptions of Theorem 3.7 and hence we get that S is a densely defined, symmetric, closed operator and $(L^2(\mathbb{R}^{d-1}), \Gamma_0, \Gamma_1)$ is a quasi boundary triple for S^*. In order to verify (5.4) it is enough to show that $\ker(\Gamma_0) = H^2(\mathbb{R}^d)$, which is already proven in Theorem 1.42. ☐

Once we have verified that $(L^2(\mathbb{R}^{d-1}), \Gamma_0, \Gamma_1)$ is indeed a quasi boundary triple, we want to derive the ranges of the boundary mappings, which will be significant in applying Theorem 3.10 later on.

Lemma 5.2. The ranges of the boundary mappings Γ_0, Γ_1 are $\operatorname{ran}(\Gamma_0) = L^2(\mathbb{R}^{d-1})$ and $\operatorname{ran}(\Gamma_1) = H^1(\mathbb{R}^{d-1})$.

Proof. Start with the proof of $\operatorname{ran}(\Gamma_0) = L^2(\mathbb{R}^{d-1})$. This means, for every $\phi \in L^2(\mathbb{R}^{d-1})$ we have to find a $g \in \operatorname{dom}(T)$ with $\Gamma_0 g = \phi$. By the surjectivity of τ_N, Theorem 1.41, there exist functions $f_+ \in H^{3/2}_\Delta(\mathbb{R}^d_+)$ and $f_- \in H^{3/2}_\Delta(\mathbb{R}^d_-)$, satisfying

$$\tau_N f_+ = \tau_N f_- = \frac{\phi}{4} \, .$$

Define $g(x) := \begin{cases} f_+(x) + f_-(-x) & \text{if } x \in \mathbb{R}^d_+ \, , \\ f_-(x) + f_+(-x) & \text{if } x \in \mathbb{R}^d_- \, , \end{cases}$ then clearly $g_\pm \in H^{3/2}_\Delta(\mathbb{R}^d_\pm)$, with continuous Dirichlet trace

$$\tau_D g_+ = \tau_D g_- \, .$$

This confirms that indeed $g \in \operatorname{dom}(T)$ and furthermore, the boundary mapping Γ_0 applied to g yields

$$\Gamma_0 g = \tau_N g_+ + \tau_N g_- = \tau_N f_+ + \tau_N f_- + \tau_N f_- + \tau_N f_+ = \phi \ .$$

In order to prove $\operatorname{ran}(\Gamma_1) = H^1(\mathbb{R}^d)$ we have to find for every $\phi \in H^1(\mathbb{R}^{d-1})$ a function $g \in \operatorname{dom}(T)$ with $\Gamma_1 g = \phi$. By the surjectivity of τ_D, Theorem 1.41, there exists a function $f_+ \in H^{3/2}_\Delta(\mathbb{R}^d_+)$ satisfying

$$\tau_D f_+ = \phi \ .$$

Define $g(x) := \begin{cases} f_+(x) & \text{if } x \in \mathbb{R}^d_+ \ , \\ f_+(-x) & \text{if } x \in \mathbb{R}^d_- \ , \end{cases}$ then clearly $g_\pm \in H^{3/2}_\Delta(\mathbb{R}^d_\pm)$, with continuous Dirichlet trace

$$\tau_D g_+ = \tau_D g_- = \phi \ .$$

This confirms that indeed $g \in \operatorname{dom}(T)$ and $\Gamma_1 g = \phi$. □

In the next theorem we want to find integral representations of the γ-field, its adjoint and the Weyl function in terms of the Green's function G_λ from (B.1). To do so we first recall the convolution mappings

a) $\mathcal{G}_\lambda f := G_\lambda * f, \qquad \forall f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),$

b) $\widetilde{\mathcal{G}}_\lambda \phi := \widetilde{G}_\lambda * \phi, \qquad \forall \phi \in L^1(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1}),$

c) $\mathcal{G}^\downarrow_\lambda f(\tilde{x}) := \int\limits_{\mathbb{R}^d} G_\lambda\left(\begin{pmatrix} \tilde{x} \\ 0 \end{pmatrix} - y\right) f(y)dy, \qquad \forall f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), \tilde{x} \in \mathbb{R}^{d-1},$

d) $\mathcal{G}^\uparrow_\lambda \phi(x) := \int\limits_{\mathbb{R}^{d-1}} G_\lambda\left(x - \begin{pmatrix} \tilde{y} \\ 0 \end{pmatrix}\right) \phi(\tilde{y})d\tilde{y}, \quad \forall \phi \in L^1(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1}), x \in \mathbb{R}^d.$

from Definition B.6. Here we used the notation that every $x \in \mathbb{R}^d$ is decomposed into $\begin{pmatrix} \tilde{x} \\ x_d \end{pmatrix}$ for some $\tilde{x} \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$.

Theorem 5.3. Let $d \geq 3$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then the γ-field, its adjoint and the Weyl function have the integral representations

$$\gamma(\lambda)\phi = \mathcal{G}^\uparrow_\lambda \phi \ , \qquad \forall \phi \in L^2(\mathbb{R}^{d-1}), \tag{5.5}$$

$$\gamma(\lambda)^* f = \mathcal{G}^\downarrow_{\bar{\lambda}} f, \qquad \forall f \in L^2(\mathbb{R}^d), \tag{5.6}$$

$$M(\lambda)\phi = \widetilde{\mathcal{G}}_\lambda \phi \ , \qquad \forall \phi \in L^2(\mathbb{R}^{d-1}). \tag{5.7}$$

Proof. For the proof of (5.6) we first consider $f \in \mathcal{C}^\infty_0(\mathbb{R}^d)$ and obtain

$$\gamma(\lambda)^* f = \Gamma_1(A_0 - \bar{\lambda})^{-1} f = \Gamma_1 \mathcal{G}_{\bar{\lambda}} f = \mathcal{G}^\downarrow_{\bar{\lambda}} f \ , \tag{5.8}$$

by Lemma B.4. Since $\gamma(\lambda)^*$ as well as $\mathcal{G}^{\downarrow}_{\bar\lambda}$ are bounded by Lemma 3.4 and Corollary B.7, the identity (5.8) can be extended to $L^2(\mathbb{R}^d)$ by continuity.

For the proof of (5.5) we first note that $\mathcal{G}^{\uparrow}_{\lambda}\phi$ is an element in $L^2(\mathbb{R}^d)$ by Corollary B.7 and we can use the already calculated representation (5.6) to obtain for every $f \in L^2(\mathbb{R}^d)$ the identity

$$\begin{aligned}\langle \phi, \gamma(\lambda)^* f\rangle_{L^2(\mathbb{R}^{d-1})} &= \int_{\mathbb{R}^{d-1}} \phi(\tilde y) \int_{\mathbb{R}^d} \overline{G_{\bar\lambda}}\left(\begin{pmatrix}\tilde y\\ 0\end{pmatrix} - x\right) \bar f(x) dx d\tilde y \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}} \phi(\tilde y) G_\lambda\left(\begin{pmatrix}\tilde y\\ 0\end{pmatrix} - x\right) d\tilde y \bar f(x) dx \\ &= \langle \mathcal{G}^{\uparrow}_{\lambda}\phi, f\rangle_{L^2(\mathbb{R}^d)} \ .\end{aligned}$$

This shows that $\mathcal{G}^{\uparrow}_{\lambda}\phi = (\gamma(\lambda)^*)^*\phi = \gamma(\lambda)\phi$, by the closedness of the γ-field.

Finally, equation (5.7) follows for $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^{d-1})$ immediately from (5.5) and the direct evaluation on the boundary

$$M(\lambda)\phi = \Gamma_1\gamma(\lambda)\phi = \Gamma_1\mathcal{G}^{\uparrow}_{\lambda}\phi = \widetilde{\mathcal{G}}_\lambda\phi \ . \tag{5.9}$$

Since $\gamma(\lambda) = \mathcal{G}^{\uparrow}_{\lambda}\restriction_{L^2(\mathbb{R}^{d-1})}$ is bounded from $L^2(\mathbb{R}^{d-1})$ into $H^s(\mathbb{R}^d)$ and the Dirichlet trace Γ_1 is bounded from $H^s(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{d-1})$, for every $\frac{1}{2} < s < \frac{3}{2}$, we obtain that $M(\lambda) = \Gamma_1\gamma(\lambda)$ is bounded in $L^2(\mathbb{R}^{d-1})$. Also the right hand side $\widetilde{\mathcal{G}}_\lambda$ is bounded by Corollary B.7 and therefore (5.9) can be extended to the whole space $L^2(\mathbb{R}^{d-1})$ by continuity. □

Once we know the representations in Theorem 5.3, we can copy the Fourier transformations and boundedness properties from Lemma B.8 and Proposition B.9.

Corollary 5.4. Let $d \geq 3$ and $\lambda \in \mathbb{C}\setminus[0,\infty)$. Then the Fourier transformation of the γ-field, its adjoint and the Weyl function have the form

a) $\mathcal{F}_d\gamma(\lambda)\phi(k) = \frac{\mathcal{F}_{d-1}\phi(\tilde k)}{\sqrt{2\pi}\,(|k|^2-\lambda)}\ , \qquad \forall k \in \mathbb{R}^d, \quad \phi \in L^2(\mathbb{R}^{d-1}),$

b) $\mathcal{F}_{d-1}\gamma(\lambda)^* f(\tilde k) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\mathcal{F}_d f(k)}{|k|^2-\lambda} dk_d\ , \quad \forall \tilde k \in \mathbb{R}^{d-1},\ f \in L^2(\mathbb{R}^d),$

c) $\mathcal{F}_{d-1}M(\lambda)\phi(\tilde k) = \frac{\mathcal{F}_{d-1}\phi(\tilde k)}{2(|\tilde k|^2-\lambda)^{\frac{1}{2}}}\ , \qquad \forall \tilde k \in \mathbb{R}^{d-1},\ \phi \in L^2(\mathbb{R}^{d-1}).$

Note, that in a) and b) the notation $k = (\tilde k, k_d)$ was used, and that the indices d and $d-1$ indicate the dimension of the Fourier transformation.

Corollary 5.5. Let $d \geq 3$, $\lambda \in \mathbb{C}\setminus[0,\infty)$ and $p \in [1,2]$. Then for every $s < \frac{3}{2} - (d-1)(\frac{1}{p} - \frac{1}{2})$, the restrictions

a) $\gamma(\lambda)\restriction_{L^p(\mathbb{R}^{d-1})} : L^p(\mathbb{R}^{d-1}) \to H^s(\mathbb{R}^d)$,

b) $\gamma(\lambda)^*\restriction_{L^p(\mathbb{R}^d)} : L^p(\mathbb{R}^d) \to H^s(\mathbb{R}^{d-1})$ and

c) $M(\lambda)\restriction_{L^p(\mathbb{R}^{d-1})} : L^p(\mathbb{R}^{d-1}) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$

are bounded operators.

Lemma 3.6 tells that already the abstract Weyl function $M(\lambda)$ has some very useful properties, like symmetry and monotonicity, if the index λ is real-valued. In our special quasi boundary triple the Weyl function is additionally everywhere defined, see (5.7), which makes the operator also self-adjoint. Moreover $M(\lambda)$ is non-negative, which can be seen in its Fourier representation in Corollary 5.4.

Lemma 5.6. Let $d \geq 3$ and $\lambda \in (-\infty, 0)$. Then the Weyl function $M(\lambda)$ is a bounded, self-adjoint and non-negative operator with an L^2-operator norm which converges like

$$\lim_{\lambda \to -\infty} \|M(\lambda)\| = 0 .$$

Furthermore, the function $\lambda \mapsto M(\lambda)$ is monotone increasing in the sense that for every $\lambda_1 \leq \lambda_2 \in (-\infty, 0)$ the quadratic form fulfils

$$\langle M(\lambda_1)\phi, \phi\rangle_{L^2(\mathbb{R}^{d-1})} \leq \langle M(\lambda_2)\phi, \phi\rangle_{L^2(\mathbb{R}^{d-1})} , \quad \forall \phi \in L^2(\mathbb{R}^{d-1}) .$$

Also a useful property is the following identity of the γ-field, which basically follows from its definition combined with Green's first formula.

Proposition 5.7. Let $d \geq 3$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then for every $\phi \in L^2(\mathbb{R}^{d-1})$ the γ-field satisfies

$$\langle \nabla\gamma(\lambda)\phi, \nabla g\rangle_{L^2(\mathbb{R}^d)} - \lambda\langle\gamma(\lambda)\phi, g\rangle_{L^2(\mathbb{R}^d)} = \langle \phi, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} , \quad \forall g \in H^1(\mathbb{R}^d) . \tag{5.10}$$

Proof. Let $\phi \in L^2(\mathbb{R}^{d-1})$ and $g \in H^1(\mathbb{R}^d)$. Then we know that $\gamma(\lambda)\phi \in \ker(T - \lambda)$ by Lemma 3.4, which in particular implies $(\gamma(\lambda)\phi)_\pm \in H^{3/2}_\Delta(\mathbb{R}^d_\pm)$. Therefore Green's first identity, Theorem 1.38, is applicable on the domains $\mathbb{R}^d_\pm$ and admits

$$\begin{aligned}\langle \nabla\gamma(\lambda)\phi, \nabla g\rangle_{L^2(\mathbb{R}^d)} &= \langle \nabla\gamma(\lambda)\phi, \nabla g\rangle_{L^2(\mathbb{R}^d_+)} + \langle \nabla\gamma(\lambda)\phi, \nabla g\rangle_{L^2(\mathbb{R}^d_-)} \\ &= -\langle \Delta\gamma(\lambda)\phi, g\rangle_{L^2(\mathbb{R}^d_+)} + \langle \tau_N(\gamma(\lambda)\phi)_+, \tau_D g_+\rangle_{L^2(\mathbb{R}^{d-1})} \\ &\qquad - \langle \Delta\gamma(\lambda)\phi, g\rangle_{L^2(\mathbb{R}^d_-)} + \langle \tau_N(\gamma(\lambda)\phi)_-, \tau_D g_-\rangle_{L^2(\mathbb{R}^{d-1})} .\end{aligned}$$

Because of $g \in H^1(\mathbb{R}^d)$, its Dirichlet traces from the upper and the lower halfplane coincide and will be denoted by $\tau_D g$. Moreover, with the definition of T in (5.1) and Γ_0 in (5.2) this equation looks like

$$\langle \nabla\gamma(\lambda)\phi, \nabla g\rangle_{L^2(\mathbb{R}^d)} = \langle T\gamma(\lambda)\phi, g\rangle_{L^2(\mathbb{R}^d)} + \langle \Gamma_0\gamma(\lambda)\phi, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} .$$

Moreover, using $\gamma(\lambda)\phi \in \ker(T-\lambda)$ and $\gamma(\lambda) = (\Gamma_0 \restriction_{\ker(T-\lambda)})^{-1}$, this formula further reduces to final result

$$\langle \nabla\gamma(\lambda)\phi, \nabla g\rangle_{L^2(\mathbb{R}^d)} = \lambda\langle \gamma(\lambda)\phi, g\rangle_{L^2(\mathbb{R}^d)} + \langle \phi, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} .$$

□

After these preparations, where we collected properties of the γ-field and the Weyl function, we are now ready to prove the main theorem of this section. It gives an explicit representation of the domain as well as of the action of the Schrödinger operator A_α, which was abstractly defined via the first representation theorem in Definition 4.3. This result is very similar to Proposition 4.4, with the only difference that here we are only allowed to consider $\alpha \in L^p(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ for $p > \frac{4}{3}(d-1)$, but get the additional regularity $f_\pm \in H^{3/2}_\Delta(\mathbb{R}^d_\pm)$ for every $f \in \mathrm{dom}(A_\alpha)$ as a consequence.

Theorem 5.8. Let $d \geq 3$ and A_α be the Schrödinger operator from Definition 4.3 for the real-valued potential $\alpha \in L^p(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ for some $p > \frac{4}{3}(d-1)$. Then this operator has the explicit representation

$$A_\alpha f = (-\Delta f_+) \oplus (-\Delta f_-) \quad \text{and}$$
$$\mathrm{dom}(A_\alpha) = \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \begin{array}{l} f_\pm \in H^{3/2}_\Delta(\mathbb{R}^d_\pm) \\ \tau_D f_+ = \tau_D f_- \\ \tau_N f_+ + \tau_N f_- = -\alpha\, \tau_D f \end{array} \right\} . \tag{5.11}$$

Moreover, the Krein resolvent formula

$$(A_\alpha - \lambda)^{-1} = (A_0-\lambda)^{-1} - \gamma(\lambda)\mathrm{sign}(\alpha)|\alpha|^{\frac{1}{3}}(1+|\alpha|^{\frac{2}{3}}M(\lambda)\mathrm{sign}(\alpha)|\alpha|^{\frac{1}{3}})^{-1}|\alpha|^{\frac{2}{3}}\gamma(\bar{\lambda})^*$$

holds for every $\lambda \in \rho(A_\alpha) \setminus [0,\infty)$.

Proof. Define the multiplication operator B in the boundary space $L^2(\mathbb{R}^{d-1})$ by

$$B\phi = -\alpha\phi \quad \text{with} \quad \mathrm{dom}(B) = H^1(\mathbb{R}^{d-1})$$

and decompose it into

$$B_1 = -\mathrm{sign}(\alpha)|\alpha|^{\frac{1}{3}} \quad \text{with} \quad \mathrm{dom}(B_1) = \left\{ \phi \in L^2(\mathbb{R}^{d-1}) \;\middle|\; |\alpha|^{\frac{1}{3}}\phi \in L^2(\mathbb{R}^{d-1}) \right\} \quad \text{and}$$
$$B_2 = |\alpha|^{\frac{2}{3}} \quad \text{with} \quad \mathrm{dom}(B_2) = \left\{ \phi \in L^2(\mathbb{R}^{d-1}) \;\middle|\; |\alpha|^{\frac{2}{3}}\phi \in L^2(\mathbb{R}^{d-1}) \right\} .$$

The operator B is symmetric because the α is real-valued and fulfils the operator inclusion $B \subseteq B_1 B_2$ by Lemma A.1.

All the statements now follow by Theorem 3.10, so it remains to choose a suitable $\lambda_0 \in \rho(A_0) \cap \mathbb{R} = (-\infty, 0)$, for which its assumptions a) - e) are satisfied.

