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Superoscillations
and their Schrödinger time evolution

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Abstract

Superoscillations are functions with the paradoxical behavior of oscillating (at least locally) faster than their largest Fourier component. For example, plane waves with small frequencies (e.g. red light) can interfere in such a way that a resulting wave with arbitrary large frequency is created (e.g. blue light or theoretically even a gamma ray).

Due to the wave-particle duality in quantum mechanics, also particles can exhibit this superoscillatory behavior. One of the main questions in this context then is: What happens when a particle with a superoscillating wave function interacts with a potential? Does this delicate interference effect persists in time, or is it destroyed by the external force? Mathematically this means: Is the solution of the time-dependent Schrödinger equation, with superoscillatory initial condition, still superoscillating at later times?

So far, answers to this question only have been given for some potentials, for which in particular the associated Green's function is known explicitly. The aim of this doctoral thesis is now to develop a general approach that proves the time persistence of superoscillations not only for individual but for whole classes of potentials. The explicit form of the Green's function will no longer be needed, only qualitative properties as holomorphicity and growth conditions will be assumed.

Another problem addressed in this thesis is, that there is still no uniform definition of a superoscillating function existing. Superoscillations have been developed by various scientific disciplines over the years and there is a certain discrepancy between the mathematical and the physical perspectives in particular. The aim of this work was to close this gap by giving a generally valid definition of a superoscillating function. Furthermore, it is also proven, that all popular variants of superoscillations from the mathematical and physical literature satisfy this definition.

Kurzfassung

Superoszillationen sind Funktionen mit dem paradoxen Verhalten, (zumindest lokal) schneller zu oszillieren als ihre größte Fourier-Komponente das vermuten lassen würde. Man kann beispielsweise ebene Wellen mit kleinen Frequenzen (zum Beispiel rotes Licht) derart interferieren lassen, so dass eine resultierende Welle mit beliebig großer Frequenz entsteht (zum Beispiel blaues Licht oder theoretisch sogar Gamma Strahlung).

Aufgrund des Welle-Teilchen-Dualismus der Quantenmechanik können auch Teilchen dieses superoszillierende Verhalten aufweisen. Eine der Hauptfragen in diesem Zusammenhang ist: Was passiert, wenn ein Teilchen mit superoszillierender Wellenfunktion mit einem Potential interagiert? Bleibt dieser empfindliche Interferenzeffekt erhalten oder wird er durch die äußere Kraft zerstört? Mathematisch bedeutet dies: Ist die Lösung der zeitabhängigen Schrödingergleichung, mit einer superoszillierenden Funktion als Anfangsbedingung, zu einem späteren Zeitpunkt immer noch superoszillierend?

Antwort auf diese Frage wurde bisher nur für einige wenige Potentiale gegeben, bei denen insbesondere die zugehörige Green'sche Funktion explizit bekannt ist. Ziel dieser Doktorarbeit ist es nun, einen allgemeinen Zugang zu diesem Problem zu entwickeln, der die zeitliche Stabilität von Superoszillationen nicht mehr nur für einzelne sondern für ganze Klassen von Potentialen beweist. Dabei soll nicht mehr die explizite Form der Green'schen Funktion verwendet, sondern lediglich qualitative Eigenschaften wie Holomorphie und Wachstumsverhalten derselben vorausgesetzt werden.

Ein weiteres Thema dieser Doktorarbeit beschäftigt sich mit dem Problem, dass es noch keine einheitliche Definition einer superoszillierenden Funktion gibt. Superoszillationen wurden im Laufe der Jahre von verschiedenen wissenschaftlichen Disziplinen entwickelt und es gibt insbesondere eine gewisse Diskrepanz zwischen der mathematischen und der physikalischen Sichtweise. Ziel dieser Arbeit ist es, diese Lücke zu schließen, indem eine allgemein gültige Definition einer superoszillierenden Funktion niedergeschrieben wird. Weiters wird auch bewiesen, dass alle in der mathematischen und physikalischen Literatur gängigen Varianten von Superoszillationen ebendieser Definition entsprechen.

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Contents

0. Introduction

0.1. What are superoscillations

Superoscillations are functions with the paradoxical behaviour to (locally) oscillate faster than their largest Fourier component. For instance, one can take plane waves, all having small frequencies, and interfere them in a way that the resulting superposition has a larger frequency than all the original waves had. What happens is an almost destructive interference, leaving a remainder with a small amplitude but an unexpected high frequency. One can even choose this interference such that the resulting frequency as well as the superoscillating region becomes arbitrary large. However, the price to pay is an exponential decrease in the amplitude, which is then of course very sensitive to noise, as pointed out in [47].

Although this effect may sound surprising, already the following simple example shows, that the above described increase in frequency is indeed possible. Consider for $n \in \mathbb{N}$ and $k > 1$ the sequence of functions

$$F_n(x) = \Big(\cos\Big(\frac{x}{n}\Big) + ik\sin\Big(\frac{x}{n}\Big)\Big)^n = \sum_{j=0}^{n} C_j(n)e^{ik_j(n)x}, \quad x \in \mathbb{R}, \tag{0.1}$$

with coefficients

$$C_j(n) = \binom{n}{j}\Big(\frac{1+k}{2}\Big)^{n-j}\Big(\frac{1-k}{2}\Big)^{j} \quad \text{and} \quad k_j(n) = 1 - \frac{2j}{n}. \tag{0.2}$$

In particular one sees, that F_n is a linear combination of plane waves with frequencies $k_j(n) \in [-1, 1]$. The superocillatory behaviour now comes from the fact, that

$$\lim_{n\to\infty} F_n(x) = e^{ikx}, \quad x \in \mathbb{R}, \tag{0.3}$$

converges to a plane wave with frequency $k > 1$. The convergence (0.3) can be understood as uniform on compact subsets of $\mathbb{R}$, or if one considers the F_n as complex functions, i.e., replaces $x \in \mathbb{R}$ by $z \in \mathbb{C}$, this sequence even converges in the space $\mathcal{A}_1(\mathbb{C})$ of entire functions with exponential growth. See Chapter 1 and in particular Proposition 1.8 for a detailed discussion.

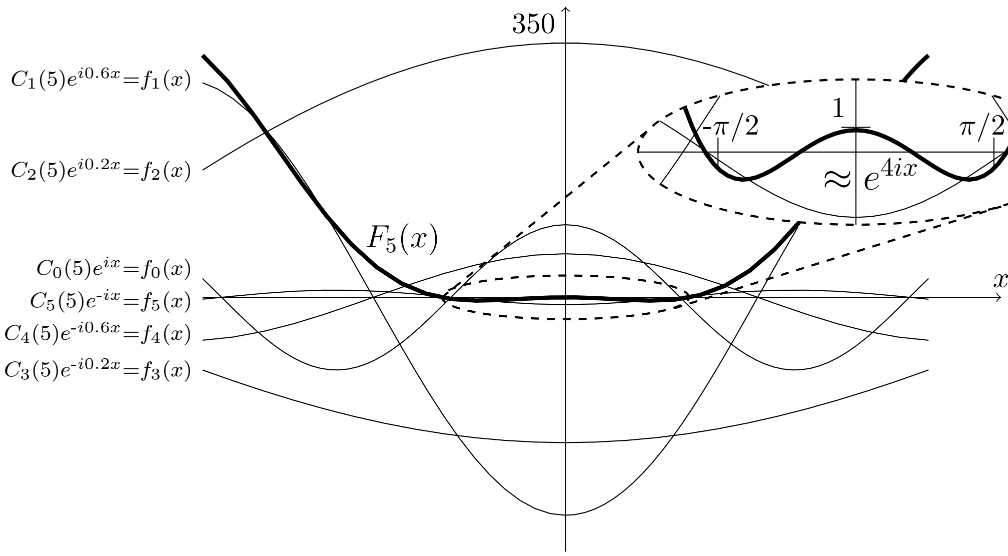


Figure 0.1.: Illustration of the superoscillatory function (0.1) for $n = 5$ and $k = 4$. The thin lines show the functions $f_j(x) = C_j(5)e^{ik_j(5)x}$, which add up to the function $F_5(x) = \sum_{j=0}^{5} f_j(x)$. Observe the almost destructive interference of the f_j, the high frequency of F_5 in the central region, and also the small amplitude inside and the larger amplitude outside the superoscillatory region.

But it is not only the limit $n \to \infty$ what makes the F_n superoscillating. In [51] for example, the authors interpret the superoscillating property as the size of the local wave number

$$\begin{aligned} k_{\text{loc}}(x) &:= \frac{d}{dx}\operatorname{Arg}(F_n(x)) = \frac{d}{dx}\operatorname{Arg}\left(\left(\cos\left(\frac{x}{n}\right) + ik\sin\left(\frac{x}{n}\right)\right)^n\right) \\ &= n\frac{d}{dx}\operatorname{Arg}\left(\cos\left(\frac{x}{n}\right) + ik\sin\left(\frac{x}{n}\right)\right) = n\frac{d}{dx}\arctan\left(k\tan\left(\frac{x}{n}\right)\right) \\ &= \frac{k}{\cos^2(\frac{x}{n}) + k^2\sin^2(\frac{x}{n})}. \end{aligned}$$

This local wave number shows, that the frequency varies in between $k_{\text{loc}}(0) = k$ and $k_{\text{loc}}(\frac{n\pi}{2}) = \frac{1}{k}$. In particular, it is larger than 1 in the region $|x| \leq n\cot^{-1}(\sqrt{k})$. Moreover, the number of oscillations in this region is

$$n_{\text{osc}} = \frac{1}{2\pi}\int_{-n\cot^{-1}(\sqrt{k})}^{n\cot^{-1}(\sqrt{k})} k_{\text{loc}}(x)dx = \frac{n}{\pi}\int_{\sqrt{k}}^{\infty} \frac{k}{u^2 + k^2}dk = \frac{n}{\pi}\tan^{-1}(\sqrt{k}),$$

which is more than $\frac{n}{\pi}\cot^{-1}(\sqrt{k})$, the number of oscillations a wave with frequency 1 would admit.

0.2. From the beginnings of superoscillations to the present state of art

The early pioneer of the concept of superoscillations was G. Toraldo di Francia who already used this effect in the 1950s to create narrow beams in antenna theory [109]. He also recognized that his research may have an impact on optics, allowing subwavelength resolution in microscopy. However, unable to produce a suitable lens system at that time, these ideas took another 60 years until they were realized in the first practical imaging aperture, built by N. I. Zheludev and collaborators in 2012 [60].

The initiator of the modern study of superoscillations was Y. Aharonov, who in his still unpublished preprint [24] from 1991 made a thought experiment considering a box only containing red light, but emitting a gamma ray. His early attempts to reformulate quantum mechanics as a time-symmetric theory [8], gave rise to the so called weak value of a quantum observable. A weak value is the result of a measurement which selects independent initial and final states [10, 11, 23, 26, 30, 65]. As a consequence, Y. Aharonov found a way *How the result of a measurement of a component of the spin of a spin-$\frac{1}{2}$ particle can turn out to be 100* [3]. It was also shown in [108] that these measurements may observe information without disturbing the system, which was against the common believe that weak values can only be determined in statistical ensembles. Continuing this line of thought even questions the present understanding and interpretation of conservation laws [25].

It was then M. Berry, who started a mathematically more rigorous treatment of superoscillations in his work *Faster than Fourier* [43]. He constructed whole families of superoscillating functions and investigated them systematically with respect to their superoscillatory behaviour. In a series of papers, [43, 44, 46, 48, 49, 51, 52] just to name some of them, he also explored various ramifications of superoscillations in different areas of physics and mathematics. This, as a consequence, made superoscillations more popular and attracted many researchers in the subsequent years to contribute in this research field.

In quantum mechanics, superoscillations appear as initial conditions of the time dependent Schrödinger equation

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(t,x) &= \Big(-\frac{\partial^2}{\partial x^2}+V(t,x)\Big)\Psi(t,x), \\ \Psi(0,x) &= F(x), \end{aligned}$$

and evolve in time accordingly. Since, as mentioned in Section 0.1, superoscillations arise from a delicate, almost destructive, interference, it is a question of central importance, whether they are destroyed or preserved when interacting with some external potential. In other words: If the initial condition F of the time dependent Schrödinger equation is superoscillatory, is this still the case for the solution $\Psi(t,\,\cdot\,)$ at later times $t>0$?

The first contribution to this problem was [43], where it was shown, that for free particles the superoscillatory behaviour occurs within a region $|x|<\mathcal{O}(n)$ and within the time $t<\mathcal{O}(n)$. In particular, in the limit $n\to\infty$ this property is preserved for all times $t>0$, as first shown in [15]. Also for nonvanishing potentials the time persistence of superoscillations was investigated, as for example the harmonic oscillator was considered in [22, 33, 35, 55, 56, 68], the electric field in [19, 22, 35, 55], the magnetic field in [22, 66], the centrifugal potential in [22, 33, 67, 68], the step potential in [20] and distributional potentials as δ and δ' in [5, 6, 36].

This time evolution property of superoscillations is of course interesting not only for the Schrödinger equation but also for other evolution equations as the Dirac or the Klein-Gordon equation. So far, only very little is known in those settings and no more than the following three basic contributions exist in the mathematical literature. In [88], the authors numerically consider the time evolution of superoscillating spin-0 wavepackets in the Klein-Gordon equation as well as spin-$\frac{1}{2}$ particles in the Dirac equation. In the work [21] again the Klein-Gordon field was considered, but this time the time persistence of superoscillations was proven from a mathematical point of view. Also for the Dirac equation, the analogue time persistence was considered in [72]. This field of relativistic evolution of superoscillations, has numerous open questions, and will for sure be subject of future research.

0.3. Topics of this thesis

The drawback of all the time evolution results mentioned in Section 0.2 is, that only specific potentials were considered, for which in particular the corresponding Green's function is known explicitly, i.e. the function G which connects the initial value F to the wave function Ψ via the integral

$$\Psi(t,x)=\int_{\mathbb{R}} G(t,x,y)F(y)dy.$$

One of the main novelties of this thesis is a generalization of the above results, considering whole classes of potentials simultaneously. This theory will be developed in Chapter 4 for potentials defined on the whole real line $\mathbb{R}$, and in Chapter 5 for potentials on $\mathbb{R}\setminus\{0\}$, having singularities at $x=0$ or being distributional as δ- and δ'-potentials. In particular, almost all potentials which were already considered separately in the existing literature, see also the list in Section 0.2, are covered by this unified approach. The key feature of this general theory is to impose qualitative properties on the corresponding Green's function, see Assumption 4.1 or Assumption 5.1, which allow to prove a continuous dependency result between the initial condition and the solution of the time dependent Schrödinger equation. Roughly speaking, for a sequence of converging initial conditions

$$\lim_{n\to\infty} F_n = F$$

we prove, that the sequence of solutions still converges as

$$\lim_{n\to\infty} \Psi(t,x;F_n)=\Psi(t,x;F),$$

see Theorem 4.6 and Theorem 5.4 for details.