Using Lemma A.1, Theorem 1.31 and Theorem 1.27, we find a constant $c_1 > 0$, such that for every $g \in H^1(\mathbb{R}^d)$ we can estimate

$$\begin{aligned}\||\alpha|^{\frac{2}{3}}\tau_D g\|^2_{L^2(\mathbb{R}^{d-1})} &\leq c^2_{\frac{2(d-1)}{3p},p,\alpha}\|\tau_D g\|^2_{H^{\frac{2(d-1)}{3p}}(\mathbb{R}^{d-1})} \\ &\leq c^2_{\frac{2(d-1)}{3p},p,\alpha}\|\tau_D\|^2\|g\|^2_{H^{\frac{2(d-1)}{3p}+\frac{1}{2}}(\mathbb{R}^d)} \\ &\leq \frac{1}{2}\|\nabla g\|^2_{L^2(\mathbb{R}^d)} + c_1\|g\|^2_{L^2(\mathbb{R}^d)} \,. \end{aligned} \tag{5.12}$$

In the last inequality it was crucial that $\frac{2(d-1)}{3p} + \frac{1}{2} < 1$, which is satisfied by the assumption $p > \frac{4}{3}(d-1)$. In the same way we find a constant $c_2 > 0$ satisfying

$$\||\alpha|^{\frac{1}{3}}\tau_D g\|^2_{L^2(\mathbb{R}^{d-1})} \leq \frac{1}{2}\|\nabla g\|^2_{L^2(\mathbb{R}^d)} + c_2\|g\|^2_{L^2(\mathbb{R}^d)} \,, \quad \forall g \in H^1(\mathbb{R}^d) \,. \tag{5.13}$$

For the choice $\lambda_0 \coloneqq -2\max\{c_1, c_2\}$, the estimates (5.12) & (5.13) turn into

$$\||\alpha|^{\frac{2}{3}}\tau_D g\|^2_{L^2(\mathbb{R}^{d-1})} \leq \frac{1}{2}\left(\|\nabla g\|_{L^2(\mathbb{R}^d)} - \lambda_0\|g\|^2_{L^2(\mathbb{R}^d)}\right) \quad \text{and} \tag{5.14}$$

$$\||\alpha|^{\frac{1}{3}}\tau_D g\|^2_{L^2(\mathbb{R}^{d-1})} \leq \frac{1}{2}\left(\|\nabla g\|_{L^2(\mathbb{R}^d)} - \lambda_0\|g\|^2_{L^2(\mathbb{R}^d)}\right) \,, \quad \forall g \in H^1(\mathbb{R}^d) \tag{5.15}$$

Assumption a) requires that $1 \in \rho(B_2\overline{M(\lambda_0)B_1})$. This will be the only part of the proof where the choice of λ_0 will be essential. Define the inner product

$$\langle f, g\rangle_{\lambda_0} \coloneqq \langle \nabla f, \nabla g\rangle_{L^2(\mathbb{R}^d)} - \lambda_0\langle f, g\rangle_{L^2(\mathbb{R}^d)} \quad \forall f, g \in H^1(\mathbb{R}^d) \,. \tag{5.16}$$

Fix now any $\phi \in \operatorname{dom}(B_2M(\lambda_0)B_1)$ and use (5.14) for $g = \gamma(\lambda_0)B_1\phi$ to obtain the estimate

$$\|B_2M(\lambda_0)B_1\phi\|^2_{L^2(\mathbb{R}^{d-1})} \leq \frac{1}{2}\|\gamma(\lambda_0)B_1\phi\|^2_{\lambda_0} = \frac{1}{2}\sup_{h\in H^1(\mathbb{R}^d)\setminus\{0\}}\frac{\langle\gamma(\lambda_0)B_1\phi, h\rangle^2_{\lambda_0}}{\|h\|^2_{\lambda_0}} \,,$$

where the supremum can be taken over all $h \in H^1(\mathbb{R}^d)$, because the norm $\|\cdot\|_{\lambda_0}$ is equivalent to the usual $\|\cdot\|_{H^1(\mathbb{R}^d)}$-norm. With the identity (5.10), the inner product can be written as

$$\begin{aligned}\|B_2M(\lambda_0)B_1\phi\|^2_{L^2(\mathbb{R}^{d-1})} &\leq \frac{1}{2}\sup_{h\in H^1(\mathbb{R}^d)\setminus\{0\}}\frac{\langle B_1\phi, \tau_D h\rangle^2_{L^2(\mathbb{R}^{d-1})}}{\|h\|^2_{\lambda_0}} \\ &\leq \frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^{d-1})}\sup_{h\in H^1(\mathbb{R}^d)\setminus\{0\}}\frac{\||\alpha|^{\frac{1}{3}}\tau_D h\|^2_{L^2(\mathbb{R}^{d-1})}}{\|h\|^2_{\lambda_0}} \,. \end{aligned}$$

Using (5.15) then gives the final estimate

$$\|B_2M(\lambda_0)B_1\phi\|^2_{L^2(\mathbb{R}^{d-1})} \leq \frac{1}{4}\|\phi\|^2_{L^2(\mathbb{R}^{d-1})} \ . \tag{5.17}$$

Since this is true for every $\phi \in \mathrm{dom}(B_2M(\lambda_0)B_1)$, the operator norm $\|\overline{B_2M(\lambda_0)B_1}\| \leq \frac{1}{2}$ is strictly smaller than 1. This implies that $(1 - \overline{B_2M(\lambda_0)B_1})^{-1}$ is an everywhere defined bounded operator and hence $1 \in \rho(\overline{B_2M(\lambda_0)B_1})$.
For the verification of Assumption a) it is left to show that $\overline{B_2M(\lambda_0)B_1} = B_2\overline{M(\lambda_0)B_1}$. From Corollary 5.5 with $s = \frac{1}{2}$ and Lemma A.1 we get the L^2-boundedness of $M(\lambda_0)B_1$ and since B_2 is a multiplication operator on its maximal domain, it is in particular closed. These two properties imply the closedness of $B_2\overline{M(\lambda_0)B_1}$ and hence the coincidence of $B_2\overline{M(\lambda_0)B_1} = \overline{B_2M(\lambda_0)B_1}$, since $\overline{B_2M(\lambda_0)B_1}$ is everywhere defined.

Assumptions b), c), d) will be treated simultaneously. First of all, Lemma 5.2 yields $\mathrm{ran}(\Gamma_0) = L^2(\mathbb{R}^{d-1})$ and $\mathrm{ran}(\Gamma_1) = H^1(\mathbb{R}^{d-1})$. From Lemma A.1 we get that $|\alpha|^{\frac{1}{3}}B_2\phi = |\alpha|\phi \in L^2(\mathbb{R}^{d-1})$ for every $\phi \in H^1(\mathbb{R}^{d-1})$ and hence

$$\mathrm{ran}(B_2\restriction_{\mathrm{ran}(\Gamma_1)}) \subseteq \mathrm{dom}(B_1) \ .$$

It remains to prove that $\mathrm{ran}(B_2\overline{M(\lambda_0)B_1}) \subseteq \mathrm{dom}(B_1)$. It is clear by Corollary B.10, that the closure of $M(\lambda_0)B_1$ is given by

$$\overline{M(\lambda_0)B_1}\ \phi = -\widetilde{\mathcal{G}}_{\lambda_0}[\mathrm{sign}(\alpha)|\alpha|^{\frac{1}{3}}\phi] \quad \text{and} \quad \mathrm{dom}(\overline{M(\lambda_0)B_1}) = L^2(\mathbb{R}^{d-1}) \ .$$

In order validate that for every $\phi \in L^2(\mathbb{R}^{d-1})$ the image $B_2\overline{M(\lambda_0)B_1}\ \phi$ is an element in $\mathrm{dom}(B_1)$, we have to check if $\alpha\ \widetilde{\mathcal{G}}_{\lambda_0}[\mathrm{sign}(\alpha)|\alpha|^{\frac{1}{3}}\phi]$ is in $L^2(\mathbb{R}^{d-1})$. Using Lemma A.1 we can estimate

$$\|\alpha\ \widetilde{\mathcal{G}}_{\lambda_0}[\mathrm{sign}(\alpha)|\alpha|^{\frac{1}{3}}\phi]\|_{L^2(\mathbb{R}^{d-1})} \leq c_{\frac{d-1}{p},p,\alpha}\|\widetilde{\mathcal{G}}_{\lambda_0}[\mathrm{sign}(\alpha)|\alpha|^{\frac{1}{3}}\phi]\|_{H^{\frac{d-1}{p}}(\mathbb{R}^{d-1})} \ ,$$

and because of the assumption $p > \frac{4}{3}(d-1)$, also Corollary B.10 is applicable for $s = \frac{d-1}{p}$ to obtain

$$\|\alpha\ \widetilde{\mathcal{G}}_{\lambda_0}[\mathrm{sign}(\alpha)|\alpha|^{\frac{1}{3}}\phi]\|_{L^2(\mathbb{R}^{d-1})} \leq c_{\frac{d-1}{p},p,\alpha}\tilde{c}_{\frac{d-1}{p}}\|\phi\|_{L^2(\mathbb{R}^{d-1})} \ ,$$

which verifies assumption d).

Assumption f) finally holds because of $\mathrm{ran}(\Gamma_1) = H^1(\mathbb{R}^{d-1}) = \mathrm{dom}(B)$.

Hence, all the assumptions of Theorem 3.10 are verified and we ensured that the extension

$$A_B f = (-\Delta f_+) \oplus (-\Delta f_-) \quad \text{and}$$
$$\operatorname{dom}(A_B) = \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \begin{array}{l} f_\pm \in H_\Delta^{\frac{3}{2}}(\mathbb{R}^d_\pm) \\ \tau_D f_+ = \tau_D f_- \\ \tau_N f_+ + \tau_N f_- = -\alpha\, \tau_D f \end{array} \right\} \tag{5.18}$$

is self-adjoint and the Krein type formula

$$(A_B - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)\operatorname{sign}(\alpha)|\alpha|^{\frac{1}{3}}(1 + |\alpha|^{\frac{2}{3}} M(\lambda)\operatorname{sign}(\alpha)|\alpha|^{\frac{1}{3}})^{-1}|\alpha|^{\frac{1}{3}}\gamma(\bar{\lambda})^*$$

holds for every $\lambda \in \rho(A_B) \setminus [0, \infty)$.

Since the operator inclusion $A_B \subseteq A_\alpha$ is obvious by Proposition 4.4 we automatically get equality $A_B = A_\alpha$, because both operators are self-adjoint. □

5.2. Quasi boundary triple for the shifted Laplacian

Since the potential strength, for which the Lieb-Thirring inequality in Chapter 7 should be derived, is allowed to contain a constant shift α_0, and we do not want to treat this shift as a perturbation, we will derive a new quasi boundary triple $(L^2(\mathbb{R}^{d-1}), \Gamma_0', \Gamma_1')$ for the shifted Laplacian $-\Delta + \alpha_0\delta_\Sigma$ in this section. However, it will turn out that this triple is closely related to the triple (W, Γ_0, Γ_1) of $-\Delta$ and many results of Section 5.1 can be reused.

Theorem 5.9. Let $\alpha_0 \in \mathbb{R}$ and S, T, Γ_0, Γ_1 as in Theorem 5.1. Define the boundary mappings of $-\Delta + \alpha_0\delta_\Sigma$ by

$$\begin{pmatrix} \Gamma_0' \\ \Gamma_1' \end{pmatrix} = \begin{pmatrix} I & \alpha_0 I \\ 0 & I \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}, \tag{5.19}$$

where I denotes the identity operator on $L^2(\mathbb{R}^{d-1})$. Then $(L^2(\mathbb{R}^{d-1}), \Gamma_0', \Gamma_1')$ is quasi boundary triple for S^* and the corresponding operator $A_0' := T\restriction_{\ker(\Gamma_0')}$ is given by the Schrödinger operator with constant potential strength α_0

$$A_0' = A_{\alpha_0} \quad \text{and} \quad \rho(A_0') = \begin{cases} \mathbb{C} \setminus [0, \infty) & \text{if } \alpha_0 \geq 0 \\ \mathbb{C} \setminus [-\frac{\alpha_0^2}{4}, \infty) & \text{if } \alpha_0 \leq 0 \end{cases}$$

Proof. Since the transformation operator in (5.19) is an isomorphism, we follow the density

$$\overline{\operatorname{ran}} \begin{pmatrix} \Gamma_0' \\ \Gamma_1' \end{pmatrix} = L^2(\mathbb{R}^{d-1}) \times L^2(\mathbb{R}^{d-1})$$

from the density of $\operatorname{ran}(\Gamma_0, \Gamma_1)$. Furthermore, the corresponding matrix has determinant

$$\det \begin{pmatrix} 1 & \alpha_0 \\ 0 & 1 \end{pmatrix} = 1 \ ,$$

which is sufficient that Γ_0', Γ_1' fulfil the abstract Green's identity as well. By the fact that $A_0' = A_{\alpha_0}$ is self-adjoint by Theorem 5.8 we proved that $(L^2(\mathbb{R}^{d-1}), \Gamma_0', \Gamma_1')$ is a quasi boundary triple. The resolvent set of A_0' is already proved in Theorem 4.5. □

Since it will be significant in applying Theorem 3.10 later on we will derive the ranges of the boundary mappings Γ_0' and Γ_1'

Lemma 5.10. The ranges of the boundary mappings Γ_0', Γ_1' are $\operatorname{ran}(\Gamma_0') = L^2(\mathbb{R}^{d-1})$ and $\operatorname{ran}(\Gamma_1') = H^1(\mathbb{R}^{d-1})$.

Proof. Since $\Gamma_1' = \Gamma_1$, Lemma 5.2 shows $\operatorname{ran}(\Gamma_1') = H^1(\mathbb{R}^{d-1})$.

In order to prove $\operatorname{ran}(\Gamma_0') = L^2(\mathbb{R}^{d-1})$, we have to find for every $\phi \in L^2(\mathbb{R}^{d-1})$ a function $g \in \operatorname{dom}(T)$, such that $\Gamma_0' g = \phi$. First of all, by the surjectivity of the Neumann trace, Theorem 1.41, there exists a function $f_+ \in H^{3/2}_{\Delta}(\mathbb{R}^d_+)$ satisfying

$$\tau_N f_+ = \frac{\phi}{2} \ .$$

The Dirichlet trace $\tau_D f_+$ of this function is then an element in $H^1(\mathbb{R}^{d-1})$. By the surjectivity of the combined Dirichlet and Neumann trace, Theorem 1.33, we also find a function $f_- \in H^{\frac{5}{2}}(\mathbb{R}^d_-)$, which satisfies

$$\begin{pmatrix} \tau_D \\ \tau_N \end{pmatrix} f_- = \begin{pmatrix} 0 \\ -\frac{\alpha_0}{2}\tau_D f_+ \end{pmatrix} \ .$$

Define $g(x) := \begin{cases} f_+(x) + f_-(-x) & \text{if } x \in \mathbb{R}^d_+ \ , \\ f_-(x) + f_+(-x) & \text{if } x \in \mathbb{R}^d_- \ , \end{cases}$, then clearly $g_\pm \in H^{3/2}_{\Delta}(\mathbb{R}^d_\pm)$, with continuous Dirichlet trace

$$\tau_D g_+ = \tau_D g_- \ .$$

This confirms that indeed $g \in \operatorname{dom}(T)$ and furthermore, the boundary mapping Γ_0' applied to g yields

$$\begin{aligned} \Gamma_0' g &= \tau_N g_+ + \tau_N g_- + \alpha_0 \tau_D g \\ &= \frac{\phi}{2} - \frac{\alpha_0}{2}\tau_D f_+ - \frac{\alpha_0}{2}\tau_D f_+ + \frac{\phi}{2} + \alpha_0 \tau_D f_+ = \phi \ . \end{aligned}$$

□

Since the boundary mapping Γ' was defined via the boundary mapping Γ, we can also find a connection between the γ-fields, its adjoints and their Weyl functions.

Lemma 5.11. Let $d \geq 3$, $\alpha_0 \in \mathbb{R}$ and $\lambda \in \rho(A_0')$. Then the operator $(1+\alpha_0 M(\lambda))^{-1}$ is an everywhere defined bounded operator in $L^2(\mathbb{R}^{d-1})$ and for every $s \geq 0$ its restriction to $H^s(\mathbb{R}^{d-1})$ is bounded as well. Furthermore, the γ-field, its adjoint and the Weyl function of $(L^2(\mathbb{R}^{d-1}), \Gamma_0', \Gamma_1')$ are connected the ones from $(L^2(\mathbb{R}^{d-1}), \Gamma_0, \Gamma_1)$ by

a) $\gamma'(\lambda) = \gamma(\lambda)(1 + \alpha_0 M(\lambda))^{-1}$,

b) $\gamma'(\lambda)^* = (1 + \alpha_0 M(\bar{\lambda}))^{-1}\gamma(\lambda)^*$,

c) $M'(\lambda) = M(\lambda)(1 + \alpha_0 M(\lambda))^{-1}$.

Proof. First note that because of $\lambda \in \rho(A_0') \subseteq \mathbb{C} \setminus [0, \infty) = \rho(A_0)$ by Theorem 5.9, the operators $\gamma(\lambda)$, $\gamma(\lambda)^*$ and $M(\lambda)$ are well-defined.

Looking at $(1 + \alpha_0 M(\lambda))$ in Fourier space, using Corollary 5.4, gives for every $\phi \in L^2(\mathbb{R}^{d-1})$ the identity

$$\mathcal{F}_{d-1}(1 + \alpha_0 M(\lambda))\phi(\tilde{k}) = \left(1 + \frac{\alpha_0}{2(|\tilde{k}|^2 - \lambda)^{\frac{1}{2}}}\right)\mathcal{F}_{d-1}\phi(\tilde{k}) , \quad \forall \tilde{k} \in \mathbb{R}^{d-1} . \tag{5.20}$$

Since $(1+\alpha_0 M(\lambda))$ is a multiplication operator in Fourier space, it is sufficient for the inverse $(1 + \alpha_0 M(\lambda))^{-1}$ to exist, be everywhere defined and H^s-bounded, that there exists constants $c_\lambda, d_\lambda > 0$, such that

$$c_\lambda \leq \left|1 + \frac{\alpha_0}{2(|\tilde{k}|^2 - \lambda)^{\frac{1}{2}}}\right| \leq d_\lambda , \quad \forall \tilde{k} \in \mathbb{R}^{d-1} . \tag{5.21}$$

But because of the choice

$$\lambda \in \rho(A_0') = \begin{cases} \mathbb{C} \setminus [0, \infty) & \text{if } \alpha_0 \geq 0 \\ \mathbb{C} \setminus [-\frac{\alpha_0^2}{4}, \infty) & \text{if } \alpha_0 \leq 0 \end{cases}$$

this is always satisfied. Knowing that $(1 + \alpha_0 M(\lambda))^{-1}$ is everywhere defined and bounded, the verification of the representations a) - c) is straight forward.

a) The operator $\gamma(\lambda)(1 + \alpha_0 M(\lambda))^{-1}$ is everywhere defined and its action

$$\begin{aligned} \Gamma_0'\gamma(\lambda)(1 + \alpha_0 M(\lambda))^{-1} &= (\Gamma_0 + \alpha_0\Gamma_1)\gamma(\lambda)(1 + \alpha_0 M(\lambda))^{-1} \\ &= (1 + \alpha_0 M(\lambda))(1 + \alpha_0 M(\lambda))^{-1} = I \end{aligned}$$

coincides with the one from $\gamma'(\lambda)$. Therefore they have to be the same operators.

b) The representation of the Weyl function follows directly from a), by

$$M'(\lambda) = \Gamma_1'\gamma'(\lambda) = \Gamma_1\gamma(\lambda)(1 + \alpha_0 M(\lambda))^{-1} = M(\lambda)(1 + \alpha_0 M(\lambda))^{-1} .$$

c) Also the representation of $\gamma'(\lambda)^*$ follows directly from a), by

$$\gamma'(\lambda)^* = (1+\alpha_0 M(\lambda)^*)^{-1}\gamma(\lambda)^* = (1+\alpha_0 M(\bar{\lambda}))^{-1}\gamma(\lambda)^* \ .$$

□

As in Corollary 5.4 we also get a Fourier representation for $\gamma'(\lambda)$, $\gamma'(\lambda)^*$ and $M'(\lambda)$ by using the representations from Lemma 5.11 and the Fourier transformation in (5.20).

Corollary 5.12. Let $d \geq 3$, $\alpha_0 \in \mathbb{R}$ and $\lambda \in \rho(A_0')$. Then the Fourier transformations of the γ-field, its adjoint and the Weyl function have the form

a) $\mathcal{F}_d\gamma'(\lambda)\phi(k) = \frac{2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}\mathcal{F}_{d-1}\phi(\tilde{k})}{\sqrt{2\pi}\,(|k|^2-\lambda)\left(2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}+\alpha_0\right)}$, $\qquad \forall \phi \in L^2(\mathbb{R}^{d-1}), k \in \mathbb{R}^d$,

b) $\mathcal{F}_{d-1}\gamma'(\lambda)^* f(\tilde{k}) = \frac{\sqrt{2}\,(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}}{\sqrt{\pi}\,\left(2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}+\alpha_0\right)}\int_{\mathbb{R}}\frac{\mathcal{F}_d f(k)}{|k|^2-\lambda}$, $\qquad \forall f \in L^2(\mathbb{R}^d), \quad \tilde{k} \in \mathbb{R}^{d-1}$,

c) $\mathcal{F}_{d-1}M'(\lambda)\phi(\tilde{k}) = \frac{\mathcal{F}_{d-1}\phi(\tilde{k})}{2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}+\alpha_0}$, $\qquad \forall \phi \in L^2(\mathbb{R}^{d-1}), \tilde{k} \in \mathbb{R}^{d-1}$.

Note that in a) and b) the notation $k = (\tilde{k}, k_d)$ was used, and that the indices d and $d-1$ indicate the dimension of the Fourier transformation.

The boundedness properties of Corollary 5.5 immediately transfer to $\gamma'(\lambda)$, $\gamma'(\lambda)^*$ and $M'(\lambda)$ by the representations in Lemma 5.11 and the boundedness (5.21).

Corollary 5.13. Let $d \geq 3$, $\alpha_0 \in \mathbb{R}$, $\lambda \in \rho(A_0')$ and $p \in [1,2]$. Then for every $s < \frac{3}{2} - (d-1)(\frac{1}{p} - \frac{1}{2})$, the restrictions

a) $\gamma'(\lambda)\restriction_{L^p(\mathbb{R}^{d-1})} : L^p(\mathbb{R}^{d-1}) \to H^s(\mathbb{R}^d)$,

b) $\gamma'(\lambda)^*\restriction_{L^p(\mathbb{R}^d)} : L^p(\mathbb{R}^d) \to H^s(\mathbb{R}^{d-1})$ and

c) $M'(\lambda)\restriction_{L^p(\mathbb{R}^{d-1})} : L^p(\mathbb{R}^{d-1}) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$

are bounded operators.

Finally, also the properties of $M(\lambda)$ in Lemma 5.6 transfer to $M'(\lambda)$ by the same reasons, namely Lemma 5.11 and (5.20).

Lemma 5.14. Let $d \geq 3$ $\alpha_0 \in \mathbb{R}$ and $\lambda \in \rho(A_0') \cap \mathbb{R}$. Then the Weyl function $M'(\lambda)$ is a bounded, self-adjoint and non-negative operator with an L^2-operator norm which converging like

$$\lim_{\lambda\to-\infty} \|M'(\lambda)\| = 0 \ .$$

Furthermore, the mapping $\lambda \mapsto M'(\lambda)$ is monotone increasing in the sense that for every $\lambda_1 \leq \lambda_2 \in (-\infty, -\frac{\alpha_0^2}{4})$ the quadratic form fulfils

$$\langle M'(\lambda_1)\phi,\phi\rangle_{L^2(\mathbb{R}^{d-1})} \leq \langle M'(\lambda_2)\phi,\phi\rangle_{L^2(\mathbb{R}^{d-1})} \ , \quad \forall \phi \in L^2(\mathbb{R}^{d-1}) \ .$$

The last property which immediately follows from the representations in Lemma 5.11 is the generalisation Proposition 5.7 to the γ-field $\gamma'(\lambda)$.

Proposition 5.15. Let $d \geq 3$, $\alpha_0 \in \mathbb{R}$ and $\lambda \in \rho(A_0')$. Then for every $\phi \in L^2(\mathbb{R}^{d-1})$ the γ-field $\gamma'(\lambda)$ fulfils for every $g \in H^1(\mathbb{R}^d)$ the identity

$$\langle \nabla\gamma'(\lambda)\phi, \nabla g\rangle_{L^2(\mathbb{R}^d)} - \lambda\langle\gamma'(\lambda)\phi, g\rangle_{L^2(\mathbb{R}^d)} + \alpha_0\langle M'(\lambda)\phi, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} = \langle\phi, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} \ . \tag{5.22}$$

Proof. Using the identities

$$\gamma'(\lambda) = \gamma(\lambda)(1+\alpha_0 M(\lambda))^{-1} \quad \text{and} \quad 1-\alpha_0 M'(\lambda) = (1+\alpha_0 M(\lambda))^{-1} \ ,$$

which follow immediately from Lemma 5.11, the identity (5.22) reduces to (5.10) by:

$$\begin{aligned}\langle \nabla\gamma'(\lambda)\phi, \nabla g\rangle_{L^2(\mathbb{R}^d)} - \lambda\langle\gamma'(\lambda)\phi, g\rangle_{L^2(\mathbb{R}^d)} &= \langle(1+\alpha_0 M(\lambda))^{-1}\phi, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} \\ &= \langle\phi, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} - \alpha_0\langle M'(\lambda)\phi, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} \ .\end{aligned}$$

□

We will end this chapter by giving an application of the Krein type formula (3.11) for the boundary triple $(L^2(\mathbb{R}^{d-1}), \Gamma_0', \Gamma_1')$. We will use it to give another version of the proof of Theorem 4.5.

Theorem 5.16. Let $d \geq 3$, $\alpha_0 \in \mathbb{R}$ and $\alpha \in L^p(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ be real-valued for some $p > \frac{4}{3}(d-1)$ and satisfying the decay property (4.18). Then the Schrödinger operator $A_{\alpha+\alpha_0}$ has the essential spectrum

$$\sigma_{\text{ess}}(A_{\alpha+\alpha_0}) = \begin{cases} [0,\infty) & \text{if } \alpha_0 \geq 0 \ , \\ [-\frac{\alpha_0^2}{4}, \infty) & \text{if } \alpha_0 \leq 0 \ . \end{cases}$$

Proof. We already know that for the operator

$$B\phi = -\alpha\phi \quad \text{and} \quad \text{dom}(B) = H^1(\mathbb{R}^{d-1}) \ ,$$

the corresponding extension $A_B' \restriction_{\text{dom}(A_B')}$ is given by

$$\begin{aligned}\text{dom}(A_B') &= \{\, f \in \text{dom}(T) \mid \Gamma_0' f = B\Gamma_1' f \,\} \\ &= \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \begin{array}{l} f_\pm \in H^{\frac{3}{2}}(\mathbb{R}^d_\pm) \\ \tau_D f_+ = \tau_D f_- \\ \tau_N f_+ + \tau_N f_- = -(\alpha+\alpha_0)\Gamma_1 f \end{array} \right\}\end{aligned}$$

and hence self-adjoint by Theorem 5.8. Therefore, we can use Theorem 3.10 and

Remark 3.11 for the decomposition

$$B_1 = -\operatorname{sign}(\alpha)|\alpha|^{\frac{1}{3}} \quad \text{with} \quad \operatorname{dom}(B_1) = \left\{ \phi \in L^2(\mathbb{R}^{d-1}) \;\middle|\; |\alpha|^{\frac{1}{3}}\phi \in L^2(\mathbb{R}^{d-1}) \right\} \quad \text{and}$$
$$B_2 = |\alpha|^{\frac{2}{3}} \quad \text{with} \quad \operatorname{dom}(B_2) = \left\{ \phi \in L^2(\mathbb{R}^{d-1}) \;\middle|\; |\alpha|^{\frac{2}{3}}\phi \in L^2(\mathbb{R}^{d-1}) \right\}$$

of B, to obtain the Krein type resolvent formula

$$(A_{\alpha+\alpha_0} - \lambda)^{-1} - (A_{\alpha_0} - \lambda)^{-1} = \underbrace{\gamma'(\lambda)B_1}_{\text{II}}\underbrace{(1 + B_2M'(\lambda)B_1)^{-1}}_{\text{I}}\underbrace{B_2\gamma'(\bar{\lambda})^*}_{\text{III}} \tag{5.23}$$

for every $\lambda \in \rho(A_{\alpha+\alpha_0}) \cap \rho(A_0')$. Note that the assumptions c) - e) of Theorem 3.10 are fulfilled by Lemma 5.10 and Lemma A.1.

In order to prove that $\sigma_{\text{ess}}(A_{\alpha+\alpha_0}) = \sigma_{\text{ess}}(A_{\alpha_0})$, it is sufficient to show that the right hand side of (5.23) is compact for some real-valued $\lambda_0 \in \rho(A_{\alpha_0}) \cap \rho(A_{\alpha+\alpha_0})$. This well be done in three steps. In the first step we will choose λ_0 such that I becomes bounded, In the second step we show that also II is bounded and in the third step we show the compactness of III.

Step I: In order to choose λ_0, we first not that by (5.20), the $H^s(\mathbb{R}^{d-1})$-operator norm of $(1 + \alpha_0 M(\lambda))^{-1}$ is bounded by

$$\|(1 + \alpha_0 M(\lambda))^{-1}\| \leq 1 + \frac{\alpha_0}{2(-\lambda)^{\frac{1}{2}}} \leq 2\,, \quad \forall \lambda \leq -\frac{\alpha_0^2}{4}\,.$$

For those $\lambda \leq -\frac{\alpha_0^2}{4}$ we then find constant $c_1, c_2 > 0$, similar to (5.12), such that for every $g \in H^1(\mathbb{R}^d)$

$$\||\alpha|^{\frac{2}{3}}\tau_D g\|^2_{L^2(\mathbb{R}^{d-1})} \leq \frac{1}{2}\|\nabla g\|^2_{L^2(\mathbb{R}^d)} + c_1\|g\|^2_{L^2(\mathbb{R}^d)} \quad \text{and} \tag{5.24}$$
$$\||\alpha|^{\frac{1}{3}}(1 + \alpha_0 M(\lambda))^{-1}\tau_D g\|^2 \leq \frac{1}{2}\|\nabla g\|^2_{L^2(\mathbb{R}^d)} + c_2\|g\|^2_{L^2(\mathbb{R}^d)}\,. \tag{5.25}$$

These are similar inequalities as (5.12) & (5.13) in the proof of Theorem 5.8. Choosing $\lambda_0 = -\max\{\frac{\alpha_0^2}{4}, 2c_1, 2c_2\}$ and following the same steps as we have done there, we will end up with

$$\|B_2M'(\lambda_0)B_1\phi\|^2 \leq \frac{1}{4}\|\phi\|^2_{L^2(\mathbb{R}^{d-1})}\,,$$

as we ended up in (5.17). This means that the operator norm of $B_2M'(\lambda_0)B_1$ is less than 1 and hence the inverse $(1 - B_2M'(\lambda_0)B_1)^{-1}$ is an everywhere defined bounded operator.

Step II: The boundedness of $\gamma'(\lambda_0)B_1 = \gamma'(\lambda_0)\operatorname{sign}(\alpha)|\alpha|^{\frac{1}{3}}$ is clear by Corollary 5.13

and Lemma A.1.

Step III: In order to prove the compactness of $|\alpha|^{2/3}\gamma'(\lambda_0)^*$ we have to ensure that for every bounded sequence $(f_n)_{n\in\mathbb{N}} \in L^2(\mathbb{R}^d)$ there exists a subsequence for which the images $(|\alpha|^{2/3}\gamma'(\lambda_0)^* f_{n_k})_{k\in\mathbb{N}}$ converge. Since by Corollary 5.13 $\gamma'(\lambda_0)^*$ is bounded as operator from $L^2(\mathbb{R}^d)$ into $H^1(\mathbb{R}^{d-1})$, there exists a weak convergent subsequence $(f_{n_k})_{k\in\mathbb{N}}$, such that

$$\phi_0 \coloneqq \underset{k\to\infty}{\operatorname{wlim}}{}_{H^1(\mathbb{R}^{d-1})} \gamma'(\lambda_0)^* f_{n_k} \ .$$

After the multiplication with the potential $|\alpha|^{\frac{2}{3}}$, this sequence then also converges in the L^2-norm

$$\lim_{k\to\infty} \| |\alpha|^{\frac{2}{3}} (\gamma'(\lambda_0)^* f_{n_k} - \phi_0) \|_{L^2(\mathbb{R}^{d-1})} = 0 \ ,$$

by Theorem A.4.

These three steps show that the right hand side of (5.23) is compact and hence the essential spectra of $A_{\alpha+\alpha_0}$ and A_{α_0} coincide. □

6. The Birman-Schwinger operator

In Theorem 4.5 we already calculated the essential spectrum of the Schrödinger operator $A_{\alpha+\alpha_0}$ for some $\alpha_0 \in \mathbb{R}$ and $\alpha \in L^{d-1}(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$, satisfying the decay property (4.18). The main content of the following two chapters will now be the description of the discrete spectrum $\sigma_{\mathrm{disc}}(A_{\alpha+\alpha_0})$. Even though for most of the results those restrictions are not necessary we will restrict ourselves to the case $\alpha_0 \leq 0$ and $\alpha \in L^p(\mathbb{R}^{d-1})$ being non-positive for some $p > d-1$. The reason for this is first of all simplicity (we will not need to distinguish different cases) and secondly we will only need the results in this special case later in Chapter 7. In Definition 6.1 we start with the Birman-Schwinger operator K_λ, which consists of the Weyl function $M'(\lambda)$ of the quasi boundary triple from Theorem 5.9, multiplied by the square root of the potential α from left and right. The main property of the Birman-Schwinger operator will be Theorem 6.3, which proves an equivalence between the discrete eigenvalues of $A_{\alpha+\alpha_0}$ and the 1-eigenvalues of $\overline{K}_\lambda$. This equivalence will then lead to Birman-Schwinger principle, Theorem 6.6, which is one of the main ingredients of the proof of the Lieb-Thirring inequality in Chapter 7.

6.1. Birman-Schwinger operator

Before we start, we recall Theorem 5.9 for the definition of the quasi boundary triple $(L^2(\mathbb{R}^{d-1}), \Gamma_0', \Gamma_1')$ of the shifted Laplacian $-\Delta + \alpha_0\delta_\Sigma$ with $M'(\lambda)$ its Weyl function and $A_0' = A_{\alpha_0}$ with the resolvent set

$$\rho(A_0') = \mathbb{C} \setminus [-\frac{\alpha_0^2}{4}, \infty)$$

in the special case $\alpha_0 \leq 0$.

Definition 6.1. Let $d \geq 3$, $\alpha_0 \leq 0$, $\lambda \in (-\infty, -\frac{\alpha_0^2}{4})$ and $\alpha \in L^p(\mathbb{R}^{d-1})$ being non-positive for some $p > d-1$. Then define the *Birman-Schwinger operator* K_λ by

$$K_\lambda \phi = |\alpha|^{\frac{1}{2}} M'(\lambda) |\alpha|^{\frac{1}{2}} \phi \quad \text{with} \quad \mathrm{dom}(K_\lambda) = H^{\frac{1}{2}}(\mathbb{R}^{d-1}) \,. \tag{6.1}$$

Note that by Lemma A.1 and Corollary 5.13, this operator is well defined.

Since the Birman-Schwinger operator is not everwhere defined, which means (for a bounded operator) that it is in particular not closed. This will be a problem when treating it as an compact operator or investigating its spectrum. However considering its everywhere defined closure $\overline{K}_\lambda$ instead will circumvent these issues.

Theorem 6.2. Let $d \geq 3$, $\alpha_0 \leq 0$, $\lambda \in (-\infty, -\frac{\alpha_0^2}{4})$ and $\alpha \in L^p(\mathbb{R}^{d-1})$ being non-positive for some $p > d-1$. Then $\overline{K}_\lambda$ is a self-adjoint, compact, non-negative operator. Moreover the mapping $\lambda \mapsto K_\lambda$ is monotone increasing in the sense that for every $\lambda_1 \leq \lambda_2 \in (-\infty, -\frac{\alpha_0^2}{4})$ the quadratic form fulfils

$$\langle K_{\lambda_1}\phi, \phi\rangle_{L^2(\mathbb{R}^{d-1})} \leq \langle K_{\lambda_2}\phi, \phi\rangle_{L^2(\mathbb{R}^{d-1})} , \quad \forall \phi \in L^2(\mathbb{R}^{d-1}) .$$

Proof. The fact that $\overline{K}_\lambda$ is self-adjoint, non-negative and monotone increasing follows directly from the respective properties of the Weyl function $M'(\lambda)$ in Lemma 5.14.

In order to prove the compactness, let $(\phi_n)_{n\in\mathbb{N}} \in \operatorname{dom}(K_\lambda)$ be a L^2-bounded sequence. In order the show the compactness of $\overline{K}_\lambda$ we have to verify the existence of a subsequence $(\phi_{n_k})_{k\in\mathbb{N}}$, such that the images $(K_\lambda\phi_{n_k})_{k\in\mathbb{N}}$ converge in $L^2(\mathbb{R}^{d-1})$. To do so, define the functions

$$\varphi_n := M'(\lambda)|\alpha|^{\frac{1}{2}}\phi_n , \quad \forall n \in \mathbb{N} ,$$

which are obviously bounded in $H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ by Lemma A.1 and Corollary 5.13. The reflexivity of the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ then ensures the existence of a weak convergent subsequence

$$\varphi_0 := \lim_{k\to\infty} \varphi_{n_k} \quad \text{in } H^{\frac{1}{2}}(\mathbb{R}^{d-1}) .$$

Since $|\alpha|^{\frac{1}{2}}$ is integrable and hence satisfies the decay property (4.18), we are allowed to apply Theorem A.4 and obtain the claimed norm convergence

$$\lim_{k\to\infty} \||\alpha|^{\frac{1}{2}}(\varphi_0 - \varphi_{n_k})\|_{L^2(\mathbb{R}^{d-1})} = 0 .$$

□

The next Theorem describes the main property of the Birman-Schwinger operator, namely the equivalence between its eigenvalues and the ones from the Schrödinger operator.

Theorem 6.3. Let $d \geq 3$, $\alpha_0 \leq 0$, $\lambda \in (-\infty, -\frac{\alpha_0^2}{4})$ and $\alpha \in L^p(\mathbb{R}^{d-1})$ be non-positive, for some $p > d-1$. Then the dimensions of the eigenspaces

$$\dim \ker(A_{\alpha+\alpha_0} - \lambda) = \dim \ker\left(\overline{K}_\lambda - 1\right)$$

are the same for the Schrödinger operator $A_{\alpha+\alpha_0}$ and the closure of the Birman-Schwinger operator $\overline{K}_\lambda$.

Proof. For the inequality " $\geq$ ", let $\phi \in \ker\left(\overline{K}_\lambda - 1\right)$. Then ϕ can be approximated by the L^2-Cauchy sequence $(\phi_n)_{n\in\mathbb{N}} \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$. By the boundedness of the operator $\gamma'(\lambda)|\alpha|^{\frac{1}{2}} : L^2(\mathbb{R}^{d-1}) \to H^1(\mathbb{R}^d)$, the functions $f_n \coloneqq \gamma'(\lambda)|\alpha|^{\frac{1}{2}}\phi_n$ form a H^1-Cauchy sequence and hence there exists an element $f_\phi \in H^1(\mathbb{R}^d)$, such that

$$\lim_{n\to\infty} \|f_\phi - f_n\|_{H^1(\mathbb{R}^d)} = 0 \ .$$

By the definition of the Birman-Schwinger operator in (6.1) and the fact that $\tau_D\gamma'(\lambda) = \Gamma_1'\gamma'(\lambda) = M'(\lambda)$, this function f_ϕ then satisfies the identity

$$\overline{K}_\lambda\phi = |\alpha|^{\frac{1}{2}}\tau_D f_\phi \ . \tag{6.2}$$

Moreover, the identity (5.22) evaluated for $|\alpha|^{\frac{1}{2}}\phi_n \in L^2(\mathbb{R}^{d-1})$ and viewed in the limit $n \to \infty$ gives

$$\langle \nabla f_\phi, \nabla g\rangle_{L^2(\mathbb{R}^d)} - \lambda\langle f_\phi, g\rangle_{L^2(\mathbb{R}^d)} + \alpha_0\langle \tau_D f_\phi, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} = \langle \phi, |\alpha|^{\frac{1}{2}}\tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} \ ,$$

for every $g \in H^1(\mathbb{R}^d)$. Using the form $a_{\alpha+\alpha_0}$ and (6.2), this equation can be rewritten as

$$a_{\alpha+\alpha_0}(f_\phi, g) - \lambda\langle f_\phi, g\rangle_{L^2(\mathbb{R}^d)} + \langle \overline{K}_\lambda\phi, |\alpha|^{\frac{1}{2}}\tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} = \langle \phi, |\alpha|^{\frac{1}{2}}\tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} \ .$$

The fact that $\phi \in \ker\left(\overline{K}_\lambda - 1\right)$ yields also the weak eigenvalue equation

$$a_{\alpha+\alpha_0}(f_\phi, g) - \lambda\langle f_\phi, g\rangle_{L^2(\mathbb{R}^d)} = 0$$

and proves that $f_\phi \in \ker(A_{\alpha+\alpha_0} - \lambda)$. So, for every element $\phi \in \ker\left(\overline{K}_\lambda - 1\right)$ we found a corresponding element $f_\phi \in \ker(A_{\alpha+\alpha_0} - \lambda)$. It remains to ensure that for linear independent $(\phi_i)_{i=1}^k \in \ker\left(\overline{K}_\lambda - 1\right)$, also the constructed $(f_{\phi_i})_{i=1}^k$ are linear independent. Let

$$(c_i)_{i=1}^k \in \mathbb{C} \ , \quad \text{with} \quad \sum_{i=1}^k c_i f_{\phi_i} = 0 \ .$$

Then by (6.2) and the fact that $\phi_i \in \ker(\overline{K}_\lambda - 1)$ this equation becomes

$$\sum_{i=1}^k c_i\phi_i = 0 \ .$$

Since the $(\phi_i)_{i=1}^k$ are linear independent, all the $c_i = 0$ have to vanish and hence also the $(f_{\phi_i})_{i=1}^n$ are linear independent as well.