However, this continuous dependency is not enough to conclude time persistence of superoscillations, i.e., the superoscillatory behaviour of the solutions $\Psi(t,x;F)$. It turns out, that the precise mathematical definition of superoscillations is too narrow to persist in time. The reason is, that if we define superoscillations via the convergence to a plane wave, as in (0.3), some potential will in general deform this plane wave and the resulting limit $\Psi(t,x;e^{ik\,\cdot})$ may still be oscillating in some sense, but in general no longer as a plane wave. This problem was already realized in [6, 20, 33, 68, 72] and inspired the notion of *supershift*, where one principally neglects the oscillatory behaviour of the plane waves $e^{ik\,\cdot}$ and substitutes them by arbitrary functions φ_k. Motivated by (0.1) this means, that the supershift in the existing literature, considers functions of the form

$$F_n(x)=\sum_{j=0}^{n}C_j(n)\varphi_{\kappa_j(n)}(x), \tag{0.4}$$

with coefficients $C_j(n)\in\mathbb{C}$ and $\kappa_j(n)\in\mathcal{U}$ in some region $\mathcal{U}$, which converge to

$$\lim_{n\to\infty}F_n(x)=\varphi_k(x), \tag{0.5}$$

for some $k\in\mathbb{C}\setminus\mathcal{U}$. One of the main results of this thesis is now Theorem 6.5, where we use the above mentioned novel unified approach to prove the time persistence of the supershift property.

The second part of this thesis addresses the problem that the theory of superoscillations was developed by different communities using different approaches throughout the years. For instance, beside the standard example (0.1), M. Berry constructed a whole family of superoscillating functions in [43], which are given by some integral of the form

$$F_n(x)=\int_{\mathbb{R}}C_n(u)e^{ik_n(u)x}du. \tag{0.6}$$

In order to unify those functions with the ones in (0.1), the notion of a *Type I superoscillating sequence* is introduced in Definition 2.1. In this definition, a sequence of functions $(F_n)_n$ has to be of the form

$$F_n(z)=\int_{-k_0}^{k_0}e^{ikz}d\mu_n(k),\quad z\in\mathbb{C}, \tag{0.7}$$

with some $k_0>0$ and complex Borel measures μ_n on $[-k_0,k_0]$. While this representation indeed covers both (0.1) and (0.6), see the choice of measures μ_n in Lemma 2.4, the required convergence

$$\lim_{n\to\infty}F_n(z)=e^{ikz}, \tag{0.8}$$

to some plane wave with frequency $k>1$, is until now only known for the functions (0.1), see [70, Lemma 2.4] or [69, Lemma 1]. In [43], complex saddle point methods were used to verify the superoscillatory properties of (0.6), but a convergence of the form (0.8) was not given there. This missing part is now added in Section 2.2, where Theorem 2.6 proves the required convergence, which makes (0.6) a Type I superoscillating sequence. The same is done in Section 2.3 for the one specific example given in [46]. Also for the functions in [12] it is proven in Section 2.4, and for the functions constructed in [16, 28] it is shown in Section 2.1, that all of them can be considered as Type I superoscillating sequences.

However, in order to agree with the new Type I superoscillations, also the notion of supershift in (0.4) is extended to the more general form

$$F_n(x) = \int_{\mathcal{U}} \varphi_\kappa(x) d\mu_n(\kappa),$$

in Definition 6.1.

Another type of superoscillatory function was constructed in [79, 91], where the functions still admit an integral representation of the form (0.7), but the superoscillatory property manifests itself no longer by the convergence (0.8), but by the number of zeros inside some interval. In Definition 2.2 we introduce the terminology *Type II superoscillating function* for these kind of functions.

The authors of [79, 91] proved, that there exist functions of the form

$$F(x) = \int_{-k_0}^{k_0} C(k) e^{ikx} dk,$$

for arbitrary small $k_0 > 0$ which have arbitrary many isolated zeros in some arbitrary small interval. Moreover, they even managed to construct a particular function having minimal L^2-norm, while passing through an arbitrary set of prescribed points. Due to the exponentially small amplitudes of superoscillations, generating them can be very energy expensive and this minimal energy result may be useful in practical applications. One result regarding these Type II superoscillating functions is now given in Theorem 2.16, where the just mentioned method from [79] is generalized to not only prescribed values of the function itself, but to also allow prescribed values of any derivative. Note, that a particular generalization to include the first derivative was already done in [84]. These additional prescribed values of the derivatives can now be used to control the shape of the resulting superoscillating function.

Summing up, in order to organize the existing literature, a mathematical precise definition of two types of superoscillations is given in Chapter 2. It is then a collection of various results, which categorizes most variants of superoscillating functions in the existing literature into those two classes. At some places missing parts were added or existing results generalized, as the mentioned convergence of Berry's superoscillating functions in [43] or [46] in Theorem 2.6 and Theorem 2.9 or the additional prescribed values of the derivatives of the Kempf-Ferreira superoscillations in Theorem 2.16. All of this is done to give a more complete picture of how the different superoscillating functions are related to each other. Closing these gaps is another step in the direction of a complete picture of the theory of superoscillations.

0.4. Applications of superoscillations

In this last part of the introduction we want to give a short overview of (possible) applications of superoscillations which are sometimes of realistic and sometimes of more academic nature.

One quantum mechanical thought experiment is the acceleration through a slit [57]. A particle with a bounded momentum range may have a superoscillatory wave function which locally oscillates with a shorter wavelength then classically allowed. If this superoscillation exactly happens at the position of a slit in some screen, only the superoscillatory part of the wave function will pass through and the rest of the wave stays on this side of the screen. Since the amplitude in the superoscillatory region is very small, this event of the particle gaining momentum is very unlikely, but still possible.

There is a large field of applications in optical superresolution, which is a way of analyzing a probe in details smaller than the diffraction limit. The theoretical basis was given by Berry and Popescu, who in [51] proved, that the superoscillatory sub-wavelength structure retained without evanescent waves. This method was then first tested in 2007 by Huang and Zheludev [89], who managed to focus light through a crystal nano-hole array. Together with coauthors they improved their technique and lenses, see [1, 61, 90, 111], until the first practical imaging apparature was built in 2012 [60]. A large progress was also made by the invention of the "metamaterial super-lens" in [104], which allows focus points of arbitrary shapes. For a rather complete overview, we refer the reader to the *Roadmap of superoscillations* [9].

Superoscillations can also be used in combination with surfaces whose reflection and transmission properties are frequency dependent, see [94]. On the one hand, if one sends a beam towards a surface which reflects low but transmits high frequencies, its distance can only be measured with accuracy according to the reflected long wavelength. A superoscillating signal instead will also be reflected completely (since it only consists of long wavelengths), but the accuracy of measuring the distance increases to the wavelength of the superoscillatory region. Conversely, if the surface only reflects high but transmits low frequencies, one can do imaging behind the surface. Classically, one can only do this with a resolution determined by the long wavelength. Using superoscillations, one can choose the superoscillatory region to be behind the surface, which means, that all the long wavelengths pass through the surface and interfere to a superoscillating high frequency behind the wall. This allows a related high resolution of images.

Finally, we also want to mention the papers [92, 93], where A. Kempf collected some (thought) experiments how one could use and apply superoscillations. Topics are the usage of superresolution for the detection of landmines, superabsorbtion in optogenetics, proving a generalization of the Shannon-Hartley theorem, recording a Beethoven symphony with a 1 Hz bandlimited signal, data compression and the trans-Planckian problem of black hole radiation.

1. The spaces $\mathcal{A}_p(\mathbb{C})$

In the introduction we mentioned the standard example (0.1) of a superoscillating sequence, which is, by its physical origin, a function of one real variable $x \in \mathbb{R}$ and converges uniformly on compact sets, but not uniformly on all of $\mathbb{R}$ [13, Theorem 4.3]. Unfortunately it turns out that in view of Chapter 6, the time persistence of superoscillations, the uniform convergence only on compact sets is not enough. However, noting that the constructed examples (2.16) and (2.20) are entire functions of the complex variable $z \in \mathbb{C}$ and also that the standard example (0.1) obviously admits an entire extension, it is a more suitable way to consider superoscillating functions as elements in the space $\mathcal{A}_p(\mathbb{C})$ of entire functions with exponential growth. The aim of this chapter is now to introduce these spaces and prove basic properties which will be of importance in the applications throughout this thesis. In the context of superoscillations it is already state of the art to consider convergences and operator continuity in these spaces, see for example [4, 5, 19, 33, 35, 66, 68]. we want to mention that these spaces are special cases of Analytically Uniform spaces (AU-spaces), introduced by Ehrenpreis in [75] and enhanced by Berenstein, Taylor and coauthors in [38, 41, 42, 107]. An overview can also be found in [16, 39, 40].