For the inverse inequality " $\leq$ ", let $f \in \ker(A_{\alpha+\alpha_0} - \lambda)$ and define $\phi_f \coloneqq |\alpha|^{\frac{1}{2}}\tau_D f$.

Then it follows from the definition of $A_{\alpha+\alpha_0}$ that for every $g \in H^1(\mathbb{R}^d)$ the equation

$$\langle \nabla f, \nabla g\rangle_{L^2(\mathbb{R}^d)} - \langle \phi_f, |\alpha|^{\frac{1}{2}}\tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} + \alpha_0\langle \tau_D f, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} = \lambda\langle f, g\rangle_{L^2(\mathbb{R}^d)} \tag{6.3}$$

holds true. Choosing $(\phi_n)_{n\in\mathbb{N}} \in H^{\frac{1}{2}}(\mathbb{R}^d)$ an L^2-approximating sequence of ϕ_f, then the identity (5.22) evaluated for $|\alpha|^{\frac{1}{2}}\phi_n$ reads

$$\begin{aligned}\langle \nabla\gamma'(\lambda)|\alpha|^{\frac{1}{2}}\phi_n, \nabla g\rangle_{L^2(\mathbb{R}^d)} &- \lambda\langle\gamma'(\lambda)|\alpha|^{\frac{1}{2}}\phi_n, g\rangle_{L^2(\mathbb{R}^d)} \\ &+ \alpha_0\langle M'(\lambda)|\alpha|^{\frac{1}{2}}\phi_n, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} = \langle|\alpha|^{\frac{1}{2}}\phi_n, \tau_D g\rangle_{L^2(\mathbb{R}^{d-1})} \ .\end{aligned} \tag{6.4}$$

Comparing (6.3) & (6.4) gives the limit

$$f = \lim_{n\to\infty} \gamma'(\lambda)|\alpha|^{\frac{1}{2}}\phi_n \quad \text{in } H^1(\mathbb{R}^d) \tag{6.5}$$

of the H^1-Cauchy sequence $(\gamma'(\lambda)|\alpha|^{\frac{1}{2}}\phi_n)_{n\in\mathbb{N}}$. After applying the Dirichlet trace and multiplying $-|\alpha|^{\frac{1}{2}}$, this equation becomes

$$\begin{aligned}\phi_f &= \lim_{n\to\infty} |\alpha|^{\frac{1}{2}}\tau_D\gamma'(\lambda)|\alpha|^{\frac{1}{2}}\phi_n \\ &= \lim_{n\to\infty} K_\lambda\phi_n \\ &= \overline{K}_\lambda\phi_f \ ,\end{aligned}$$

which proves that $\phi_f \in \ker(\overline{K}_\lambda - 1)$. It remains to ensure that for linear independent $(f_i)_{i=1}^k \in \ker(A_{\alpha+\alpha_0} - \lambda)$ also the corresponding $(\phi_{f_i})_{i=1}^k$ are linear independent. Let

$$(c_i)_{i=1}^k \in \mathbb{C} \ , \quad \text{with} \quad \sum_{i=1}^k c_i\phi_{f_i} = 0 \ .$$

Then for every $i \in \{1, \ldots, k\}$ there exists an L^2-approximating sequence $(\phi_{i,n})_{n\in\mathbb{N}} \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$ of ϕ_{f_i}. By (6.5) we then observe that also the sum

$$\sum_{i=1}^k c_i f_i = \sum_{i=1}^k c_i \lim_{n\to\infty} \gamma'(\lambda)|\alpha|^{\frac{1}{2}}\phi_n = \gamma'(\lambda)|\alpha|^{\frac{1}{2}} \sum_{i=1}^k c_i\phi_{f_i} = 0$$

also vanishes. Since the $(f_i)_{i=1}^k$ were assumed to be linear independent all the coefficients c_i have to vanish and $(\phi_{f_i})_{i=1}^k$ are linear independent as well. □

6.2. Birman-Schwinger principle

The main theorem of this chapter will be the Birman-Schwinger principle, Theorem 6.6. In order to prove this, it is not enough to have this one-to-one correspondence of the eigenvalues from Theorem 6.3. In addition we will need the monotonicity and continuity properties of the eigenvalues from Lemma 6.5. The first step to prove this, is the characterisation of the eigenvalues by the min-max-principle.

Lemma 6.4 (Min-max-principle)**.** Let $d \geq 3$, $\alpha_0 \leq 0$, $\lambda \in (-\infty, -\frac{\alpha_0^2}{4})$ and $\alpha \in L^p(\mathbb{R}^{d-1})$ be non-positive for some $p > d-1$. Then the countably many positive eigenvalues $(\mu_n(\lambda))_{n\in\mathbb{N}}$ of $\overline{K}_\lambda$ in decreasing order (counted with multiplicity) can be represented by

$$\mu_n(\lambda) = \max_{\substack{S\leq L^2(\mathbb{R}^{d-1}) \\ \dim(S)=n}} \min_{\substack{\phi\in S \\ \|\phi\|=1}} \langle \overline{K}_\lambda \phi, \phi\rangle_{L^2(\mathbb{R}^{d-1})} = \min_{\substack{S\leq L^2(\mathbb{R}^{d-1}) \\ \dim(S)=n-1}} \max_{\substack{\phi\in S^\perp \\ \|\phi\|=1}} \langle \overline{K}_\lambda \phi, \phi\rangle_{L^2(\mathbb{R}^{d-1})} . \tag{6.6}$$

Note that $S \leq L^2(\mathbb{R}^{d-1})$ means that $S \subseteq L^2(\mathbb{R}^{d-1})$ is a linear subspace of $L^2(\mathbb{R}^{d-1})$.

Lemma 6.5. Let $d \geq 3$, $\alpha_0 \leq 0$ and $\alpha \in L^p(\mathbb{R}^{d-1})$ be non-positive for some $p > d-1$. Then $\mu_n(\cdot)$ is a monotone increasing, continuous function with $\lim_{\lambda\to-\infty} \mu_n(\lambda) = 0$.

Proof. The monotonicity of $\mu_n(\cdot)$ follows directly from the monotonicity of the Birman-Schwinger operator in Theorem 6.2 and the min-max-principle (6.6).

In the proof of the continuity we will prove the left- and the right-continuity separately. We will start with the left-continuity at some fixed point $\lambda_0 \in (-\infty, -\frac{\alpha_0^2}{4})$. Then for every $\lambda < \lambda_0$ and $\phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1})$, the weak difference between the two Birman-Schwinger operators has the representation

$$\begin{aligned}
&\langle (K_{\lambda_0} - K_\lambda)\phi, \phi\rangle_{L^2(\mathbb{R}^{d-1})} \\
&\qquad = \langle \mathcal{F}_{d-1}(M'(\lambda_0) - M'(\lambda))|\alpha|^{\frac{1}{2}}\phi, \mathcal{F}_{d-1}|\alpha|^{\frac{1}{2}}\phi\rangle_{L^2(\mathbb{R}^{d-1})} \\
&\qquad = \int_{\mathbb{R}^{d-1}} \left(\frac{1}{2(|\tilde{k}|^2-\lambda_0)^{\frac{1}{2}} + \alpha_0} - \frac{1}{2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}} + \alpha_0} \right) |\mathcal{F}_{d-1}|\alpha|^{\frac{1}{2}}\phi(\tilde{k})|^2 d\tilde{k} ,
\end{aligned} \tag{6.7}$$

where in the last equality we used Corollary 5.12. We will now estimate the bracket-term of the integrand by using that for every $x_0 > 1$ the basic inequality

$$\frac{1}{\sqrt{x_0}-1} - \frac{1}{\sqrt{x}-1} \leq \frac{x-x_0}{2(\sqrt{x_0}-1)^2\sqrt{x_0}} , \qquad \forall x \geq x_0 \tag{6.8}$$

holds true. Using this inequality for the values $x_0 = \frac{4(|\tilde{k}|^2-\lambda_0)}{\alpha_0^2}$ and $x = \frac{4(|\tilde{k}|^2-\lambda)}{\alpha_0^2}$, we find some constant $c_{\alpha_0,\lambda_0} > 0$ and obtain for every $\tilde{k} \in \mathbb{R}^{d-1}$ the estimate

$$\frac{1}{2(|\tilde{k}|^2-\lambda_0)^{\frac{1}{2}}+\alpha_0} - \frac{1}{2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}+\alpha_0} \leq c_{\lambda_0,\alpha_0} \frac{\lambda_0-\lambda}{2(|\tilde{k}|^2-\lambda_0)^{\frac{1}{2}}+\alpha_0} \,. \tag{6.9}$$

Using this estimate in (6.7) we get

$$\begin{aligned}\langle (K_{\lambda_0}-K_\lambda)\phi,\phi\rangle_{L^2(\mathbb{R}^{d-1})} &\leq c_{\lambda_0,\alpha_0}(\lambda_0-\lambda)\int_{\mathbb{R}^{d-1}} \frac{|\mathcal{F}_{d-1}|\alpha|^{\frac{1}{2}}\phi(\tilde{k})|^2}{2(|\tilde{k}|^2-\lambda_0)^{\frac{1}{2}}+\alpha_0} d\tilde{k} \\ &= c_{\lambda_0,\alpha_0}(\lambda_0-\lambda)\langle K_{\lambda_0}\phi,\phi\rangle_{L^2(\mathbb{R}^{d-1})} \\ &\leq c_{\lambda_0,\alpha_0}(\lambda_0-\lambda)\, \|K_{\lambda_0}\|\, \|\phi\|^2_{L^2(\mathbb{R}^{d-1})} \,.\end{aligned}$$

Since this inequality is true for every $\phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}) = \mathrm{dom}(K_\lambda)$, it can be extended by continuity to closure of the Birman-Schwinger operator, to obtain

$$\langle (\overline{K}_{\lambda_0}-\overline{K}_\lambda)\phi,\phi\rangle_{L^2(\mathbb{R}^{d-1})} \leq c_{\lambda_0,\alpha_0}(\lambda_0-\lambda)\, \|K_{\lambda_0}\|\, \|\phi\|^2_{L^2(\mathbb{R}^{d-1})} \,, \qquad \forall \phi \in L^2(\mathbb{R}^{d-1}) \,. \tag{6.10}$$

In order to expand this inequality to the function μ_n by the min-max-principle (6.6) we consider subspaces $S,T \subseteq L^2(\mathbb{R}^{d-1})$ with $\dim(S) = n-1$ and $\dim(T) = n$. Then obviously $S^\perp \cap T \neq \{0\}$ by their dimensions and there exists some $\phi_0 \in S^\perp \cap T$ with $\|\phi_0\|_{L^2(\mathbb{R}^{d-1})} = 1$. This ϕ_0 then clearly satisfies (6.10) and therefore

$$\min_{\substack{\phi\in T\\ \|\phi\|=1}} \langle \overline{K}_{\lambda_0}\phi,\phi\rangle_{L^2(\mathbb{R}^{d-1})} - \max_{\substack{\phi\in S^\perp\\ \|\phi\|=1}} \langle \overline{K}_{\lambda}\phi,\phi\rangle_{L^2(\mathbb{R}^{d-1})} \leq c_{\lambda_0,\alpha_0}(\lambda_0-\lambda)\, \|K_{\lambda_0}\| \,.$$

Since the subspaces S,T were arbitrary, we validate the convergence

$$\mu_n(\lambda_0)-\mu_n(\lambda) \leq c_{\lambda_0,\alpha_0}\, \|K_{\lambda_0}\|\, (\lambda_0-\lambda) \overset{\lambda\to\lambda_0^-}{\longrightarrow} 0 \,,$$

by the min-max-principle (6.6), which proofs the left-continuity of $\mu_n(\cdot)$.

The right-continuity follows the same steps if one replaces (6.8) by the following inequality: For every $y_0 > 1$ there exists some $c \in (0,y_0)$, such that the basic inequality

$$\frac{1}{\sqrt{x}-1} - \frac{1}{\sqrt{x_0}-1} \leq \frac{x_0-x}{(\sqrt{x_0}-1)^2\sqrt{x_0}} \,, \qquad \forall x_0 \geq y_0 \,, x \in (x_0-c,x_0)$$

holds true.

The third and final part of the proof is the limit $\lim\limits_{\lambda\to-\infty} \mu_n(\lambda)$. As in (6.7) we get

$$\langle K_\lambda\phi,\phi\rangle_{L^2(\mathbb{R}^{d-1}} = \int_{\mathbb{R}^{d-1}} \frac{|\mathcal{F}_{d-1}|\alpha|^{\frac{1}{2}}\phi(\tilde{k})|^2}{2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}+\alpha_0} d\tilde{k} \,, \qquad \forall \phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}) \,. \tag{6.11}$$

Defining the constant

$$c_\lambda := \inf_{\tilde{k}\in\mathbb{R}^{d-1}} \frac{(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}}{2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}+\alpha_0} = \frac{(-\lambda)^{\frac{1}{2}}}{2(-\lambda)^{\frac{1}{2}}+\alpha_0} ,$$

we can estimate (6.11) by

$$\begin{aligned}\langle K_\lambda\phi,\phi\rangle_{L^2(\mathbb{R}^{d-1}} &\leq c_\lambda \int_{\mathbb{R}^{d-1}} \frac{|\mathcal{F}_{d-1}|\alpha|^{\frac{1}{2}}\phi(\tilde{k})|^2}{(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}} d\tilde{k} \\ &\leq c_\lambda \left(\int_{\mathbb{R}^{d-1}} \frac{1}{(|\tilde{k}|^2-\lambda)^{\frac{p}{2}}} d\tilde{k}\right)^{\frac{1}{p}} \|\mathcal{F}_{d-1}|\alpha|^{\frac{1}{2}}\phi\|^2_{L^{\frac{2p}{p-1}}(\mathbb{R}^{d-1})} ,\end{aligned}$$

where in the second estimate the Hölder inequality was applied. By Theorem 1.18 and Lemma A.1 the norms of the Fourier transformations on the right hand side are finite. Moreover, the integral converges as well, because of $p > d-1$, but since we want to investigate the limit $\lambda \to -\infty$ we also need its λ-dependence, which is given by

$$\int_{\mathbb{R}^{d-1}} \frac{1}{(|\tilde{k}|^2-\lambda)^{\frac{p}{2}}} d\tilde{k} \propto (-\lambda)^{-\frac{p-d+1}{2}} .$$

Alltogether we end up with the inequality

$$\langle K_\lambda\phi,\phi\rangle_{L^2(\mathbb{R}^{d-1})} \leq c_\alpha c_\lambda (-\lambda)^{-\frac{p-d+1}{2p}} \|\phi\|^2_{L^2(\mathbb{R}^{d-1})} , \quad \forall \phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}) .$$

To conclude the convergence of μ_n from this inequality, we first extend it by continuity to $\overline{K}_\lambda$ and then by the min-max-principle also to

$$\mu_n(\lambda) \leq c_\alpha c_\lambda (-\lambda)^{-\frac{p-d+1}{2p}} .$$

Since $p > d-1$ and $\lim_{\lambda\to-\infty} c_\lambda = 1$, this proves that $\lim_{\lambda\to-\infty} \mu_n(\lambda) = 0$. □

With Theorem 6.3 and Lemma 6.5 we are now ready to prove the important Birman-Schwinger principle, which allows us to count the eigenvalues of the Birman-Schwinger operator instead the ones from the Schrödinger operator.

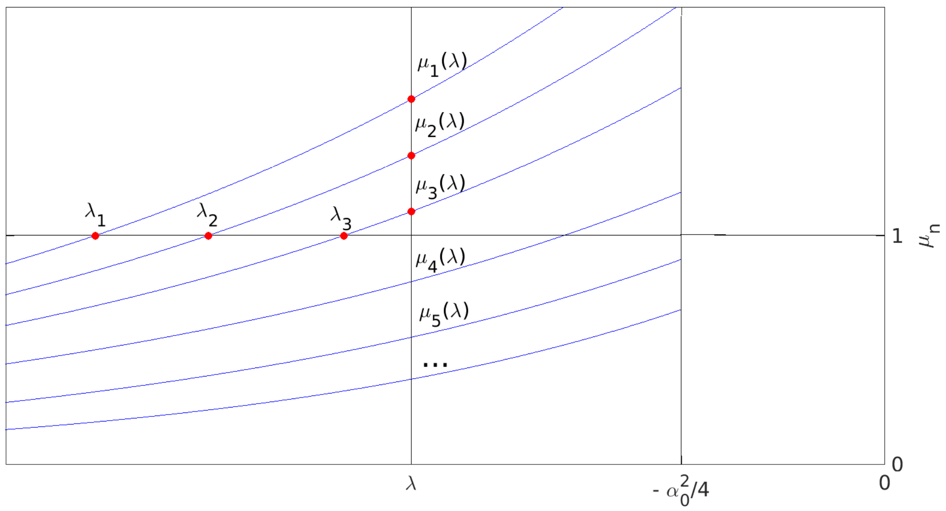
Theorem 6.6. Let $d \geq 3$, $\alpha_0 \leq 0$ and $\alpha \in L^p(\mathbb{R}^{d-1})$ non-positive for some $p > d-1$. If we define for every $\lambda \in (-\infty, -\frac{\alpha_0^2}{4})$,

a) $N_\lambda(\alpha+\alpha_0)$ as the number of eigenvalues of $A_{\alpha+\alpha_0}$ smaller or equal then λ, and

b) $B_\lambda(\alpha)$ as the number of eigenvalues of $\overline{K}_\lambda$ larger or equal than 1.

Then we get $N_\lambda(\alpha+\alpha_0) = B_\lambda(\alpha)$.

Proof. From Lemma 6.5 it follows, that for every $n \in \mathbb{N}$ with $\mu_n(\lambda) \geq 1$, there exists exactly one $\lambda_n \leq \lambda$, such that $\mu_n(\lambda_n) = 1$. From Theorem 6.3 we know, that λ_n is then an eigenvalue of $A_{\alpha+\alpha_0}$ with the same multiplicity. Altogether we get that the number of eigenvalues of $\overline{K}_\lambda$ greater or equal than 1 equals to the number of eigenvalues of $A_{\alpha+\alpha_0}$ which are smaller or equal than λ.



□

7. Lieb-Thirring inequality on the hyperplane

The goal of this chapter and also the main goal of the whole thesis, is, to derive an Lieb-Thirring type inequality for the discrete spectrum of the Schrödinger operator $A_{\alpha+\alpha_0}$ with some integrable potential α and some negative constant $\alpha_0 < 0$. This means that we want to obtain an upper bound for the sum over all discrete eigenvalues of $A_{\alpha+\alpha_0}$ to some power γ. Furthermore, this upper bound should only consist of some constant and the integral over the potential strength α to some power.

It is clear that if one adds a positive constant δ, the potential becomes larger and hence all eigenvalues of the Schrödinger operator shift to the right. The following Lemma gives an upper bound for this shift in terms of the corresponding sesquilinear form of the operator.

Lemma 7.1. Let $d \geq 3$, $\delta \geq 0$ and $\alpha \in L^p(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ be real-valued for some $p > d-1$. Let a_α and $a_{\alpha+\delta}$ be the forms (4.2) with respect to the potentials α and $\alpha + \delta$. Then for every $\xi \in (0, \xi_\alpha)$ and $f \in H^1(\mathbb{R}^d)$ we get the inequality

$$a_{\alpha+\delta}(f,f) \leq \left(1 + \frac{\xi\delta}{1 - \frac{\xi}{\xi_\alpha}}\right) a_\alpha(f,f) + \frac{\delta}{1 - \frac{\xi}{\xi_\alpha}} \left(\frac{c}{\xi^\theta} + \xi\right) \|f\|^2_{L^2(\mathbb{R}^d)} ,$$

for some constant $c > 0$, $\theta = \frac{p+d-1}{p-d+1}$ and $\frac{1}{\xi_\alpha} = c^2_{\frac{d-1}{2p}, \frac{2p}{p-1}} \|u\|_p + \|v\|_\infty$ from Corollary 1.9 and some decomposition $\alpha = u + v$ for $u \in L^p(\mathbb{R}^{d-1}), v \in L^\infty(\mathbb{R}^{d-1})$.

Proof. First of all, notice that by Lemma A.1 and Corollary 1.9 we obtain the inequality

$$\||\alpha|^{\frac{1}{2}}\phi\|^2_{L^2(\mathbb{R}^{d-1})} \leq \frac{1}{\xi_\alpha}\|\phi\|^2_{H^{\frac{d-1}{2p}}(\mathbb{R}^{d-1})} , \quad \forall \phi \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}) . \tag{7.1}$$

For some arbitrary constant $a \geq 0$ one can now estimate for every $f \in H^1(\mathbb{R}^d)$ the sesquilinear form

$$\begin{aligned} a_{\alpha+\delta}(f,f) &= \|\nabla f\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^{d-1}} \alpha\, |\tau_D f|^2 d\tilde{x} + \delta \|\tau_D f\|^2_{L^2(\mathbb{R}^{d-1})} \\ &\leq \|\nabla f\|^2_{L^2(\mathbb{R}^d)} + (1+a) \int_{\mathbb{R}^{d-1}} \alpha\, |\tau_D f|^2 d\tilde{x} \\ &\qquad + a \||\alpha|^{\frac{1}{2}} \tau_D f\|^2_{L^2(\mathbb{R}^{d-1})} + \delta \|\tau_D f\|^2_{L^2(\mathbb{R}^{d-1})} \end{aligned}$$

With the estimate (7.1) and Lemma 1.30, we obtain

$$\begin{aligned} a_{\alpha+\delta}(f,f) &\leq \|\nabla f\|^2_{L^2(\mathbb{R}^d)} + (1+a) \int_{\mathbb{R}^{d-1}} \alpha\, |\tau_D f|^2 d\tilde{x} \\ &\qquad + \|\tau_D\|^2 \left(\frac{a}{\xi_\alpha} + \delta \right) \|f\|^2_{H^{\frac{d-1}{2p}+\frac{1}{2}}(\mathbb{R}^d)} . \end{aligned}$$

The Sobolev norm of f will be estimated by Theorem 1.27 and give

$$\begin{aligned} a_{\alpha+\delta}(f,f) &\leq \|\nabla f\|^2_{L^2(\mathbb{R}^d)} + (1+a) \int_{\mathbb{R}^{d-1}} \alpha\, |\tau_D f|^2 d\tilde{x} \\ &\qquad + \left(\frac{a}{\xi_\alpha} + \delta \right) \left(\xi \|f\|^2_{H^1(\mathbb{R}^d)} + \frac{c}{\xi^\theta} \|f\|^2_{L^2(\mathbb{R}^d)} \right) \\ &= \left(1 + \xi \left(\frac{a}{\xi_\alpha} + \delta \right) \right) \|\nabla f\|^2_{L^2(\mathbb{R}^d)} + (1+a) \int_{\mathbb{R}^{d-1}} \alpha\, |\tau_D f|^2 d\tilde{x} \\ &\qquad + \left(\frac{a}{\xi_\alpha} + \delta \right) \left(\frac{c}{\xi^\theta} + \xi \right) \|f\|^2_{L^2(\mathbb{R}^d)} . \end{aligned}$$

Choosing now $a = \frac{\xi\delta}{1-\frac{\xi}{\xi_\alpha}}$, which is allowed because of the restrictions $\delta \geq 0$ and $\xi \in (0, \xi_\alpha)$, the estimate of the form finally becomes

$$a_{\alpha+\delta}(f,f) \leq \left(1 + \frac{\xi\delta}{1-\frac{\xi}{\xi_\alpha}} \right) a_\alpha(f,f) + \frac{\delta}{1-\frac{\xi}{\xi_\alpha}} \left(\frac{c}{\xi^\theta} + \xi \right) \|f\|^2_{L^2(\mathbb{R}^d)} .$$

□

Once we have bounded the shift of the eigenvalues in Lemma 7.1 we can translate this bound into an inequality for the number of eigenvalues below some bound. This means in detail that if consider all the eigenvalues of $A_{\alpha+\alpha_0}$ below some λ and choose some $\tilde{\lambda} \geq \lambda$, then we find some shift $\delta \geq 0$, which is linear in $\tilde{\lambda} - \lambda$, such that the respective eigenvalues of $A_{\alpha+\alpha_0+\delta}$ stay below $\tilde{\lambda}$.

Lemma 7.2. Let $d \geq 3$, $\alpha_0 \leq 0$ and $\alpha \in L^p(\mathbb{R}^{d-1})$ be real-valued for some $p > d-1$.

For $\lambda < -\frac{\alpha_0^2}{4}$ and $\tilde{\lambda} \in [\lambda, -\frac{\alpha_0^2}{4})$ choose

$$\delta = (\tilde{\lambda} - \lambda)\frac{\xi_{\text{sol}}^{\theta}}{c(\theta+1)} \ ,$$

where $c, \xi_{\alpha+\alpha_0}$ and θ are from Lemma 7.1 and $\xi_{\text{sol}} \in (0, \xi_{\alpha+\alpha_0})$ is the unique solution of

$$\xi_{\text{sol}}^{1+\theta} + \frac{c(1+\theta)}{\xi_{\alpha+\alpha_0}}\xi_{\text{sol}} - c\theta = 0 \ . \tag{7.2}$$

With N_λ form Theorem 6.6 we get the estimate

$$N_\lambda(\alpha + \alpha_0) \leq N_{\tilde{\lambda}}(\alpha + \alpha_0 + \delta) \ .$$

Proof. First we have to check that (7.2) has indeed a unique solution in the interval $(0, \xi_\alpha)$. The function $f(\xi) = \xi^{1+\theta} + \frac{c(1+\theta)}{\xi_{\alpha+\alpha_0}}\xi - \theta c$ evaluated at 0 and ξ_0 has the values

$$f(0) = -\theta c < 0 \quad \text{and} \quad f(\xi_{\alpha+\alpha_0}) = \xi_{\alpha+\alpha_0}^{1+\theta} + c > 0 \ .$$

Moreover, its derivative

$$f'(\xi) = (1+\theta)\xi^{\theta} + \frac{c(1+\theta)}{\xi_{\alpha+\alpha_0}} > 0$$

is strictly positive, which confirms that there exists exactly one zero of (7.2) in the interval $(0, \xi_{\alpha+\alpha_0})$.

To begin with the actual proof, let $E_n(\alpha + \alpha_0) \leq \lambda$ be the n-th discrete eigenvalue of $A_{\alpha+\alpha_0}$, counted with multiplicity. Then Lemma 7.1 and the min-max-principle [64, Theorem XIII.1] gives for every $\xi \in (0, \xi_{\alpha+\alpha_0})$ the upper bound

$$E_n(\alpha + \alpha_0 + \delta) \leq \left(1 + \frac{\xi\delta}{1 - \frac{\xi}{\xi_{\alpha+\alpha_0}}}\right)\lambda + \frac{\delta}{1 - \frac{\xi}{\xi_{\alpha+\alpha_0}}}\left(\frac{c}{\xi^{\theta}} + \xi\right) =: M(\xi)$$

for the n-th eigenvalue $E_n(\alpha + \alpha_0 + \delta)$ of $A_{\alpha+\alpha_0+\delta}$. If we now manage to find a $\xi \in (0, \xi_{\alpha+\alpha_0})$, for which $M(\xi) \leq \tilde{\lambda}$, then we are finished, because in this case we ensured that for every $E_n(\alpha + \alpha_0) \leq \lambda$ the corresponding $E_n(\alpha + \alpha_0 + \delta) \leq \tilde{\lambda}$ and the stated inequality of the number of eigenvalues $N_\lambda(\alpha + \alpha_0) \leq N_{\tilde{\lambda}}(\alpha + \alpha_0 + \delta)$ holds true. To find the minimum of the bound $M(\xi)$, we first calculate its limits as $\xi \to 0$ and $\xi \to \xi_{\alpha+\alpha_0}$.

Consider some $f \in H^1(\mathbb{R}^d)$. Since $\delta \geq 0$, we trivially obtain that $a_{\alpha+\alpha_0}(f, f) \leq$

$a_{\alpha+\alpha_0+\delta}(f,f)$. Together with Lemma 7.1 this gives the estimate

$$0 \leq \xi a_{\alpha+\alpha_0}(f,f) + \left(\frac{c}{\xi^\theta} + \xi\right) \|f\|^2_{L^2(\mathbb{R}^d)} , \quad \forall \xi \in (0, \xi_{\alpha+\alpha_0}) .$$

The min-max-principle [64, Theorem XIII.1] and the choice $E_n(\alpha+\alpha_0) \leq \lambda$ makes this inequality

$$\frac{c}{\xi^\theta} + \xi(1+\lambda) \geq 0 , \quad \forall \xi \in (0, \xi_{\alpha+\alpha_0}) , \tag{7.3}$$

which ensures that

$$\lim_{\xi \to 0} M(\xi) = \infty \quad \text{and} \quad \lim_{\xi \to \xi_{\alpha+\alpha_0}} M(\xi) = \infty .$$

Since $M(\xi)$ converges to $+\infty$ on both ends of the interval $(0, \xi_{\alpha+\alpha_0})$, there has to be a minimum $\xi_{\min}$ somewhere in the interior, which we will calculate by differentiating

$$\frac{d}{d\xi} M(\xi) = \frac{\delta \xi^{\theta+1}}{c(1-\frac{\xi}{\xi_{\alpha+\alpha_0}})^2} \left(\frac{1+\lambda}{c}\xi^{1+\theta} + \frac{1+\theta}{\xi_{\alpha+\alpha_0}}\xi - \theta\right) = 0 .$$

The minimum $\xi_{\min}$ therefore has to satisfy the equation

$$\frac{1+\lambda}{c}\xi_{\min}^{1+\theta} + \frac{1+\theta}{\xi_{\alpha+\alpha_0}}\xi_{\min} - \theta = 0 \tag{7.4}$$

and the minimal bound $M(\xi_{\min})$ has the value

$$M(\xi_{\min}) = \lambda + \frac{\delta c(1+\theta)}{\xi_{\min}^\theta} = \lambda + (\tilde{\lambda} - \lambda)\left(\frac{\xi_{\text{sol}}}{\xi_{\min}}\right)^\theta \tag{7.5}$$

by the choice of δ. Define the functions $f_{\min}(\xi) = \frac{1+\lambda}{c}\xi^{1+\theta} + \frac{1+\theta}{\xi_0}\xi - \theta$ and $f_{\text{sol}}(\xi) = \frac{1}{c}\xi^{1+\theta} + \frac{1+\theta}{\xi_0}\xi - \theta$.

Using that (7.3) also holds in the limit $\xi \to \xi_{\alpha+\alpha_0}$, we can estimate the derivatives of these functions by

$$f'_{\min}(\xi) = \frac{1+\lambda}{c}(1+\theta)\xi^\theta + \frac{1+\theta}{\xi_{\alpha+\alpha_0}} \geq \frac{1+\theta}{\xi_{\alpha+\alpha_0}}\left(1 - \frac{\xi^\theta}{\xi_{\alpha+\alpha_0}^\theta}\right) \geq 0$$
$$f'_{\text{sol}}(\xi) = \frac{1+\theta}{c}\xi^\theta + \frac{1+\theta}{\xi_{\alpha+\alpha_0}} \geq 0 ,$$

and obtain that both of them are non-negative. So we are faced with functions $f_{\min}$ and f_{sol} which are monotone increasing and satisfy $f_{\min} \leq f_{\text{sol}}$. This in particular implies that the zero ξ_{sol} of f_{sol} has to be smaller than the zero $\xi_{\min}$ of $f_{\min}$, which proves that $M(\xi_{\min}) \leq \tilde{\lambda}$ in (7.5). $\square$

Theorem 7.3. Let $d \geq 3$, $\gamma > \frac{d-1}{2}$, $0 < \eta < \frac{1}{2}(\gamma - \frac{d-1}{2})$ and $\alpha \in L^{d-1+\gamma+\eta}(\mathbb{R}^{d-1}) \cap L^{\frac{d-1}{2}+\gamma}(\mathbb{R}^{d-1})$ be real-valued. Furthermore, let $\alpha_0 < 0$ and $(E_i)_i$ the (at most countable many) discrete eigenvalues of $A_{\alpha+\alpha_0}$, which are smaller than $-\frac{\alpha_0^2}{4}$. Then the following estimate holds true:

$$\sum_i \left| E_i + \frac{\alpha_0^2}{4} \right|^\gamma \leq L_{\gamma,\eta}(\delta) \left(\int_{\mathbb{R}^{d-1}} \alpha_-^{d-1+\gamma+\eta} d\tilde{x} + (-\alpha_0)^{d-1+2\eta} \int_{\mathbb{R}^{d-1}} \alpha_-^{\frac{d-1}{2}+\gamma} d\tilde{x} \right) , \quad (7.6)$$

where ξ_{sol} is the solution of (7.2) and depends on $\|\alpha\|_{L^{d-1+\gamma+\eta}(\mathbb{R}^{d-1})}$ as well as on α_0.

Proof. Let $\varepsilon > 0$, $\lambda = -\frac{\alpha_0^2}{4} - \varepsilon$ and $\tilde{\lambda} = -\frac{\alpha_0^2}{4} - \frac{\varepsilon}{2}$. Furthermore, let $\delta = \frac{\xi_{\text{sol}}^\theta}{c(\theta+1)}$ with the constants ξ_{sol}, c and θ from Lemma 7.2, which then shows that

$$N_\lambda(\alpha + \alpha_0) \leq N_{\tilde{\lambda}} \left(\alpha + \alpha_0 + \frac{\varepsilon}{2}\delta \right) . \quad (7.7)$$

Furthermore, define

$$\alpha_\varepsilon := \min \left\{ \alpha + \frac{\varepsilon}{2}\delta, 0 \right\} \quad (7.8)$$

the negative part of the shifted potential. The Birman-Schwinger principle, Theorem 6.6 then gives

$$B_{\tilde{\lambda}}(\alpha_\varepsilon) = N_{\tilde{\lambda}}(\alpha_\varepsilon + \alpha_0) . \quad (7.9)$$

As a third relation related to this number of eigenvalues we notice that cutting off the positive part of the potential in (7.8) makes the potential smaller and by the min-max principle [64, Theorem XIII.2], this means that in this case all the eigenvalues become smaller as well. Consequently, the number of eigenvalues below $\tilde{\lambda}$ increases:

$$N_{\tilde{\lambda}} \left(\alpha + \alpha_0 + \frac{\varepsilon}{2}\delta \right) \leq N_{\tilde{\lambda}}(\alpha_\varepsilon + \alpha_0) . \quad (7.10)$$

Combining now (7.7), (7.9) & (7.10) gives the estimate

$$N_\lambda(\alpha + \alpha_0) \leq B_{\tilde{\lambda}}(\alpha_\varepsilon) , \quad (7.11)$$

from which we will now investigate both sides seperately.

Step 1: Denote the Birman-Schwinger operator (6.1) corresponding to the potential α_ε by K_λ. Since $\overline{K}_\lambda$ is self-adjoint, compact and non-negative by Theorem 6.2, it has at most countably many non-negative eigenvalues $(\mu_n(\lambda))_{n\in\mathbb{N}}$. For these eigenvalues we obtain the estimate

$$B_\lambda(\alpha_\varepsilon) = \sum_{\substack{n=1 \\ \mu_n(\lambda)\geq 1}}^\infty 1 \leq \sum_{\substack{n=1 \\ \mu_n(\lambda)\geq 1}}^\infty \mu_n(\lambda) \leq \sum_{\substack{n=1 \\ \mu_n(\lambda)\geq 1}}^\infty \mu_n(\lambda)^{d-1+2\eta} \leq \sum_{n=1}^\infty \mu_n(\lambda)^{d-1+2\eta} .$$

The last sum in this chain of estimates is the sum over the $(d-1+2\eta)$-th powers of the eigenvalues of the non-negative, self-adjoint operator $\overline{K}_\lambda$. Using the functional calculus, this is the same as the sum over the eigenvalues of the respective power $\overline{K}_\lambda^{d-1+2\eta}$, which can be written as the trace

$$B_\lambda(\alpha_\varepsilon) \leq \operatorname{tr}\left(\overline{K}_\lambda^{d-1+2\eta}\right) .$$

By a theorem of Araki [[10], Theorem 1], applied on $K_\lambda = |\alpha_\varepsilon|^{\frac{1}{2}} M'(\lambda) |\alpha_\varepsilon|^{\frac{1}{2}}$, we get the inequality

$$\operatorname{tr}\left(\overline{K}_\lambda^{d-1+2\eta}\right) \leq \operatorname{tr}\left(\overline{|\alpha_\varepsilon|^{\frac{d-1}{2}+\eta} M'(\lambda)^{d-1+2\eta} |\alpha_\varepsilon|^{\frac{d-1}{2}+\eta}}\right) .$$

With this inequality the problem of calculating the power of the Birman-Schwinger operator $\overline{K}_\lambda$ is reduced to the problem of calculating the power of the Weyl function $M'(\lambda)$. We will now ensure that $M'(\lambda)^{d-1+2\eta}$ admits an integral representation with some integral kernel $\widetilde{G}_\lambda^{(d-1+2\eta)}$.