Definition 1.1. Let $\mathcal{H}(\mathbb{C})$ denote the set of all entire functions on $\mathbb{C}$. Then for every $p > 0$ define the *space of entire functions with exponential growth of order* p as

$$\mathcal{A}_p(\mathbb{C}) := \left\{ F \in \mathcal{H}(\mathbb{C}) \;\middle|\; \exists A, B \geq 0 \text{ such that } |F(z)| \leq Ae^{B|z|^p} \text{ for all } z \in \mathbb{C} \right\}. \tag{1.1}$$

A sequence of functions $(F_n)_n \in \mathcal{A}_p(\mathbb{C})$ converges to $F_0 \in \mathcal{A}_p(\mathbb{C})$ in $\mathcal{A}_p(\mathbb{C})$, if and only if there exists some $B \geq 0$, such that

$$\lim_{n\to\infty} \sup_{z\in\mathbb{C}} |F_n(z) - F_0(z)|e^{-B|z|^p} = 0. \tag{1.2}$$

We will write $F_n \xrightarrow{\mathcal{A}_p} F_0$ for this type of convergence.

For more details about the topology of the space $\mathcal{A}_p(\mathbb{C})$ which leads to the convergence (1.2) we refer to [28, Section 2] and [39, Section 2.1]. For the purpose of this thesis, the detailed structure of the topology will not be of importance.

The following lemma shows, that the spaces $\mathcal{A}_p(\mathbb{C})$ are continuously included in each other, and also give some relation to the uniform convergence.

Lemma 1.2. For every $0 < p \leq q$ we have $\mathcal{A}_p(\mathbb{C}) \subseteq A_q(\mathbb{C})$ and for any sequence

$F_0, (F_n)_n \subseteq A_p(\mathbb{C})$ we get the following implications of convergences

$$\begin{array}{ll} F_n \overset{n\to\infty}{\longrightarrow} F_0 & \text{in } \mathcal{A}_p(\mathbb{C}) \\ \Downarrow & \\ F_n \overset{n\to\infty}{\longrightarrow} F_0 & \text{in } \mathcal{A}_q(\mathbb{C}) \\ \Downarrow & \\ F_n \overset{n\to\infty}{\longrightarrow} F_0 & \text{uniformly on every compact } K \subseteq \mathbb{C} \end{array}$$

Proof. Let $F \in \mathcal{A}_p(\mathbb{C})$. Then by definition there exist $A, B \geq 0$ such that

$$|F(z)| \leq Ae^{B|z|^p}, \quad z \in \mathbb{C}.$$

It then immediately follows from $p \leq q$ that also

$$|F(z)| \leq A \begin{cases} e^{B|z|^q}, & \text{if } |z| \geq 1, \\ e^B, & \text{if } |z| \geq 1, \end{cases} \leq Ae^B e^{B|z|^q}, \quad z \in \mathbb{C}, \tag{1.3}$$

which proves $F \in A_q(\mathbb{C})$. Let now $F_n \overset{n\to\infty}{\longrightarrow} F_0$ in $\mathcal{A}_p(\mathbb{C})$. By definition this means

$$\lim_{n\to\infty} \sup_{z\in\mathbb{C}} |F_n(z) - F_0(z)|e^{-B|z|^p} = 0,$$

for some $B \geq 0$. By the same estimate as in (1.3) we get

$$\sup_{z\in\mathbb{C}} |F_n(z) - F_0(z)|e^{-B|z|^q} \leq e^B \sup_{z\in\mathbb{C}} |F_n(z) - F_0(z)|e^{-B|z|^p} \overset{n\to\infty}{\longrightarrow} 0,$$

and hence the convergence $F_n \overset{n\to\infty}{\longrightarrow} F_0$ in $A_q(\mathbb{C})$ follows. Furthermore, for every $r > 0$ we get

$$\begin{aligned} \sup_{|z|\leq r} |F_n(z) - F(z)| &\leq e^{Br^q} \sup_{|z|\leq r} |F_n(z) - F(z)|e^{-B|z|^q} \\ &\leq e^{Br^q} \sup_{z\in\mathbb{C}} |F_n(z) - F(z)|e^{-B|z|^q} \overset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Hence $F_n \overset{n\to\infty}{\longrightarrow} F_0$ uniformly on every ball of radius $r > 0$ and hence also on every compact subset $K \subseteq \mathbb{C}$. □

One standard example of an $\mathcal{A}_p$-function is the following Mittag-Leffler function (1.4). This function will also play a central role in the upcoming estimates of this chapter.

Definition 1.3 (Mittag-Leffler function)**.** For every $\alpha, \beta > 0$, we define the *Mittag-Leffler function* as the power series

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}, \tag{1.4}$$

where Γ denotes the well known Gamma function.

Lemma 1.4. Let $\alpha, \beta > 0$. Then there exists some $A_{\alpha,\beta} \geq 0$, such that

$$|E_{\alpha,\beta}(z)| \leq A_{\alpha,\beta} e^{|z|^{\frac{1}{\alpha}}}, \quad z \in \mathbb{C}. \tag{1.5}$$

In particular we have $E_{\alpha,\beta} \in A_{\frac{1}{\alpha}}(\mathbb{C})$.

Proof. Due to [99, Eq. (1.1) & (1.2)], the Mittag-Leffler function admits the asymptotic behaviour

$$E_{\alpha,\beta}(x) = \frac{1}{\alpha} \sum_{n=-[\frac{\alpha-1}{2}]}^{[\frac{\alpha-1}{2}]} e^{x^{\frac{1}{\alpha}} e^{\frac{2\pi i n}{\alpha}}}, \quad \text{as } x \to \infty,$$

where $[\frac{\alpha-1}{2}]$ denotes the nearest integer part, i.e. $[\frac{\alpha-1}{2}] = N \in \mathbb{N}_0$, if $\frac{\alpha-1}{2} \in (-\frac{N}{2}, \frac{N}{2}]$. This sum can now be estimated as

$$E_{\alpha,\beta}(x) \leq \frac{1}{\alpha} \sum_{n=-[\frac{\alpha-1}{2}]}^{[\frac{\alpha-1}{2}]} e^{x^{\frac{1}{\alpha}} \cos(\frac{2\pi n}{\alpha})} \leq \frac{1}{\alpha}\Big(2\Big[\frac{\alpha-1}{2}\Big] + 1\Big) e^{x^{\frac{1}{\alpha}}}, \quad \text{as } x \to \infty.$$

Since $E_{\alpha,\beta}$ is an entire function, it is in particular bounded on any compact interval and hence there exists some $A_{\alpha,\beta} \geq 0$ such that

$$E_{\alpha,\beta}(x) \leq A_{\alpha,\beta} e^{x^{\frac{1}{\alpha}}}, \quad x \geq 0.$$

For the complex argument then immediately follows the desired

$$|E_{\alpha,\beta}(z)| \leq E_{\alpha,\beta}(|z|) \leq A_{\alpha,\beta} e^{|z|^{\frac{1}{\alpha}}}, \quad z \in \mathbb{C}. \qquad \square$$

With this result about the Mittag-Leffler function, we can now characterize the space $\mathcal{A}_p(\mathbb{C})$ and the corresponding convergence (1.2) in terms of the power series coefficients of the respective functions. See also [35, Lemma 2.2].