From Corollary 5.12 we know that in Fourier space the Weyl function looks like

$$\mathcal{F}_{d-1} M'(\lambda)\phi(\tilde{k}) = \frac{\mathcal{F}_{d-1}\phi(\tilde{k})}{2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}+\alpha_0} , \quad \forall \phi \in L^2(\mathbb{R}^{d-1}), \tilde{k} \in \mathbb{R}^{d-1} .$$

Since the Fourier transformation is unitary in $L^2(\mathbb{R}^{d-1})$, it can be interchanged with the operation of taking the power of the operator:

$$\mathcal{F}_{d-1} M'(\lambda)^{d-1+2\eta}\phi(\tilde{k}) = \frac{\mathcal{F}_{d-1}\phi(\tilde{k})}{\left(2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}+\alpha_0\right)^{d-1+2\eta}} , \quad \forall \phi \in L^2(\mathbb{R}^{d-1}), \tilde{k} \in \mathbb{R}^{d-1} .$$

Since $\left(2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}+\alpha_0\right)^{-(d-1+2\eta)} \in L^1(\mathbb{R}^{d-1})$, we can define the integral kernel

$$\widetilde{G}_\lambda^{(d-1+2\eta)} := \frac{1}{(2\pi)^{\frac{d-1}{2}}} \mathcal{F}_{d-1}^{-1}\left[\frac{1}{\left(2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}+\alpha_0\right)^{d-1+2\eta}}\right] \in L^\infty(\mathbb{R}^{d-1})$$

and obtain the representation

$$M'(\lambda)^{d-1+2\eta}\phi = \widetilde{G}_\lambda^{(d-1+2\eta)} * \phi , \quad \forall \phi \in L^2(\mathbb{R}^{d-1})$$

of the Weyl function as a convolution.
Since $\alpha_\varepsilon \in L^{d-1+2\eta}(\mathbb{R}^{d-1})$, its power $|\alpha_\varepsilon|^{\frac{d-1}{2}+\eta}$ is in $L^2(\mathbb{R}^{d-1})$. Moreover, because of

$\widetilde{G}_\lambda^{(d-1+2\eta)} \in L^\infty(\mathbb{R}^{d-1})$, the convolution $\varphi \mapsto \widetilde{G}_\lambda^{(d-1+2\eta)} * \varphi$ is bounded from $L^1(\mathbb{R}^{d-1})$ into $L^\infty(\mathbb{R}^{d-1})$. These two regularity properties ensure the representation

$$\overline{|\alpha_\varepsilon|^{\frac{d-1}{2}+\eta} M'(\lambda)^{d-1+2\eta} |\alpha_\varepsilon|^{\frac{d-1}{2}+\eta}}\, \phi = |\alpha_\varepsilon|^{\frac{d-1}{2}+\eta} \left(\widetilde{G}_\lambda^{(d-1+2\eta)} * (|\alpha_\varepsilon|^{\frac{d-1}{2}+\eta}\phi) \right)$$

of the closure, since the right hand side is an everywhere defined bounded operator in $L^2(\mathbb{R}^{d-1})$. The trace of such an integral operator is then given by the integral over the diagonal of the kernel and looks like

$$\begin{aligned} B_\lambda(\alpha_\varepsilon) &\le \int_{\mathbb{R}^{d-1}} |\alpha_\varepsilon(\tilde{x})|^{\frac{d-1}{2}+\eta} \widetilde{G}_\lambda^{(d-1+2\eta)}(0) |\alpha_\varepsilon(\tilde{x})|^{\frac{d-1}{2}+\eta} d\tilde{x} \\ &= c_1 \int_{\mathbb{R}^{d-1}} \frac{1}{\left((|\tilde{k}|^2 - \lambda)^{\frac{1}{2}} + \frac{\alpha_0}{2} \right)^{d-1+2\eta}} d\tilde{k} \int_{\mathbb{R}^{d-1}} |\alpha_\varepsilon(\tilde{x})|^{d-1+2\eta} d\tilde{x} \ , \end{aligned}$$

for some constant $c_1 > 0$. After changing the notation from λ to ε, defined in the very beginning of the proof, and using the basic inequality

$$\frac{1}{\sqrt{x+1}-1} \le \frac{\sqrt{x_0}}{\sqrt{x_0+1}-1} \frac{1}{\sqrt{x}} \ , \qquad \forall x \ge x_0 > 0 \ ,$$

with $x_0 = \frac{4\varepsilon}{\alpha_0^2}$ and $x = \frac{4}{\alpha_0^2}(|\tilde{k}|^2 + \varepsilon)$ for the integrand, one can calculate the integral and gets the estimate

$$B_\lambda(\alpha_\varepsilon) \le c_\eta \frac{\varepsilon^{\frac{d-1}{2}}}{\left((\varepsilon + \frac{\alpha_0^2}{4})^{\frac{1}{2}} + \frac{\alpha_0}{2} \right)^{d-1+2\eta}} \int_{\mathbb{R}^{d-1}} |\alpha_\varepsilon(\tilde{x})|^{d-1+2\eta} d\tilde{x} \ ,$$

for some $c_\eta > 0$. Using the inequality

$$\frac{x}{\sqrt{x+1}-1} \le \sqrt{x} + 2 \ , \qquad \forall x > 0 \ ,$$

with $x = \frac{4\varepsilon}{\alpha_0^2}$ leads to the final estimate of first step:

$$B_\lambda(\alpha_\varepsilon) \le \tilde{c}_\eta \left(\frac{1}{\varepsilon^\eta} + \frac{(-\alpha_0)^{d-1+2\eta}}{\varepsilon^{\frac{d-1}{2}+2\eta}} \right) \int_{\mathbb{R}^{d-1}} |\alpha_\varepsilon(\tilde{x})|^{d-1+2\eta} d\tilde{x} \ . \tag{7.12}$$

Step 2 gives an integral representation of the left hand side of the Lieb-Thirring inequality (7.6) using $N_\lambda(\alpha + \alpha_0)$. Since $A_{\alpha+\alpha_0}$ is self-adjoint and semibounded by Definition 4.3, with essential spectrum $\sigma_{\text{ess}}(A_{\alpha+\alpha_0}) = [-\frac{\alpha_0^2}{4}, \infty)$ by Theorem 4.5, it has at most countably many ascending discrete eigenvalues $(E_i)_{i=1}^M$ below $-\frac{\alpha_0^2}{4}$, for some $M \in \mathbb{N}_0 \cup \{\infty\}$. In the case $M = \infty$ the following calculation has to be understood in the sense that one defines $E_\infty := -\frac{\alpha_0^2}{4}$, which works because in this case the eigenvalues $(E_i)_{i\in\mathbb{N}}$ converge towards the bottom of the essential spectrum $-\frac{\alpha_0^2}{4}$.

$$\begin{aligned}\int_0^\infty & \varepsilon^{\gamma-1} N_{-\alpha_0^2/4-\varepsilon}(\alpha+\alpha_0) d\varepsilon \\ &= \int_0^{-E_M-\alpha_0^2/4} \varepsilon^{\gamma-1} M d\varepsilon + \sum_{j=1}^{M-1} \int_{-E_{j+1}-\alpha_0^2/4}^{-E_j-\alpha_0^2/4} \varepsilon^{\gamma-1} j d\varepsilon \\ &= \frac{M}{\gamma}\left(-E_M - \frac{\alpha_0^2}{4}\right)^\gamma + \sum_{j=1}^{M-1} \frac{j}{\gamma}\left(\left(-E_j - \frac{\alpha_0^2}{4}\right)^\gamma - \left(-E_{j+1} - \frac{\alpha_0^2}{4}\right)^\gamma\right) \\ &= \frac{1}{\gamma}\sum_{j=1}^{M}\left(-E_j - \frac{\alpha_0^2}{4}\right)^\gamma \qquad (7.13)\end{aligned}$$

Combining now (7.11), (7.12) & (7.13) gives the estimate

$$\sum_{j=1}^{M}\left|E_j + \frac{\alpha_0^2}{4}\right|^\gamma \leq \gamma\tilde{c}_\gamma \int_0^\infty \varepsilon^{\gamma-1}\left(\frac{1}{\varepsilon^\eta} + \frac{(-\alpha_0)^{d-1+2\eta}}{\varepsilon^{\frac{d-1}{2}+2\eta}}\right)\int_{\mathbb{R}^{d-1}} |\alpha_\varepsilon(\tilde{x})|^{d-1+2\eta} d\tilde{x} d\varepsilon\ .$$

Using the definition (7.8) of α_ε and interchanging the order of integration makes this inequality

$$\sum_{j=1}^{M}\left|E_j + \frac{\alpha_0^2}{4}\right|^\gamma \leq \bar{c}_\gamma \int\limits_{\{\alpha<0\}} \int\limits_0^{\frac{-2\alpha(\tilde{x})}{\delta}} \left(\frac{1}{\varepsilon^{1-\gamma+\eta}} + \frac{(-\alpha_0)^{d-1+2\eta}}{\varepsilon^{\frac{d+1}{2}-\gamma+2\eta}}\right)\left(-\alpha(\tilde{x}) - \frac{\varepsilon}{2}\delta\right)^{d-1+2\eta} d\varepsilon d\tilde{x}\ .$$

After the integral substitution $\sigma = \frac{\delta}{-2\alpha(\tilde{x})}\varepsilon$ we end up with

$$\begin{aligned}\sum_{j=1}^{M}\left|E_j + \frac{\alpha_0^2}{4}\right|^\gamma \leq c(\xi_{\mathrm{sol}}) \int_{\{\alpha<0\}} \Bigg(& (-\alpha(\tilde{x}))^{d-1+\gamma-\eta} \int_0^1 \sigma^{-1+\gamma-\eta}(1-\sigma)^{d-1+2\eta} d\sigma \\ & + (-\alpha(\tilde{x}))^{\frac{d-1}{2}+\gamma} \int_0^1 \sigma^{-\frac{d+1}{2}+\gamma-2\eta}(1-\sigma)^{d-1+2\eta} d\sigma \Bigg) d\tilde{x}\ .\end{aligned}$$

The σ-integrals exists by the assumption $\eta < \frac{1}{2}(\gamma - \frac{d-1}{2})$ and give the final inequality

$$\sum_{j=1}^{M}\left|E_j + \frac{\alpha_0^2}{4}\right|^\gamma \leq L_{\gamma,\eta}(\xi_{\mathrm{sol}})\left(\int_{\{\alpha<0\}} (-\alpha(\tilde{x}))^{d-1+\gamma+\eta} + \int_{\{\alpha<0\}} (-\alpha(\tilde{x}))^{\frac{d-1}{2}+\gamma} d\tilde{x}\right)\ .$$

□

A. Regularity and convergence properties of the potential

In this chapter we will investigate the regularity and convergence properties of the multiplication with some function α. In Lemma A.1 we will start with $\alpha \in L^p(\mathbb{R}^{d-1})$ and derive basic boundedness properties between different Lebesgue and Sobolev spaces. The second important result of this section is Theorem A.4, which states that if $\alpha \in L^p(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ fulfils some decay property at infinity, then already the weak H^s-convergence of $(\phi_n)_{n\in\mathbb{N}}$ implies the L^2-norm convergence of $(\alpha\phi_n)_{n\in\mathbb{N}}$. We will need this theorem for the calculation of the essential spectrum of the Schrödinger operator in Theorem 5.16 as well as for the verification of the compactness of the Birman-Schwinger operator in Theorem 6.2.

Lemma A.1. For every $q \in [1,\infty]$ and $p \in [\frac{q}{q-1},\infty]$, the multiplication with some function $\alpha \in L^p(\mathbb{R}^{d-1})$, has the boundedness property

$$\|\alpha\phi\|_{L^{\frac{pq}{p+q}}(\mathbb{R}^{d-1})} \leq \|\alpha\|_{L^p(\mathbb{R}^{d-1})}\|\phi\|_{L^q(\mathbb{R}^{d-1})}\ , \quad \forall \phi \in L^q(\mathbb{R}^{d-1})\ . \tag{A.1}$$

Furthermore, for every $p \in (d-1,\infty)$, $s \in [\frac{d-1}{p},1)$ and $\alpha \in L^p(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$ there exists some constant $c_{s,p,\alpha}$ such that the multiplication with α has the boundedness property

$$\|\alpha\phi\|_{L^2(\mathbb{R}^{d-1})} \leq c_{s,p,\alpha}\|\phi\|_{H^s(\mathbb{R}^{d-1})}\ , \quad \forall \phi \in H^s(\mathbb{R}^{d-1})\ . \tag{A.2}$$

Proof. In order to prove the first inequality (A.1), first note that because of $p \geq \frac{q}{q-1}$ we have $\frac{pq}{p+q} \geq 1$ and the norm $\|\cdot\|_{L^{\frac{pq}{p+q}}(\Sigma)}$ is well-defined. The estimate itself follows immediately using the Hölder inequality with $\tilde{p} = 1 + \frac{p}{q}$ and $\tilde{q} = 1 + \frac{q}{p}$.

For the boundedness (A.2) we split up $\alpha = u + v$ for some $u \in L^p(\mathbb{R}^{d-1})$ and $v \in L^\infty(\mathbb{R}^{d-1})$ and estimate both terms separately.

The bounded part v can simply be estimated by

$$\|v\phi\|_{L^2(\mathbb{R}^{d-1})} \leq \|v\|_{L^\infty(\mathbb{R}^{d-1})}\|\phi\|_{L^2(\mathbb{R}^{d-1})} \leq \|v\|_{L^\infty(\mathbb{R}^{d-1})}\|\phi\|_{H^s(\mathbb{R}^{d-1})}\ . \tag{A.3}$$

Because of $p > d-1$ and $d \geq 3$, we especially have $\frac{p}{2} > 1$ and the Hölder inequality with $\tilde{p} = \frac{p}{2}$ and $\tilde{q} = \frac{p}{p-2}$ can be applied and yields

$$\|u\phi\|_{L^2(\mathbb{R}^{d-1})} \leq \|u\|_{L^p(\mathbb{R}^{d-1})} \|\phi\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^{d-1})} .$$

Since $2 \leq \frac{2p}{p-2} \leq \frac{2(d-1)}{d-1-2s}$ is satisfied by $s \in [\frac{d-1}{p}, 1)$, Theorem 1.8 ensures the boundedness

$$\|u\phi\|_{L^2(\mathbb{R}^{d-1})} \leq c_{s,\frac{2p}{p-2}} \|u\|_{L^p(\mathbb{R}^{d-1})} \|\phi\|_{H^s(\mathbb{R}^{d-1})} . \tag{A.4}$$

Combining (A.3) & (A.4) proves the stated boundedness (A.2). □

After this boundedness properties of the multiplication with some potential, we will now state some (weak) convergence properties. It is obvious that norm convergence in H^s implies norm convergence in L^2, but in terms of weak convergences this is no longer obvious. However, the next Lemma will proves this fact.

Lemma A.2. Let $s \in [0, \frac{d}{2})$, $p \in [2, \frac{2d}{d-2s}]$ and $f_0, (f_n)_{n\in\mathbb{N}} \in H^s(\mathbb{R}^d)$. Then $f_0, (f_n)_{n\in\mathbb{N}}$ are also contained in $L^p(\mathbb{R}^d)$ and

$$\text{if} \quad f_0 = \operatorname*{wlim}_{n\to\infty} {}_{H^s(\mathbb{R}^d)} f_n , \quad \text{then also} \quad f_0 = \operatorname*{wlim}_{n\to\infty} {}_{L^p(\mathbb{R}^d)} f_n . \tag{A.5}$$

Proof. The fact that $f_0, (f_n)_{n\in\mathbb{N}}$ are contained in $L^p(\mathbb{R}^d)$ follows from Corollary 1.9. For the proof of the weak convergence, note first that according to the weak convergence, the sequence $(f_n)_{n\in\mathbb{N}}$ is in particular bounded in $H^s(\mathbb{R}^d)$ and consequently also bounded in $L^p(\mathbb{R}^d)$ by Corollary 1.9.

Assume now that f_0 is not the weak L^p-limit of $(f_n)_{n\in\mathbb{N}}$. This means that there exists some $F_0 \in L^p(\mathbb{R}^d)'$ in the dual space, some $\varepsilon_0 > 0$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$, such that

$$|F_0(f_{n_k}) - F_0(f_0)| \geq \varepsilon_0 , \quad \forall k \in \mathbb{N} . \tag{A.6}$$

According to the L^p-boundedness of $(f_{n_k})_{k\in\mathbb{N}}$ and the reflexivity of the Lebesgue space, there exists a weak convergent subsequence $(f_{n_{k_l}})_{l\in\mathbb{N}}$

$$g_0 = \operatorname*{wlim}_{l\to\infty} {}_{L^p(\mathbb{R}^d)} f_{n_{k_l}} , \tag{A.7}$$

converging to some element $g_0 \in L^p(\mathbb{R}^d)$. Because of the bounded embedding of $H^s(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$, their dualspaces $L^p(\mathbb{R}^d)' \subseteq H^s(\mathbb{R}^d)'$ are reversed embedded and together with the two weak convergences (A.5) and (A.7) this gives the identity

$$F(g_0) = \lim_{n\to\infty} F(f_{n_{k_l}}) = F(f_0) \tag{A.8}$$

for every $F \in L^p(\mathbb{R}^d)'$. This implies that also the functions $f_0 = g_0$ itself have to coincide, which makes (A.7) a contradiction to (A.6). □

The next lemma treats the transition from weak Sobolev convergence to convergence in the Lebesgue norm, at least on sets of finite measures.

Lemma A.3. Let $s \in [0, \frac{d}{2})$ and $p \in [1, \frac{2d}{d-2s})$. Then for $f_0, (f_n)_{n\in\mathbb{N}} \in H^s(\mathbb{R}^d)$, the weak convergence

$$f_0 = \underset{n\to\infty}{\mathrm{wlim}}_{H^s(\mathbb{R}^d)} f_n \quad \text{implies the norm convergence} \quad \lim_{n\to\infty} \|f_n - f\|_{L^p(A)} = 0 \ ,$$

for any Borel set $A \subseteq \mathbb{R}^d$ with finite measure.

Proof. In Lemma A.2 we proved that the weak Sobolev convergence implies the weak Lebesgue convergence $f_0 = \underset{n\to\infty}{\mathrm{wlim}}_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} f_n$ and consequently the boundedness

$$\|f_n\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \le M \ , \quad \forall n \in \mathbb{N} \ , \tag{A.9}$$

for some $M > 0$. We will first consider the special case $p = 2$ and generalise afterwards to $p \in [1, 2)$ and $p \in (2, \frac{2d}{d-2s})$ by reducing it to the L^2-case.

For every $t > 0$, define the mollifier function

$$\varphi_t(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} \ , \quad \forall x \in \mathbb{R}^d \ .$$

With this function φ_t, the strategy will be, to first ensure the convergence (A.12) of the approximated functions $\varphi_t * f_n$ and then the convergence of the approximation (A.14) in t.

Fix now any $t > 0$. Then for every $x \in \mathbb{R}^d$ the function $\varphi_t(x - \ \cdot\ )$ is contained in $L^2(\mathbb{R}^d)$. By the weak L^2-convergence of $(f_n)_{n\in\mathbb{N}}$ from Lemma A.2, we can conclude the pointwise convergence of the convolution

$$\begin{aligned} \lim_{n\to\infty} (\varphi_t * f_n)(x) &= \lim_{n\to\infty} \langle \varphi_t(x - \ \cdot\ ), f_n \rangle_{L^2(\mathbb{R}^d)} \\ &= \langle \varphi_t(x - \ \cdot\ ), f_0 \rangle_{L^2(\mathbb{R}^d)} = (\varphi_t * f_0)(x) \ . \end{aligned} \tag{A.10}$$

Furthermore, by the boundedness of the convolution (1.14) and the $L^{\frac{2d}{d-2s}}$-boundedness (A.9) of $(f_n)_{n\in\mathbb{N}}$, we also conclude the uniform boundedness

$$\|\varphi_t * f_n\|_{L^\infty(\mathbb{R}^d)} \le \|\varphi_t\|_{L^{\frac{2d}{d+2s}}(\mathbb{R}^d)} \|f_n\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \le \|\varphi_t\|_{L^{\frac{2d}{d+2s}}(\mathbb{R}^d)} M \tag{A.11}$$

of the sequence $(\varphi_t * f_n)_{n\in\mathbb{N}}$. Because A is a set of finite measure, the equations (A.10) & (A.11) are sufficient to apply the dominated convergence theorem and leads to the norm convergence

$$\lim_{n\to\infty} \|\varphi_t * f_n - \varphi_t * f_0\|_{L^2(A)} = 0. \tag{A.12}$$

Next we want to verify the inequality (A.14) for any $f \in L^2(\mathbb{R}^d)$. Using Plancherel's theorem, we are allowed to do this in Fourier space, where the convolution reduces to a simple multiplication:

$$\|f - \varphi_t * f\|_{L^2(\mathbb{R}^d)} = \|(1 - (2\pi)^{\frac{d}{2}} \mathcal{F}\varphi_t)\mathcal{F}f\|_{L^2(\mathbb{R}^d)} . \tag{A.13}$$

The functions φ_t admit the explicit Fourier transformations $\mathcal{F}\varphi_t(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-t|k|^2}$ and with the estimate $1 - e^{-x} \le x^r$, which holds for every $x \ge 0$ and $r \in [0, 1]$, we can estimate the integrand of (A.13) by

$$\left|1 - (2\pi)^{\frac{d}{2}} \mathcal{F}\varphi_t(k)\right| = 1 - e^{-t|k|^2} \le (t|k|^2)^{\min\{\frac{s}{2},1\}} \le t^{\min\{\frac{s}{2},1\}}(1 + |k|^2)^{\frac{s}{2}} , \quad \forall k \in \mathbb{R}^d$$

and consequently the whole L^2-norm by

$$\|f - \varphi_t * f\|_{L^2(\mathbb{R}^d)} \le t^{\min\{\frac{s}{2},1\}}\|(1 + |k|^2)^{\frac{s}{2}}\mathcal{F}f\|_{L^2(\mathbb{R}^d)} = t^{\min\{\frac{s}{2},1\}}\|f\|_{H^s(\mathbb{R}^d)} . \tag{A.14}$$

The inequality (A.14) can now be used once for f_n and once for f_0 to validate the estimate

$$\|f_n - f\|_{L^2(A)} \le t^{\min\{\frac{s}{2},1\}}\|f_n\|_{H^s(\mathbb{R}^d)} + \|\varphi_t * (f_n - f)\|_{L^2(A)} + t^{\min\{\frac{s}{2},1\}}\|f\|_{H^s(\mathbb{R}^d)}$$

for every $n \in \mathbb{N}$ and $t > 0$. By the H^s-boundedness of the weak convergent sequence $(f_n)_{n\in\mathbb{N}}$, the first and third term can be made arbitrary small by the choice of $t > 0$ and the second term converges by (A.14), which proves the lemma for $p = 2$.

The generalisation to $p \in [1, 2)$ follows from the fact that, since A has finite measure, $L^2(A)$ is continuously embedded in $L^p(A)$.