Lemma 1.5. For every $p > 0$, the space $\mathcal{A}_p(\mathbb{C})$ can be characterized by

$$\mathcal{A}_p(\mathbb{C}) = \left\{ F \in \mathcal{H}(\mathbb{C}) \;\middle|\; \exists A, B \geq 0 \text{ such that } |f_k| \leq A \frac{B^k}{\Gamma(\frac{k}{p}+1)} \text{ for all } k \in \mathbb{N}_0 \right\}, \tag{1.6}$$

where $(f_k)_k \in \mathbb{C}$ are the coefficients of the power series representation $F(z) = \sum_{k=0}^{\infty} f_k z^k$. Moreover, any sequence $(F_n)_n \in \mathcal{A}_p(\mathbb{C})$ converges to $F_0 \in \mathcal{A}_p(\mathbb{C})$ in $\mathcal{A}_p(\mathbb{C})$ if and only if there exist $(A_n)_n, B \geq 0$, such that

(i) $|f_{n,k} - f_{0,k}| \leq A_n \frac{B^k}{\Gamma(\frac{k}{p}+1)}, \quad k, n \in \mathbb{N}_0,$

(ii) $\lim_{n\to\infty} A_n = 0.$

Proof. We start with the inclusion " $\subseteq$ " of (1.6). Let $F \in \mathcal{A}_p(\mathbb{C})$. By (1.1), there exist $A, B \geq 0$ such that $|F(z)| \leq Ae^{B|z|^p}$, for every $z \in \mathbb{C}$. Without loss of generality we will assume $B > 0$. Using the Cauchy integral formula, the power series coefficients are given by

$$f_k = \frac{1}{2\pi i}\int_\gamma \frac{F(\xi)}{\xi^{k+1}}d\xi = \frac{1}{2\pi r^k}\int_0^{2\pi} F(re^{i\varphi})e^{-ik\varphi}d\varphi, \quad k \in \mathbb{N},$$

where γ is a circle around the origin with a not yet specified radius $r > 0$. This integral can now be estimated as

$$|f_k| \leq \frac{A}{2\pi r^k}\int_0^{2\pi} e^{Br^p}d\varphi = \frac{A}{r^k}e^{Br^p}, \quad k \in \mathbb{N}.$$

Since the right hand side depends on the free parameter $r > 0$, we can minimize it by choosing $r = (\frac{k}{Bp})^{\frac{1}{p}}$, which is possible since we assumed $B > 0$ and only considered $k \in \mathbb{N}$. This gives the upper bound

$$|f_k| \leq A\Big(\frac{Bep}{k}\Big)^{\frac{k}{p}}, \quad k \in \mathbb{N}. \tag{1.7}$$

Due to the asymptotic behaviour $\Gamma(\frac{k}{p}+1) \sim \sqrt{2\pi\frac{k}{p}}(\frac{k}{ep})^{\frac{k}{p}}$, as $k \to \infty$ of the Gamma function [2, Eq.(6.1.37)], we get

$$\Big(\frac{ep}{k}\Big)^{\frac{k}{p}}\frac{\Gamma(\frac{k}{p}+1)}{2^{\frac{k}{p}}} \sim \frac{\sqrt{2\pi\frac{k}{p}}}{2^{\frac{k}{p}}}, \quad \text{as } k \to \infty.$$

Since the right hand side tends to zero as $k \to \infty$, there exists some $A_\Gamma \geq 1$, such that

$$\Big(\frac{ep}{k}\Big)^{\frac{k}{p}}\frac{\Gamma(\frac{k}{p}+1)}{2^{\frac{k}{p}}} \leq A_\Gamma, \quad k \in \mathbb{N}. \tag{1.8}$$

Using this in (1.7) gives

$$|f_k| \leq AA_\Gamma\frac{(2B)^{\frac{k}{p}}}{\Gamma(\frac{k}{p}+1)}, \quad k \in \mathbb{N}. \tag{1.9}$$

The missing term $k = 0$ can be estimated as $|f_0| = |F(0)| \leq A$, and since we chose $A_\Gamma \geq 1$, it is also covered by (1.9).

For the inverse inclusion " $\supseteq$ " let the coefficients be bounded as $|f_k| \leq A\frac{B^k}{\Gamma(\frac{k}{p}+1)}$, for some $A, B \geq 0$. Then we can estimate the function as

$$|F(z)| \leq A\sum_{k=0}^\infty \frac{(B|z|)^k}{\Gamma(\frac{k}{p}+1)} = AE_{\frac{1}{p},1}(B|z|), \quad z \in \mathbb{C},$$

using the Mittag-Leffler function (1.4). From Lemma 1.4 we then conclude the estimate

$$|F(z)| \leq AA_{\frac{1}{p},1}e^{B^p|z|^p}, \quad z \in \mathbb{C}, \tag{1.10}$$

which proves, that $F \in \mathcal{A}_p(\mathbb{C})$.

For the equivalence of the $\mathcal{A}_p$-convergence, let us first assume, that $F_n \overset{n\to\infty}{\longrightarrow} F_0$ in $\mathcal{A}_p(\mathbb{C})$. If we define

$$A_n := \sup_{z\in\mathbb{C}} |F_n(z) - F_0(z)| e^{-B|z|^p},$$

then $\lim\limits_{n\to\infty} A_n = 0$ by (1.2) and we get the estimate

$$|F_n(z) - F_0(z)| \leq A_n e^{B|z|^p}, \quad z \in \mathbb{C}.$$

In the same way as we derived (1.9) we then also get

$$|f_{n,k} - f_{0,k}| \leq A_n A_\Gamma \frac{(2B)^{\frac{k}{p}}}{\Gamma(\frac{k}{p}+1)}.$$

Note here, that the coefficient A_Γ from (1.8) does not depend on n. Hence we proved the conditions (i) and (ii).

Conversely, assume that the conditions (i) and (ii) are satisfied. Then analogously as we derived (1.10), we get

$$|F_n(z) - F_0(z)| \leq A_n A_{\frac{1}{p},1} e^{B^p|z|^p}, \quad z \in \mathbb{C}.$$

Also here we note, that the constant $A_{\frac{1}{p},1}$ comes from the estimate (1.5) and in particular does not depend on n. Since $\lim_{n\to\infty} A_n = 0$ by assumption we conclude the $\mathcal{A}_p$-convergence

$$\sup_{z\in\mathbb{C}} |F_n(z) - F_0(z)| e^{-BB'|z|^p} \leq A_n A_{\frac{1}{p},1} \overset{n\to\infty}{\longrightarrow} 0. \qquad \square$$

A similar argument as for the power series coefficients in Lemma 1.5 shows, that for functions in $\mathcal{A}_p(\mathbb{C})$ all its derivatives are contained in $\mathcal{A}_p(\mathbb{C})$ as well.

Lemma 1.6. Let $p > 0$ and $F \in \mathcal{A}^p(\mathbb{C})$. Then also $F^{(n)} \in \mathcal{A}^p(\mathbb{C})$ for every $n \in \mathbb{N}_0$.

Proof. Let $F \in \mathcal{A}^p(\mathbb{C})$ and $A, B \geq 0$ such that $|F(z)| \leq Ae^{B|z|^p}$, for every $z \in \mathbb{C}$. Without loss of generality we will assume $B > 0$ and $n \in \mathbb{N}$. By the Cauchy integral formula, we can write the n-th derivative as

$$F^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{F(\xi)}{(\xi - z)^{n+1}} d\xi = \frac{n!}{2\pi r^n} \int_0^{2\pi} F(z + re^{i\varphi}) e^{-in\varphi} d\varphi,$$

where γ is a circle around z with a not yet specified radius $r > 0$. This integral can now be estimated as

$$|F^{(n)}(z)| \leq \frac{An!}{2\pi r^n} \int_0^{2\pi} e^{B|z+re^{i\varphi}|^p} d\varphi \leq \frac{An!}{r^n} e^{B2^p(|z|^p + r^p)}.$$

Since the right hand side still depends on the free parameter $r > 0$, we minimize it by choosing $r = \frac{1}{2}(\frac{n}{Bp})^{\frac{1}{p}}$. This gives

$$|F^{(n)}(z)| \leq An!2^n \Big(\frac{eBp}{n}\Big)^{\frac{n}{p}} e^{B2^p|z|^p} \leq AA_\Gamma \frac{n! 2^{n(1+\frac{1}{p})} B^{\frac{n}{p}}}{\Gamma(\frac{n}{p}+1)} e^{B2^p|z|^p},$$

where in the last inequality we used the constant A_Γ from (1.8). $\square$

By definition, the space $\mathcal{A}_p(\mathbb{C})$ consists of entire functions which admit some additional exponential growth condition. From standard analysis we know, that the power series $F(z)=\sum_{k=0}^\infty f_kz^k$ of any entire function uniformly converges on compact subsets of $\mathbb{C}$. However, the next lemma shows, that the additional exponential growth constraint of $\mathcal{A}_p(\mathbb{C})$ ensures, that the series even converges in the stronger $\mathcal{A}_p$-sense.

Lemma 1.7. Let $F\in\mathcal{A}_p(\mathbb{C})$. Then the power series representation

$$F(z)=\sum_{k=0}^\infty f_kz^k \quad \text{converges in } \mathcal{A}_p(\mathbb{C}).$$

Proof. For every $z\neq 0$, we can use the Cauchy integral formula to write for every $N\in\mathbb{N}_0$ the function as,

$$\begin{aligned}
F(z)&=\frac{1}{2\pi i}\int_{|\xi|=2|z|}\frac{F(\xi)}{\xi-z}d\xi\\
&=\frac{1}{2\pi i}\int_{|\xi|=2|z|}F(\xi)\sum_{k=0}^\infty\frac{z^k}{\xi^{k+1}}d\xi\\
&=\sum_{k=0}^N\frac{z^k}{2\pi i}\int_{|\xi|=2|z|}\frac{F(\xi)}{\xi^{k+1}}d\xi+\frac{1}{2\pi i}\int_{|\xi|=2|z|}F(\xi)\sum_{k=N+1}^\infty\frac{z^k}{\xi^{k+1}}d\xi\\
&=\sum_{k=0}^N f_kz^k+\frac{1}{2\pi i}\int_{|\xi|=2|z|}F(\xi)\sum_{k=N+1}^\infty\frac{z^k}{\xi^{k+1}}d\xi,\quad z\in\mathbb{C}\setminus\{0\}.
\end{aligned}$$

where in the last line we used the Cauchy integral formula for the coefficients of the power series. Using the exponential bound $|F(z)|\leq Ae^{B|z|^p}$ of functions in $\mathcal{A}_p(\mathbb{C})$, we can estimate the difference between F and its partial sum as

$$\Big|F(z)-\sum_{k=0}^N f_kz^k\Big|\leq\frac{1}{2\pi}\int_0^{2\pi}|F(2|z|e^{i\varphi})|\sum_{k=N+1}^\infty\frac{1}{2^k}d\varphi\leq\frac{A}{2^N}e^{B|2z|^p},\quad z\in\mathbb{C}\setminus\{0\}.$$

Since this estimate is obviously also true for $z=0$, we conclude the $\mathcal{A}_p$-convergence

$$\sup_{z\in\mathbb{C}}\Big|F(z)-\sum_{k=0}^N f_kz^k\Big|e^{-B2^p|z|^p}\leq\frac{A}{2^N}\overset{N\to\infty}{\longrightarrow}0. \qquad \square$$

As an application of the $\mathcal{A}_p$-spaces, we verify, that the example superoscillating functions (0.1) indeed converge in $\mathcal{A}_1(\mathbb{C})$. It is to mention, that a very similar proof is already given in [68, Lemma 2.4] or [69, Lemma 1].

Proposition 1.8. For every $k\in\mathbb{C}$, the sequence

$$F_n(z)=\sum_{j=0}^n C_j(n)e^{ik_j(n)z},\quad z\in\mathbb{C},$$

with the coefficients

$$C_j(n)=\binom{n}{j}\Big(\frac{1+k}{2}\Big)^{n-j}\Big(\frac{1-k}{2}\Big)^j\quad\text{and}\quad k_j(n)=1-\frac{2j}{n},$$

converges to

$$\lim_{n\to\infty} F_n(z) = e^{ikz}, \quad \text{in } \mathcal{A}_1(\mathbb{C}). \tag{1.11}$$

Proof. Using the binomial formula, we write the function $F_n(z)$ as

$$\begin{aligned} F_n(z) &= \sum_{j=0}^{n} \binom{n}{j} \Big(\frac{1+k}{2}\Big)^{n-j} e^{i\frac{n-j}{n}z} \Big(\frac{1-k}{2}\Big)^{j} e^{-i\frac{j}{n}z} \\ &= \Big(\frac{1+k}{2} e^{i\frac{z}{n}} + \frac{1-k}{2} e^{-i\frac{z}{n}}\Big)^n \\ &= \Big(\cos\Big(\frac{z}{n}\Big) + ik\sin\Big(\frac{z}{n}\Big)\Big)^n. \end{aligned}$$

Using the bounds

$$|\sin(\xi)| \leq |\xi| e^{|\xi|} \quad \text{and} \quad |\cos(\xi)| \leq e^{|\xi|}, \quad \xi \in \mathbb{C}, \tag{1.12}$$

which immediately follow from the respective power series representation of $\sin(\xi)$ and $\cos(\xi)$, we can estimate the functions F_n as

$$|F_n(z)| = \Big|\cos\Big(\frac{z}{n}\Big) + ik\sin\Big(\frac{z}{n}\Big)\Big|^n \leq \Big(1 + \frac{|k||z|}{n}\Big)^n e^{|z|} \leq e^{(|k|+1)|z|}, \quad z \in \mathbb{C}. \tag{1.13}$$

Next, we estimate the difference of the cos-terms

$$\begin{aligned} \Big|\cos\Big(\frac{z}{n}\Big) - \cos\Big(\frac{kz}{n}\Big)\Big| &= \Big|\int_{\frac{z}{n}}^{\frac{kz}{n}} \sin(\xi) d\xi\Big| \leq \frac{|k-1||z|}{n} \sup_{\xi\in[\frac{z}{n},\frac{kz}{n}]} |\sin(\xi)| \\ &\leq \frac{(|k|+1)|z|}{n} \sup_{\xi\in[\frac{z}{n},\frac{kz}{n}]} |\xi| e^{|\xi|} \leq \frac{(|k|+1)^2|z|^2}{n^2} e^{\frac{(1+|k|)|z|}{n}}, \end{aligned} \tag{1.14}$$

where we again used the estimate (1.12). In a similar way we also derive an inequality for the difference of sin-terms, namely we use

$$\Big|\frac{\cos(\xi)}{\xi} - \frac{\sin(\xi)}{\xi^2}\Big| = \Big|\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2n+2}{(2n+3)!} \xi^{2n+1}\Big| \leq \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} |\xi|^{2n+1} \leq e^{|\xi|}, \quad \xi \in \mathbb{C},$$

to estimate

$$\begin{aligned} \Big|k\sin\Big(\frac{z}{n}\Big) - \sin\Big(\frac{kz}{n}\Big)\Big| &= \frac{|k||z|}{n} \Big|\int_{\frac{z}{n}}^{\frac{kz}{n}} \Big(\frac{\cos(\xi)}{\xi} - \frac{\sin(\xi)}{\xi^2}\Big) d\xi\Big| \\ &\leq \frac{|k||k-1||z|^2}{n^2} \sup_{\xi\in[\frac{z}{n},\frac{kz}{n}]} \Big|\frac{\cos(\xi)}{\xi} - \frac{\sin(\xi)}{\xi^2}\Big| \\ &\leq \frac{(|k|+1)^2|z|^2}{n^2} \sup_{\xi\in[\frac{z}{n},\frac{kz}{n}]} e^{|\xi|} \leq \frac{(|k|+1)^2|z|^2}{n^2} e^{\frac{(|k|+1)|z|}{n}}. \end{aligned} \tag{1.15}$$