For the generalisation to $p \in (2, \frac{2d}{d-2s})$ we can use the Hölder inequality with the exponents $\tilde{p} = \frac{4s}{2d-p(d-2s)}$ and $\tilde{q} = \frac{4s}{(p-2)(d-2s)}$, where the restriction $p < \frac{2d}{d-2s}$ is needed to make $\tilde{p}$ finite.

$$\begin{aligned}
\|f - f_n\|_{L^p(A)} &\le \left(\int_A |f - f_n|^{\tilde{p}(p-\frac{d}{2s}(p-2))}\right)^{\frac{\tilde{p}}{p}} \left(\int_A |f - f_n|^{\tilde{q}\frac{d}{2s}(p-2)} dx\right)^{\frac{\tilde{q}}{p}} \\
&= \|f - f_n\|^{\frac{\tilde{p}}{p}}_{L^2(A)} \|f - f_n\|^{\frac{\tilde{q}}{p}}_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)}
\end{aligned}$$

The norm $L^{\frac{2d}{d-2s}}$ is bounded by (A.9) and the first term yields the convergence, which finishes the proof. $\square$

Theorem A.4. Let $d \geq 3$, $s \in (0, \frac{d-1}{2})$ and $p \in [\frac{d-1}{s}, \infty)$. Consider some potential $\alpha \in L^p(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$, with the additional decay property:

$$\left\{ \tilde{x} \in \mathbb{R}^{d-1} \;\middle|\; |\alpha(\tilde{x})| > \varepsilon \right\} \text{ has finite measure for every } \varepsilon > 0 \ . \tag{A.15}$$

Then for all functions $\phi_0, (\phi_n)_{n\in\mathbb{N}} \in H^s(\mathbb{R}^{d-1})$, the weak H^s-convergence

$$\phi_0 = \operatorname*{wlim}_{n\to\infty}{}_{H^s(\mathbb{R}^{d-1})} \phi_n$$

implies the existance of a subsequence $(\phi_{n_k})_{k\in\mathbb{N}}$ which converges in the L^2-norm

$$\lim_{k\to\infty} \|\alpha(\phi_{n_k} - \phi_0)\|_{L^2(\mathbb{R}^{d-1})} = 0 \ , \tag{A.16}$$

after one multiplied the potential α.

Proof. By assumption the potential has the representation

$$\alpha = u + v \ , \quad \text{for some} \quad u \in L^p(\mathbb{R}^{d-1}) \text{ and } v \in L^\infty(\mathbb{R}^{d-1}) \ .$$

In order to make also the potential u bounded we cut it off at some bound $b > 0$ and define

$$u_b := \begin{cases} u & \text{if } |u| \leq b \ , \\ 0 & \text{if } |u| > b \end{cases} \qquad \text{and} \qquad \alpha_b := u_b + v \ .$$

The proof of the convergence (A.16) is now splitted into two parts. First we will confirm the convergence according to the approximated potential α_b and then the convergence according to a subsequence $(\phi_{n_k})_{k\in\mathbb{N}}$.

By the integrability of $u \in L^p(\mathbb{R}^{d-1})$, we find for every $\varepsilon > 0$ some bound $b_\varepsilon > 0$ which satisfies

$$\|u - u_{b_\varepsilon}\|_{L^p(\mathbb{R}^{d-1})} < \varepsilon \ .$$

The Hölder inequality with $\tilde{p} = \frac{p}{2}$ and $\tilde{q} = \frac{p}{p-2}$ as well as Corollary 1.9 give for every $\phi \in H^s(\mathbb{R}^{d-1})$ the bound

$$\begin{aligned} \|(u - u_{b_\varepsilon})\phi\|_{L^2(\mathbb{R}^{d-1})} &\leq \|u - u_{b_\varepsilon}\|_{L^p(\mathbb{R}^{d-1})} \|\phi\|_{L^{\frac{2p}{p-2}}(\mathbb{R}^{d-1})} \\ &\leq \varepsilon \, c_{s,\frac{2p}{p-2}} \|\phi\|_{H^s(\mathbb{R}^{d-1})} \ , \end{aligned} \tag{A.17}$$

for the u-part of the potential. By the definition of α_{b_ε} we get $\alpha - \alpha_{b_\varepsilon} = u - u_{b_\varepsilon}$ and the bound (A.17) transfers to

$$\|(\alpha - \alpha_{b_\varepsilon})\phi\|_{L^2(\mathbb{R}^{d-1})} \leq \varepsilon \, c_{s.\frac{2p}{p-2}} \|\phi\|_{H^s(\mathbb{R}^{d-1})} \ , \quad \forall \phi \in H^s(\mathbb{R}^{d-1}) \ . \tag{A.18}$$

In the next step we will check the L^2-convergence of $(\phi_n)_{n\in\mathbb{N}}$, but not on the whole space $\mathbb{R}^{d-1}$, just on the set

$$A_\varepsilon = \left\{ \tilde{x} \in \mathbb{R}^{d-1} \,\middle|\, |u_{b_\varepsilon}(\tilde{x})| > \frac{\varepsilon}{3} \right\} \cup \left\{ \tilde{x} \in \mathbb{R}^{d-1} \,\middle|\, |v(\tilde{x})| > \frac{2\varepsilon}{3} \right\} , \tag{A.19}$$

on whose complement the potential α_{b_ε} becomes smaller than ε. Note that, because of the inclusion $A_\varepsilon \subseteq \{|u| > \frac{\varepsilon}{3}\} \cup \{|\alpha| > \frac{\varepsilon}{3}\}$ as well as the integrability of u and the decay property (A.15) of α, this set A_ε has finite measure. Lemma A.3 then confirms the convergence

$$\lim_{n\to\infty} \|\phi_n - \phi_0\|_{L^2(A_\varepsilon)} = 0 \, .$$

This convergence in particular gives an index $N_\varepsilon \in \mathbb{N}$, such that

$$\|\phi_n - \phi_0\|_{L^2(A_\varepsilon)} \leq \frac{\varepsilon}{b_\varepsilon + \|v\|_{L^\infty}} \, , \qquad \forall n \geq N_\varepsilon \, . \tag{A.20}$$

The equations (A.18) & (A.20) can now be combined to get for every $n \geq N_\varepsilon$ the final estimate

$$\begin{aligned} \|\alpha(\phi_n - \phi_0)\|_{L^2(\mathbb{R}^{d-1})} &\leq \varepsilon \, c_{s,\frac{2p}{p-2}} \|\phi_n - \phi_0\|_{H^s(\mathbb{R}^{d-1})} + \varepsilon + \varepsilon \|\phi_n - \phi_0\|_{L^2(\mathbb{R}^{d-1}\setminus A_\varepsilon)} \leq \\ &\leq \varepsilon \left(\left(c_{s,\frac{2p}{p-2}} + 1 \right) \|\phi_n - \phi_0\|_{H^s(\mathbb{R}^{d-1})} + 1 \right) \, . \end{aligned}$$

The norm $\|\phi_n - \phi_0\|_{H^s(\mathbb{R}^{d-1})}$ on the right hand side is bounded by the weak H^s-convergence and so this inequality proves the stated convergence (A.16). □

B. Green's function of the Laplace operator

The second part of the appendix deals with the integral kernel G_λ of the resolvent $(-\Delta - \lambda)^{-1}$. We will define the integral operators $\mathcal{G}_\lambda$, $\widetilde{\mathcal{G}}_\lambda$, $\mathcal{G}_\lambda^\uparrow$, $\mathcal{G}_\lambda^\downarrow$ in Definition B.6, which will in Chapter 5 turn out to be representations of the γ-field and the Weyl function. The main task of this section is now to prove the boundedness of these integral operators in Proposition B.9.

Definition B.1. In $d \geq 3$ dimensions define for every $\lambda \in \mathbb{C} \setminus [0, \infty)$ the *Green's function* G_λ by

$$G_\lambda(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{\sqrt{-\lambda}}{|x|} \right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}\left(\sqrt{-\lambda}\, |x| \right) , \quad \forall x \in \mathbb{R}^d \setminus \{0\} , \tag{B.1}$$

where K_α are the modified Bessel functions of the second kind. Moreover, define the *Restricted Green's function*

$$\widetilde{G}_\lambda(\tilde{x}) = G_\lambda(\tilde{x}, 0) , \quad \forall \tilde{x} \in \mathbb{R}^{d-1} \setminus \{0\} . \tag{B.2}$$

In the following Lemma we will collect basic properties of the modified Bessel functions of the second kind, which for example can be found in [1, 9.6.1, 9.6.9, 9.7.2].

Lemma B.2. For every $\alpha > 0$, the modified Bessel function of the second kind $K_\alpha : (0, \infty) \to \mathbb{R}$ has the following properties:

a) $K_\alpha \in \mathcal{C}^\infty(0, \infty)$,
b) $(t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} - t^2 - \alpha^2) K_\alpha(t) = 0 , \quad \forall t \in (0, \infty)$,
c) $K_\alpha(t) \sim \frac{\Gamma(\alpha)}{2} (\frac{2}{t})^\alpha$ as $t \to 0$,
d) $K_\alpha(t) \sim \sqrt{\frac{\pi}{2t}}\, e^{-t}$ as $t \to \infty$,
e) K_α monotone decreasing.

From these properties of K_α, similar properties of the Green's function G_λ follow immediately by using its definition in (B.1).

Lemma B.3. Let $d \geq 3$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then the Green's function G_λ has the following properties:

a) $G_\lambda \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$,

b) $G_\lambda(x) \sim \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}} \frac{1}{|x|^{d-2}}$ as $x \to 0$,

c) $G_\lambda(x) \sim \frac{\left(\sqrt{-\lambda}\right)^{\frac{d-3}{2}}}{2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}} \frac{e^{-\sqrt{-\lambda}|x|}}{|x|^{\frac{d-1}{2}}}$ as $x \to \infty$,

d) $|G_\lambda(x)| \geq |G_\lambda(y)|\,, \quad \forall\, |x| \leq |y|$.

The following lemma states the main property of G_λ, as it is the integral kernel of the resolvent of the self-adjoint free Laplace operator

$$A_{\text{free}} = -\Delta \quad \text{and} \quad \text{dom}(A_{\text{free}}) = H^2(\mathbb{R}^d)\,.$$

Lemma B.4. Let $d \geq 3$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then the resolvent $(A_{\text{free}} - \lambda)^{-1}$ can be expressed as a convolution with the Green's function G_λ by

$$(A_{\text{free}} - \lambda)^{-1} f = G_\lambda * f\,, \quad \forall f \in L^2(\mathbb{R}^d)\,. \tag{B.3}$$

Lemma B.5. Let $d \geq 3$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then for every $p \in [1, \frac{d}{d-2})$ the Green's function G_λ is in $L^p(\mathbb{R}^d)$ and its Fourier transformation has the explicit form

$$\mathcal{F}_d G_\lambda(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{|k|^2 - \lambda}\,. \tag{B.4}$$

Moreover, for every $p \in [1, \frac{d-1}{d-2})$ the restricted Green's functions $\widetilde{G}_\lambda$ is in $L^p(\mathbb{R}^{d-1})$ and its Fourier transformation has the explicit form

$$\mathcal{F}_{d-1} \widetilde{G}_\lambda(\tilde{k}) = \frac{1}{2(2\pi)^{\frac{d-1}{2}}} \frac{1}{(|\tilde{k}|^2 - \lambda)^{\frac{1}{2}}}\,. \tag{B.5}$$

Proof. From Lemma B.3 we get the asymptotic behaviour

$$G_\lambda(x) \sim c_0 \frac{1}{|x|^{d-2}} \quad \text{as} \quad x \to 0 \quad \text{and} \quad G_\lambda(x) \sim c_\infty \frac{e^{-\sqrt{-\lambda}|x|}}{|x|^{\frac{d-1}{2}}} \quad \text{as} \quad x \to \infty\,.$$

Because of the exponential decay at infinity, the only restriction according to integrability is the singularity at $x = 0$. For arbitrary $\varepsilon > 0$ we obtain

$$\int_{U_\varepsilon(0)} |G_\lambda(x)|^p dx \sim c_0^p \int_{U_\varepsilon(0)} \frac{1}{|x|^{p(d-2)}} dx = c_0^p \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^\varepsilon \frac{r^{d-1}}{r^{p(d-2)}} dr < \infty\,,$$

where the last integral is finite if and only if $p < \frac{d}{d-2}$.

By the definition $\widetilde{G}_\lambda(\tilde{x}) = G_\lambda(\tilde{x}, 0)$, its integrability is also restricted by the singularity at $\tilde{x} = 0$. For arbitrary $\varepsilon > 0$ we get

$$\int_{U_\varepsilon(0)} |\widetilde{G}_\lambda(\tilde{x})|^p d\tilde{x} \sim c_0^p \int_{U_\varepsilon(0)} \frac{1}{|\tilde{x}|^{p(d-2)}} d\tilde{x} = c_0^p \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int_0^\varepsilon \frac{r^{d-2}}{r^{p(d-2)}} dr < \infty ,$$

where the last integral is finite if and only if $p < \frac{d-1}{d-2}$.

Once we have calculated the L^p-regularities of the Green's functions, we will now calculate its Fourier transformations. From Lemma B.4 we know that G_λ is the integral kernel of the resolvent $(A_{\text{free}} - \lambda)^{-1}$. If we use (1.16) on the left hand side for the convolution and on the right hand side the known fact that A_{free} acts in Fourier space as the multiplication with $|k|^2$, the equation (B.3) admits in Fourier space the form

$$(2\pi)^{\frac{d}{2}} \mathcal{F}_d G_\lambda(k) \mathcal{F}_d f(k) = \frac{1}{|k|^2 - \lambda} \mathcal{F}_d f(k) , \quad \forall f \in L^2(\mathbb{R}^d) .$$

Since this is true for every $f \in L^2(\mathbb{R}^d)$ one can cancel $\mathcal{F}_d f(k)$ on both sides and the stated Fourier transformation

$$\mathcal{F}_d G_\lambda(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{|k|^2 - \lambda} \tag{B.6}$$

of the Green's function G_λ remains.

Similar as in (1.22) we reduce the $(d-1)$-dimensional Fourier transformation of $\widetilde{G}_\lambda$ to the d-dimensional Fourier transformation of G_λ by

$$\mathcal{F}_{d-1} \widetilde{G}_\lambda(\tilde{k}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}_d G_\lambda(\tilde{k}, k_d) dk_d .$$

After inserting (B.6) into this equation one can do the integral analytically and ends up with

$$\mathcal{F}_{d-1} \widetilde{G}_\lambda(\tilde{k}) = \frac{1}{\sqrt{2\pi}(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{1}{k_d^2 + |\tilde{k}|^2 - \lambda} dk_d = \frac{1}{2(2\pi)^{\frac{d-1}{2}}} \frac{1}{(|\tilde{k}|^2 - \lambda)^{\frac{1}{2}}} .$$

□

Definition B.6. Let $d \geq 3$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then define the *convolution mappings*

a) $\mathcal{G}_\lambda f := G_\lambda * f, \qquad \forall f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d),$

b) $\widetilde{\mathcal{G}}_\lambda \phi := \widetilde{G}_\lambda * \phi, \qquad \forall \phi \in L^1(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1}),$

c) $\mathcal{G}_\lambda^\downarrow f(\tilde{x}) := \int\limits_{\mathbb{R}^d} G_\lambda \left(\begin{pmatrix} \tilde{x} \\ 0 \end{pmatrix} - y \right) f(y) dy, \qquad \forall f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), \tilde{x} \in \mathbb{R}^{d-1},$

d) $\mathcal{G}_\lambda^{\uparrow}\phi(x) := \int\limits_{\mathbb{R}^{d-1}} G_\lambda\left(x - \begin{pmatrix} \tilde{y} \\ 0 \end{pmatrix}\right)\phi(\tilde{y})d\tilde{y}, \quad \forall\phi \in L^1(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1}), x \in \mathbb{R}^d.$

Corollary B.7. Let $d \geq 3$, $\lambda \in \mathbb{C} \setminus [0, \infty)$ and $p \in [1, \infty]$. Then the convolution mappings from Definition B.6 generate bounded operators in L^p, with the estimates

a) $\|\mathcal{G}_\lambda f\|_{L^p(\mathbb{R}^d)} \leq \|G_\lambda\|_{L^1(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d)}, \qquad \forall f \in L^p(\mathbb{R}^d),$

b) $\|\widetilde{\mathcal{G}}_\lambda\phi\|_{L^p(\mathbb{R}^{d-1})} \leq \|\widetilde{G}_\lambda\|_{L^1(\mathbb{R}^{d-1})}\|\phi\|_{L^p(\mathbb{R}^{d-1})}, \qquad \forall\phi \in L^p(\mathbb{R}^{d-1}),$

c) $\|\mathcal{G}_\lambda^{\downarrow} f\|_{L^p(\mathbb{R}^{d-1})} \leq \|\widetilde{G}_\lambda\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{p}}\|G_\lambda\|_{L^1(\mathbb{R}^d)}^{\frac{p-1}{p}}\|f\|_{L^p(\mathbb{R}^d)}, \quad \forall f \in L^p(\mathbb{R}^d),$

d) $\|\mathcal{G}_\lambda^{\uparrow}\phi\|_{L^p(\mathbb{R}^d)} \leq \|G_\lambda\|_{L^1(\mathbb{R}^d)}^{\frac{1}{p}}\|\widetilde{G}_\lambda\|_{L^1(\mathbb{R}^{d-1})}^{\frac{p-1}{p}}\|\phi\|_{L^p(\mathbb{R}^{d-1})}, \quad \forall\phi \in L^p(\mathbb{R}^{d-1}).$

Proof. The estimates of a) and b) follow directly from (1.14). In order to proof the boundedness d), distinguish three cases

- In the case $p = 1$ we can estimate

$$\|\mathcal{G}_\lambda^{\uparrow}\phi\|_{L^1(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}} \left|G_\lambda\left(x - \begin{pmatrix} \tilde{y} \\ 0 \end{pmatrix}\right)\right| |\phi(\tilde{y})|d\tilde{y}dx = \|G_\lambda\|_{L^1(\mathbb{R}^d)}\|\phi\|_{L^1(\mathbb{R}^{d-1})} .$$

- In the case $p = \infty$ we estimate for every $x \in \mathbb{R}^d$

$$\begin{aligned} |\mathcal{G}_\lambda^{\uparrow}\phi(x)| &\leq \int_{\mathbb{R}^{d-1}} \left|G_\lambda\left(x - \begin{pmatrix} \tilde{y} \\ 0 \end{pmatrix}\right)\right| |\phi(\tilde{y})|d\tilde{y} \\ &\leq \|\phi\|_{L^\infty(\mathbb{R}^{d-1})}\int_{\mathbb{R}^{d-1}} \left|G_\lambda\left(x - \begin{pmatrix} \tilde{y} \\ 0 \end{pmatrix}\right)\right| d\tilde{y} \\ &\leq \|\phi\|_{L^\infty(\mathbb{R}^{d-1})}\|\widetilde{G}_\lambda\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{q}} , \end{aligned}$$

  where in the last inequality it was used that

$$\left|G_\lambda\begin{pmatrix} \tilde{x} - \tilde{y} \\ x_d \end{pmatrix}\right| \leq \left|G_\lambda\begin{pmatrix} \tilde{x} - \tilde{y} \\ 0 \end{pmatrix}\right| = |\widetilde{G}_\lambda(\tilde{x} - \tilde{y})| \tag{B.7}$$

  because of the monotonicity of G_λ from Lemma B.3.