Using now the identity $A^n - B^n = (A-B)\sum_{j=0}^{n-1} A^j B^{n-1-j}$, together with the estimates (1.13), (1.14) and (1.15), gives for every $n \geq |k|+1$ the estimate

$$\begin{aligned}
|F_n(z) - e^{ikz}| &= \Big| \Big(\cos\Big(\frac{z}{n}\Big) + ik\sin\Big(\frac{z}{n}\Big)\Big)^n - \big(e^{i\frac{kz}{n}}\big)^n \Big| \\
&= \Big| \cos\Big(\frac{z}{n}\Big) + ik\sin\Big(\frac{z}{n}\Big) - e^{i\frac{kz}{n}} \Big| \Big| \sum_{j=0}^{n-1} \Big(\cos\Big(\frac{z}{n}\Big) + ik\sin\Big(\frac{z}{n}\Big)\Big)^j e^{i\frac{n-1-j}{n}kz} \Big| \\
&\leq \frac{2(|k|+1)^2|z|^2}{n^2} e^{\frac{(|k|+1)|z|}{n}} \sum_{j=0}^{n-1} e^{\frac{j}{n}(|k|+1)|z|} e^{\frac{n-1-j}{n}|k||z|} \\
&\leq \frac{2(|k|+1)^2|z|^2}{n} e^{(|k|+1)|z|} \\
&\leq \frac{8(|k|+1)^2}{ne^2} e^{(|k|+2)|z|},
\end{aligned}$$

where in the last line we used $|z|^2 \leq \frac{4}{e^2}e^{|z|}$. This estimate now shows, that

$$\lim_{n\to\infty} |F_n(z) - e^{ikz}| e^{-(|k|+2)|z|} \leq \lim_{n\to\infty} \frac{8(|k|+1)^2}{ne^2} = 0,$$

and hence $\lim_{n\to\infty} F_n(z) = e^{ikz}$ in $\mathcal{A}_1(\mathbb{C})$. □

In these spaces $\mathcal{A}_p(\mathbb{C})$ we can now define different kind of infinite order differential operators as in (1.17), (1.21) and (1.29). In the literature, those kind of operators play the role of a time evolution operator for the time dependent Schrödinger equation, and were used to prove continuity results similar to the one in Theorem 4.6 and Theorem 5.4, see for example [5, 19, 20, 22, 32, 34, 35, 55, 66, 67, 72]. However, our methods in Chapter 4 and Chapter 5 use more direct arguments and avoid these operators. Nevertheless, they will still be used in Theorem 2.5 for the construction of superoscillating functions, similar as in [16, 28]. The first very large family of operators is introduced in the following Theorem 1.9 and is for $p \geq 1$ already stated in [35, Theorem 2.4]. Also the proof is basically taken from there.

Theorem 1.9. Let $p > 0$ and $(a_n)_n : \mathbb{C} \to \mathbb{C}$ be entire functions, such that there exists some $C \geq 0$ and for every $\varepsilon > 0$ some $B_\varepsilon \geq 0$, with

$$|a_n(z)| \leq B_\varepsilon \frac{\varepsilon^n}{\Gamma(\frac{n}{q}+1)} e^{C|z|^p}, \quad z \in \mathbb{C}, \tag{1.16}$$

where $\frac{1}{q} := \begin{cases} 1 - \frac{1}{p}, & \text{if } 1 < p < \infty, \\ 0, & \text{if } 0 < p \leq 1. \end{cases}$ Then the infinite order differential operator

$$U := \sum_{n=0}^{\infty} a_n \frac{d^n}{dz^n}, \tag{1.17}$$

acts continuously as an operator $U : \mathcal{A}_p(\mathbb{C}) \to \mathcal{A}_p(\mathbb{C})$.

Proof. Let $F \in \mathcal{A}_p(\mathbb{C})$ admitting the power series representation $F(z) = \sum_{k=0}^{\infty} f_k z^k$. In this form the action of U on F is

$$(UF)(z) = \sum_{n=0}^{\infty} a_n(z) \frac{d^n}{dz^n} \sum_{k=0}^{\infty} f_k z^k = \sum_{n=0}^{\infty} a_n(z) \sum_{k=0}^{\infty} f_{n+k} \frac{(n+k)!}{k!} z^k, \quad z \in \mathbb{C}. \tag{1.18}$$

By the boundedness of the functions a_n in (1.16) and the bound $|f_k| \leq A \frac{B^k}{\Gamma(\frac{k}{p}+1)}$ of the coefficients in Lemma 1.5, we can estimate this expression for every $\varepsilon > 0$ by

$$|(UF)(z)| \leq AB_\varepsilon e^{C|z|^p} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n+k)!(B\varepsilon)^n (B|z|)^k}{\Gamma(\frac{n}{q}+1)\Gamma(\frac{n+k}{p}+1)k!}, \quad z \in \mathbb{C}.$$

Next we will use the following basic inequalities of the Gamma function and factorial

$$\frac{(n+k)!}{n!k!} = \binom{n+k}{k} \leq \sum_{j=0}^{n+k} \binom{n+k}{j} = 2^{n+k},$$

$$\frac{\Gamma(\frac{n}{p}+\frac{1}{2})\Gamma(\frac{k}{p}+\frac{1}{2})}{\Gamma(\frac{n+k}{p}+1)} = B\Big(\frac{n}{p}+\frac{1}{2}, \frac{k}{p}+\frac{1}{2}\Big) \leq B\Big(\frac{1}{2}, \frac{1}{2}\Big) = \pi,$$

to further estimate the double sum as

$$|(UF)(z)| \leq AB_\varepsilon \pi e^{C|z|^p} \sum_{n=0}^{\infty} \frac{n!(2B\varepsilon)^n}{\Gamma(\frac{n}{q}+1)\Gamma(\frac{n}{p}+\frac{1}{2})} \sum_{k=0}^{\infty} \frac{(2B|z|)^k}{\Gamma(\frac{k}{p}+\frac{1}{2})}, \quad z \in \mathbb{C}.$$

Due to the asymptotic behaviour $\Gamma(x) \sim \sqrt{2\pi}\, \frac{x^{x-\frac{1}{2}}}{e^x}$, as $x \to \infty$, see [2, Eq.(6.1.37)], the coefficients of the first sum for $1 < p < \infty$, where we have $\frac{1}{p} + \frac{1}{q} = 1$ asymptotically behave as

$$\frac{n!}{\Gamma(\frac{n}{q}+1)\Gamma(\frac{n}{p}+\frac{1}{2})} \leq \frac{n!}{\Gamma(\frac{n}{q})\Gamma(\frac{n}{p})} \sim \frac{n^{n+\frac{1}{2}}}{\sqrt{2\pi}\, (\frac{n}{q})^{\frac{n}{q}-\frac{1}{2}} (\frac{n}{p})^{\frac{n}{p}-\frac{1}{2}}} = \frac{n^{\frac{3}{2}} (p^{\frac{1}{p}} q^{\frac{1}{q}})^n}{\sqrt{2\pi pq}}, \quad \text{as } n \to \infty.$$

However, also for $0 < p \leq 1$, where $\frac{1}{q} := 0$, we have the inequality

$$\frac{n!}{\Gamma(\frac{n}{p}+\frac{1}{2})} \leq \frac{n!}{\Gamma(n)} = n+1, \quad n \geq 1.$$

Hence, if we fix $0 < \varepsilon < \frac{p^{\frac{1}{p}} q^{\frac{1}{q}}}{2B}$, the sum

$$C_\varepsilon := \sum_{n=0}^{\infty} \frac{n!(2B\varepsilon)^n}{\Gamma(\frac{n}{q}+1)\Gamma(\frac{n}{p}+\frac{1}{2})} < \infty$$

is finite. Using the Mittag-Leffler function (1.4), with its exponential bound (1.5), we can finally estimate $(Uf)(z)$ as

$$|(UF)(z)| \leq AB_\varepsilon C_\varepsilon \pi e^{C|z|^p} E_{\frac{1}{p},\frac{1}{2}}(2B|z|) \leq AA_{\frac{1}{p},\frac{1}{2}} B_\varepsilon C_\varepsilon \pi e^{(C+(2B)^p)|z|^p}, \quad z \in \mathbb{C}. \tag{1.19}$$

On the one hand, this estimate now proves, that the sum (1.18) uniformly converges compact subsets of $\mathbb{C}$, and hence UF is entire. On the other hand, it also gives the exponential boundedness $UF \in \mathcal{A}_p(\mathbb{C})$.