- In the case $p \in (1, \infty)$, let $q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and use the Hölder inequality to obtain for every $x \in \mathbb{R}^d$ the estimate

$$\begin{aligned} |\mathcal{G}_\lambda^{\uparrow}\phi(x)| &\leq \int_{\mathbb{R}^{d-1}} \left|G_\lambda\left(x - \begin{pmatrix} \tilde{y} \\ 0 \end{pmatrix}\right)\right| |\phi(\tilde{y})|d\tilde{y} \\ &\leq \|\widetilde{G}_\lambda\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{q}}\left(\int_{\mathbb{R}^{d-1}} \left|G_\lambda\left(x - \begin{pmatrix} \tilde{y} \\ 0 \end{pmatrix}\right)\right| |\phi(\tilde{y})|^p d\tilde{y}\right)^{\frac{1}{p}} , \end{aligned}$$

where in the last line we again used (B.7) to get $\|\widetilde{G}_\lambda\|_{L^1(\mathbb{R}^{d-1})}$. Integration over $\mathbb{R}^d$ yields the L^p-norm

$$\|\mathcal{G}_\lambda^\uparrow \phi\|_{L^p(\mathbb{R}^d)}^p \leq \|\widetilde{G}_\lambda\|_{L^1(\mathbb{R}^{d-1})}^{\frac{p}{q}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-1}} \left| G_\lambda \left(x - \begin{pmatrix} \tilde{y} \\ 0 \end{pmatrix} \right) \right| |\phi(\tilde{y})|^p d\tilde{y} dx$$
$$= \|\widetilde{G}_\lambda\|_{L^1(\mathbb{R}^{d-1})}^{\frac{p}{q}} \|G_\lambda\|_{L^1(\mathbb{R}^d)} \|\phi\|_{L^p(\mathbb{R}^{d-1})}^p .$$

With obvious replacements of the space dimension, the same steps can be used to obtain the boundedness in c). □

Lemma B.8. Let $d \geq 3$, $\lambda \in \mathbb{C}\backslash[0,\infty)$ and $p \in [1,2]$. Then the Fourier transformation of the convolution mappings have the form

a) $\mathcal{F}_d \mathcal{G}_\lambda f(k) = \frac{\mathcal{F}_d f(k)}{|k|^2-\lambda}$, $\quad \forall f \in L^p(\mathbb{R}^d), \quad k \in \mathbb{R}^d$

b) $\mathcal{F}_{d-1} \widetilde{\mathcal{G}}_\lambda \phi(\tilde{k}) = \frac{\mathcal{F}_{d-1}\phi(\tilde{k})}{2(|\tilde{k}|^2-\lambda)^{\frac{1}{2}}}$, $\quad \forall \phi \in L^p(\mathbb{R}^{d-1}), \tilde{k} \in \mathbb{R}^{d-1}$,

c) $\mathcal{F}_{d-1} \mathcal{G}_\lambda^\downarrow f(\tilde{k}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\mathcal{F}_d f(k)}{|k|^2-\lambda} dk_d$, $\quad \forall f \in L^p(\mathbb{R}^d), \quad \tilde{k} \in \mathbb{R}^{d-1}$,

d) $\mathcal{F}_d \mathcal{G}_\lambda^\uparrow \phi(k) = \frac{\mathcal{F}_{d-1}\phi(\tilde{k})}{\sqrt{2\pi}\,(|k|^2-\lambda)}$, $\quad \forall \phi \in L^p(\mathbb{R}^{d-1}), k \in \mathbb{R}^d$.

Note that in c) and d) the notation $k = (\tilde{k}, k_d)$ was used, and that the indices d and $d-1$ indicate the dimension of the Fourier transformation.

Proof. The identities a) and b) are clear from Lemma B.5 and (1.16).

In order to prove c), let $f \in L^p(\mathbb{R}^d)$. Then by definition we have

$$\mathcal{G}_\lambda^\downarrow f(\tilde{x}) = \mathcal{G}_\lambda f(\tilde{x}, 0) , \quad \forall \tilde{x} \in \mathbb{R}^{d-1}$$

and with (1.21) its Fourier transformation looks like

$$\mathcal{F}_{d-1} \mathcal{G}_\lambda^\downarrow f(\tilde{k}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}_d \mathcal{G}_\lambda f(k) dk_d .$$

Using a) in this equation gives the stated Fourier transformation c).

In d) we will calculate $\mathcal{F}_d \mathcal{G}_\lambda^\uparrow \phi$ first for $\phi \in L^1(\mathbb{R}^{d-1}) \cap L^p(\mathbb{R}^{d-1})$. Using Corollary B.7 shows that in this case also $\mathcal{G}_\lambda^\uparrow \phi \in L^1(\mathbb{R}^d)$ and we are allowed to use the integral representation (1.11) for $\mathcal{F}_{d-1}\phi$ as well as for $\mathcal{F}_d \mathcal{G}_\lambda^\uparrow \phi$. If we define for every $x_d \in \mathbb{R}$ the function

$$G_\lambda^{(x_d)}(\tilde{x}) = G_\lambda \begin{pmatrix} \tilde{x} \\ x_d \end{pmatrix} , \quad \forall \tilde{x} \in \mathbb{R}^{d-1} ,$$

we can write

$$\mathcal{G}_\lambda^\uparrow \phi(x) = (G_\lambda^{(x_d)} * \phi)(\tilde{x})$$

in the form of a convolution. With the help of (1.16), we can now calculate the Fourier integral

$$\begin{aligned}\mathcal{F}_d\mathcal{G}_\lambda^\uparrow\phi(k) &= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} e^{-ik_dx_d}\mathcal{F}_{d-1}[G_\lambda^{(x_d)} * \phi](\tilde{k})dx_d \\ &= (2\pi)^{\frac{d-2}{2}}\int_{\mathbb{R}} e^{-ik_dx_d}\mathcal{F}_{d-1}G_\lambda^{(x_d)}(\tilde{k})\mathcal{F}_{d-1}\phi(\tilde{k})dx_d \\ &= (2\pi)^{\frac{d-1}{2}}\mathcal{F}_dG_\lambda(k)\mathcal{F}_{d-1}\phi(\tilde{k}) \\ &= \frac{\mathcal{F}_{d-1}\phi(\tilde{k})}{\sqrt{2\pi}\,(|k|^2-\lambda)}\ ,\end{aligned} \tag{B.8}$$

where in the last equality the explicit form (B.4) of $\mathcal{F}_dG_\lambda$ was used.

Let now $\phi \in L^p(\mathbb{R}^{d-1})$. Then there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \in L^1(\mathbb{R}^{d-1})\cap L^p(\mathbb{R}^{d-1})$, such that $\lim_{n\to\infty}\|\phi-\phi_n\|_{L^p(\mathbb{R}^{d-1})} = 0$. By the boundedness of the $\mathcal{G}_\lambda^\uparrow$ from Corollary B.7 also the images $\lim_{n\to\infty}\|\mathcal{G}_\lambda^\uparrow(f-f_n)\|_{L^p(\mathbb{R}^d)}$ converge. These convergences in the Lebesgue norm in particular imply the existence of a subsequence $(\phi_{n_l})_{l\in\mathbb{N}}$, for which

$$\mathcal{F}_{d-1}\phi(\tilde{k}) = \lim_{l\to\infty}\mathcal{F}_{d-1}\phi_{n_l}(\tilde{k}) \quad \text{and} \quad \mathcal{F}_d\mathcal{G}_\lambda^\uparrow\phi(k) = \lim_{l\to\infty}\mathcal{F}_d\mathcal{G}_\lambda^\uparrow\phi_{n_l}(k)$$

converges for almost every $\tilde{k} \in \mathbb{R}^{d-1}$ and almost every $k \in \mathbb{R}^d$. This finally confirms that (B.8) holds for $\phi \in L^p(\mathbb{R}^{d-1})$ almost everywhere. □

In Corollary B.7 we already stated boundedness properties of the convolution mappings in L^p spaces. But with the explicit Fourier transformations from Lemma B.8 we can also derive bounds with respect to Sobolev norms.

Proposition B.9. Let $d \geq 3$, $\lambda \in \mathbb{C}\setminus[0,\infty)$ and $p \in [1,2]$. Then for every $s < \frac{3}{2}-(d-1)(\frac{1}{p}-\frac{1}{2})$, the convolution mappings

a) $\mathcal{G}_\lambda : L^p(\mathbb{R}^d) \to H^{s+\frac{1}{2}}(\mathbb{R}^d)$,
b) $\widetilde{\mathcal{G}}_\lambda : L^p(\mathbb{R}^{d-1}) \to H^{s-\frac{1}{2}}(\mathbb{R}^{d-1})$,
c) $\mathcal{G}_\lambda^\downarrow : L^p(\mathbb{R}^d) \to H^s(\mathbb{R}^{d-1})$ and
d) $\mathcal{G}_\lambda^\uparrow : L^p(\mathbb{R}^{d-1}) \to H^s(\mathbb{R}^d)$

are bounded operators.

Proof.

a) In order to verify that $\mathcal{G}_\lambda f \in H^s(\mathbb{R}^d)$, for every $f \in L^p(\mathbb{R}^d)$, we will use the Fourier transformation from Lemma B.8 and estimate the resulting expression

$$\|\mathcal{G}_\lambda f\|^2_{H^{s+\frac{1}{2}}(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\frac{(1+|k|^2)^{s+\frac{1}{2}}}{||k|^2-\lambda|^2}|\mathcal{F}_df(k)|^2dk\ . \tag{B.9}$$

Choosing $r > 0$ large enough, then there exists some constant $c > 0$, such that inside and outside the cube $Q := [-r, r]^d$ the integrand can be estimated by

$$\frac{(1+|k|^2)^{s+\frac{1}{2}}}{||k|^2-\lambda|^2} \le c \begin{cases} 1 & \text{if } k \in Q \ , \\ \frac{1}{|k|^{3-2s}} & \text{if } k \in \mathbb{R}^d \setminus Q \ . \end{cases} \tag{B.10}$$

Using (B.10) in (B.9) gives

$$\|\mathcal{G}_\lambda f\|^2_{H^{s+\frac{1}{2}}(\mathbb{R}^d)} \le c \left(\int_Q |\mathcal{F}_d f(k)|^2 dk + \int_{\mathbb{R}^d \setminus Q} \frac{|\mathcal{F}_d f(k)|^2}{|k|^{3-2s}} dk \right) \ . \tag{B.11}$$

For further estimates distinguish now the cases $p = 2$ and $p < 2$.

- In the $p = 2$ case we can simply estimate the denominator $\frac{1}{|k|^{3-2s}}$ of the second integral and enlarge the domain of integration $\mathbb{R}^d$ in both integrals, to end up with the boundedness

$$\|\mathcal{G}_\lambda f\|^2_{H^{s+\frac{1}{2}}(\mathbb{R}^d)} \le c \left(1 + \frac{1}{r^{3-2s}} \right) \|f\|^2_{L^2(\mathbb{R}^d)} \ . \tag{B.12}$$

- In the $p < 2$ case we have to use Hölder's inequality with $\tilde{p} = \frac{p}{2-p}$ and $\tilde{q} = \frac{p}{2(p-1)}$ first, to get the correct powers of the Fourier transformation. In the first integral of (B.11) this means that

$$\int_Q |\mathcal{F}_d f(k)|^2 dk \le (2r)^{\frac{d(2-p)}{p}} \|\mathcal{F}_d f\|^2_{L^{\frac{p}{p-1}}(\mathbb{R}^d)} \ , \tag{B.13}$$

by using that the cube Q has finite measure $(2r)^d$ and the second integral becomes

$$\int_{\mathbb{R}^d \setminus Q} \frac{|\mathcal{F}_d f(k)|^2}{|k|^{3-2s}} dk \le \left(\int_{\mathbb{R}^d \setminus Q} \frac{1}{|k|^{\frac{(3-2s)p}{2-p}}} dk \right)^{\frac{2-p}{p}} \|\mathcal{F}_d f\|^2_{L^{\frac{p}{p-1}}(\mathbb{R}^d)} \ , \tag{B.14}$$

where the integral converges if and only if $\frac{(3-2s)p}{2-p} > d$, which is exactly our assumption $s < \frac{3}{2} - d(\frac{1}{p} - \frac{1}{2})$. Also the $L^{\frac{p}{p-1}}$-norm of $\mathcal{F}_d f$ can be estimated by (1.13). Using (B.13) & (B.14) in (B.11) finally gives also in this $p < 2$ case the boundedness

$$\|\mathcal{G}_\lambda f\|^2_{H^{s+\frac{1}{2}}(\mathbb{R}^d)} \le \tilde{c} \, \|f\|^2_{L^p(\mathbb{R}^d)} \ ,$$

for some constant $\tilde{c}$.

b) The same calculation as in a) can be used to obtain b).

d) In order to verify that $\mathcal{G}_\lambda^{\uparrow}\phi \in H^s(\mathbb{R}^d)$, for every $\phi \in L^p(\mathbb{R}^{d-1})$, we will use the Fourier transformation from Lemma B.8 and estimate the resulting expression

$$\|\mathcal{G}_\lambda^{\uparrow}\phi\|^2_{H^s(\mathbb{R}^d)} = \frac{1}{2\pi}\int_{\mathbb{R}^d} \frac{(1+|k|^2)^s}{||k|^2-\lambda|^2}|\mathcal{F}_{d-1}\phi(\tilde{k})|^2 dk \,. \tag{B.15}$$

Using again the estimate (B.10) and the additional $\tilde{Q} := [-r,r]^{d-1}$, we can split up $\mathbb{R}^d$ into

$$\begin{aligned} Q &= \tilde{Q}\times[-r,r] \quad \text{and} \\ \mathbb{R}^d\setminus Q &= \tilde{Q}\times(\mathbb{R}\setminus[-r,r]) \uplus \left(\mathbb{R}^{d-1}\setminus\tilde{Q}\right)\times\mathbb{R} \end{aligned} \tag{B.16}$$

and integrate over k_d analytically. This gives the inequality

$$\begin{aligned} \|\mathcal{G}_\lambda^{\uparrow}\phi\|^2_{H^s(\mathbb{R}^d)} \le \frac{c}{2\pi}\Bigg(& 2r\int_{\tilde{Q}} |\mathcal{F}_{d-1}\phi(\tilde{k})|^2 d\tilde{k} + \frac{2}{(3-2s)r^{3-2s}}\int_{\tilde{Q}} |\mathcal{F}_{d-1}\phi(\tilde{k})|^2 d\tilde{k} \\ &+ B\left(\frac{1}{2},\frac{3}{2}-s\right)\int_{\mathbb{R}^{d-1}\setminus\tilde{Q}} \frac{|\mathcal{F}_{d-1}\phi(\tilde{k})|^2}{|\tilde{k}|^{3-2s}} d\tilde{k}\Bigg) \,, \end{aligned} \tag{B.17}$$

where the second and third integral converge if and only if $s < \frac{3}{2}$, which is obviously satisfied by our assumptions on s and p.

As in a) distinguish now the cases $p = 2$ and $p < 2$ and also use the same estimates as in (B.12), (B.13) & (B.14) to obtain the boundedness

$$\|\mathcal{G}_\lambda^{\uparrow}\phi\|^2_{H^s(\mathbb{R}^d)} \le \tilde{c}\, \|\phi\|^2_{L^p(\mathbb{R}^{d-1})} \,.$$

c) The same calculation can now be used to obtain c). Again, with the representation of its Fourier transformation in Lemma B.8 the H^s-norm of $\mathcal{G}_\lambda^{\downarrow}f$ looks like

$$\begin{aligned} \|\mathcal{G}_\lambda^{\downarrow}f\|_{H^s(\mathbb{R}^{d-1})} &= \frac{1}{2\pi}\int_{\mathbb{R}^{d-1}} (1+|\tilde{k}|^2)^s \left|\int_{\mathbb{R}} \frac{\mathcal{F}_d f(k)}{|k|^2-\lambda} dk_d\right|^2 d\tilde{k} \\ &\le \frac{1}{2\pi}\int_{\mathbb{R}^d} \frac{(1+|k|^2)^s}{||k|^2-\lambda|^2}\int_{\mathbb{R}} |\mathcal{F}_d f(\tilde{k},\xi_d)|^2 d\xi_d dk \,, \end{aligned} \tag{B.18}$$

where in the second line the Hölder inequality was applied. The right hand side of (B.18) looks now the same as the right hand side of (B.15) if one

$$\text{replaces} \quad |\mathcal{F}_{d-1}\phi(\tilde{k})|^2 \quad \text{by} \quad \int_{\mathbb{R}} |\mathcal{F}_d f(\tilde{k},\xi_d)|^2 d\xi_d \,.$$

With this replacement all the steps from the first part of the proof are the same until one ends up with the boundedness

$$\|\mathcal{G}_\lambda^\downarrow f\|^2_{H^s(\mathbb{R}^{d-1})} \leq \tilde{c}\ \|f\|^2_{L^p(\mathbb{R}^d)}\ .$$

□

At the end of this section we will state a useful Corollary with respect to the boundedness of $\widetilde{\mathcal{G}}_\lambda$ in combination with the multiplication of a potential α.

Corollary B.10. Let $d \geq 3$, $\lambda \in \mathbb{C}\setminus[0,\infty)$, $p \in [1,\infty)$ and $\alpha \in L^p(\mathbb{R}^{d-1}) + L^\infty(\mathbb{R}^{d-1})$. Then for every $s < 1 - \frac{d-1}{p}$, there exists some constant $\tilde{c}_s > 0$, such that

$$\|\widetilde{\mathcal{G}}_\lambda[\alpha\phi]\|_{H^s(\mathbb{R}^{d-1})} \leq \tilde{c}_s \|\phi\|_{L^2(\mathbb{R}^{d-1})}\ , \quad \forall \phi \in L^2(\mathbb{R}^{d-1})\ . \tag{B.19}$$

Furthermore, for every $s < \frac{3}{2} - \frac{d-1}{p}$, there exists another constant $c_s^\uparrow > 0$, such that

$$\|\mathcal{G}_\lambda^\uparrow[\alpha\phi]\|_{H^s(\mathbb{R}^d)} \leq c_s^\uparrow\ \|\phi\|_{L^2(\mathbb{R}^{d-1})}\ , \quad \forall \phi \in L^2(\mathbb{R}^{d-1})\ . \tag{B.20}$$

Proof. By assumption we can decompose $\alpha = u + v$, for some $u \in L^p(\mathbb{R}^{d-1})$ and $v \in L^\infty(\mathbb{R}^{d-1})$. Therefore also the norm

$$\|\widetilde{\mathcal{G}}_\lambda[\alpha\phi]\|_{H^s(\mathbb{R}^{d-1})} \leq \|\widetilde{\mathcal{G}}_\lambda[u\phi]\|_{H^s(\mathbb{R}^{d-1})} + \|\widetilde{\mathcal{G}}_\lambda[v\phi]\|_{H^s(\mathbb{R}^{d-1})} \tag{B.21}$$

splits up and we can estimate both terms separately.

Because of $s < 1 - \frac{d-1}{p}$ we can estimate the first term by the boundedness of $\widetilde{\mathcal{G}}_\lambda : L^{\frac{2p}{p+2}}(\mathbb{R}^{d-1}) \to H^s(\mathbb{R}^{d-1})$ from Proposition B.9 and by Lemma A.1:

$$\begin{aligned}\|\widetilde{\mathcal{G}}_\lambda[u\phi]\|_{H^s(\mathbb{R}^{d-1})} &\leq \|\widetilde{\mathcal{G}}_\lambda\|_{L^{\frac{2p}{p+2}},H^s} \|u\phi\|_{L^{\frac{2p}{p+2}}(\mathbb{R}^{d-1})} \\ &\leq \|\widetilde{\mathcal{G}}_\lambda\|_{L^{\frac{2p}{p+2}},H^s} \|u\|_{L^p(\mathbb{R}^{d-1})} \|\phi\|_{L^2(\mathbb{R}^{d-1})}\ .\end{aligned} \tag{B.22}$$

The second term can be estimated by the boundedness $\widetilde{\mathcal{G}}_\lambda : L^2(\mathbb{R}^{d-1}) \to H^s(\mathbb{R}^{d-1})$ and the L^∞-norm of the potential v:

$$\begin{aligned}\|\widetilde{\mathcal{G}}_\lambda[v\phi]\|_{H^s(\mathbb{R}^{d-1})} &\leq \|\widetilde{\mathcal{G}}_\lambda\|_{L^2,H^s} \|v\phi\|_{L^2(\mathbb{R}^{d-1})} \\ &\leq \|\widetilde{\mathcal{G}}_\lambda\|_{L^2,H^s} \|v\|_{L^\infty(\mathbb{R}^{d-1})} \|\phi\|_{L^2(\mathbb{R}^{d-1})}\ .\end{aligned} \tag{B.23}$$

Inserting (B.22) & (B.23) into (B.21) finishes the first part of the proof. The same strategy can be used to estimate $\mathcal{G}_\lambda^\uparrow$ in (B.20). □

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