In order to prove the continuity of the operator U, let $F_n \overset{\mathcal{A}_p}{\longrightarrow} F_0$. According to Lemma 1.5, this means that there exist constants $A_n, B \geq 0$ with $\lim_{n\to\infty} A_n = 0$, such that the power series coefficients can be estimated as

$$|f_{n,k} - f_{0,k}| \leq A_n \frac{B^k}{\Gamma(\frac{k}{p}+1)}, \quad k, n \in \mathbb{N}_0.$$

The same estimate (1.19) done for the difference $UF_n - UF_0$, gives

$$|(UF_n)(z) - (UF_0)(z)| \leq A_n A_{\frac{1}{p},\frac{1}{2}} B_\varepsilon C_\varepsilon \pi e^{(C+(2B)^p)|z|^p}.$$

Rewriting this inequality gives

$$|(UF_n)(z) - (UF_0)(z)| e^{-(C+(2B)^p)|z|^p} \leq A_n A_{\frac{1}{p},\frac{1}{2}} B_\varepsilon C_\varepsilon \pi \overset{n\to\infty}{\longrightarrow} 0,$$

which is exactly the $\mathcal{A}_p$-convergence (1.2). Hence $UF_n \overset{\mathcal{A}_p}{\longrightarrow} UF_0$ and we proved the $\mathcal{A}_p$-continuity of the operator U. □

For the particular $\mathcal{A}_1$-space the previous Theorem 1.9 can still be generalized in the sense that the simple derivative $\frac{d}{dz}$ can be replaced by some operator $H(\frac{d}{dz})$.

Theorem 1.10. Let $p > 0$ and $H, (a_n)_n : \mathbb{C} \to \mathbb{C}$ be entire functions, such that there exists some $C \geq 0$ and for every $\varepsilon > 0$ some $B_\varepsilon \geq 0$, with

$$|a_n(z)| \leq B_\varepsilon \varepsilon^n e^{C|z|^p}, \quad z \in \mathbb{C},\ \varepsilon > 0. \tag{1.20}$$

Then the differential operator

$$U \coloneqq \sum_{n=0}^{\infty} a_n \Big(H\Big(\frac{d}{dz}\Big)\Big)^n, \tag{1.21}$$

acts continuously as an operator $U : \mathcal{A}_1(\mathbb{C}) \to \mathcal{A}_1(\mathbb{C})$. Note, that for $H(z) = \sum_{l=0}^{\infty} h_l z^l$ we consider the corresponding infinite order differential operator $H(\frac{d}{dz}) \coloneqq \sum_{l=0}^{\infty} h_l \frac{d^l}{dz^l}$.

Proof. For any $F \in \mathcal{A}_1(\mathbb{C})$, having the power series representation $F(z) = \sum_{k=0}^{\infty} f_k z^k$, the action of $H(\frac{d}{dz})$ is given by

$$\Big(H\Big(\frac{d}{dz}\Big)F\Big)(z) = \sum_{l=0}^{\infty} h_l \frac{d^l}{dz^l} \sum_{k=0}^{\infty} f_k z^k = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} h_l f_{k+l} \frac{(k+l)!}{k!} z^k, \tag{1.22}$$

where we already interchanged the order of summation, which will be justified by the upcoming estimate (1.24). This shows, that $H(\frac{d}{dz})F$ is again a power series with coefficients

$$\Big(H\Big(\frac{d}{dz}\Big)F\Big)_k = \frac{1}{k!} \sum_{l=0}^{\infty} h_l f_{k+l} (k+l)!, \quad k \in \mathbb{N}_0. \tag{1.23}$$

Using now the estimate $|f_k| \leq A\frac{B^k}{k!}$ from Lemma 1.5, we can estimate the coefficients (1.23) as

$$\Big|\Big(H\Big(\frac{d}{dz}\Big)F\Big)_k\Big| \leq A\frac{B^k}{k!}\sum_{l=0}^{\infty}|h_l|B^l = AD\frac{B^k}{k!}, \tag{1.24}$$

where we used the constant $D := \sum_{l=0}^{\infty}|h_l|B^l < \infty$. With this estimate we verified, that the double sum in (1.22) is absolutely convergent and we were indeed allowed to interchange the sums. Since $H(\frac{d}{dz})F$ is an everywhere convergent power series, it is again entire, and since its coefficients admit the bound (1.24), even $H(\frac{d}{dz})F \in \mathcal{A}_1(\mathbb{C})$. Moreover, equation (1.24) tells us, that whenever we apply $H(\frac{d}{dz})$, the estimate of the corresponding power series coefficients gets multiplied by D. Which means, that if we apply $H(\frac{d}{dz})$ n-times, we get the estimate

$$\Big|\Big(\Big(H\Big(\frac{d}{dz}\Big)\Big)^n F\Big)_k\Big| \leq AD^n\frac{B^k}{k!}, \quad k \in \mathbb{N}_0, \tag{1.25}$$

of the corresponding power series coefficients. Next, the action of U on the function F is given by

$$(UF)(z) = \sum_{n=0}^{\infty}a_n(z)\sum_{k=0}^{\infty}\Big(\Big(H\Big(\frac{d}{dz}\Big)\Big)^n F\Big)_k z^k, \quad z \in \mathbb{C}. \tag{1.26}$$

Using (1.20) and (1.25), we can estimate this function for every $\varepsilon > 0$ as

$$|(UF)(z)| \leq AB_\varepsilon e^{C|z|}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\varepsilon D)^n(B|z|)^k}{k!} = \frac{AB_\varepsilon}{1-\varepsilon D}e^{(B+C)|z|}, \quad z \in \mathbb{C}, \tag{1.27}$$

where we chose $0 < \varepsilon < \frac{1}{D}$. This estimate shows, that the sum in (1.26) is absolutely convergent as well as uniform on compact subsets of $\mathbb{C}$, hence UF is entire. On the other hand it also gives the exponential boundedness to make $UF \in \mathcal{A}_1(\mathbb{C})$.

In order to prove the continuity of the operator U, let $F_n \overset{\mathcal{A}_1}{\longrightarrow} F_0$. According to Lemma 1.5, this means that there exist constants $A_n, B \geq 0$ with $\lim_{n\to\infty}A_n = 0$, such that the power series coefficients can be estimated as

$$|f_{n,k} - f_{0,k}| \leq A_n\frac{B^k}{k!}, \quad k, n \in \mathbb{N}_0.$$

The same estimate (1.27) done for the difference $UF_n - UF_0$, gives

$$|(UF_n)(z) - (UF_0)(z)| \leq \frac{A_nB_\varepsilon}{1-\varepsilon D}e^{(B+C)|z|}, \quad z \in \mathbb{C}. \tag{1.28}$$

Rewriting this inequality gives the convergence

$$\sup_{z\in\mathbb{C}}|(UF_n)(z) - (UF_0)(z)|e^{-(C+B)|z|} \leq \frac{A_nB_\varepsilon}{1-\varepsilon D} \overset{n\to\infty}{\longrightarrow} 0,$$

which is exactly the $\mathcal{A}_1$-convergence (1.2). Hence $UF_n \overset{\mathcal{A}_1}{\longrightarrow} UF_0$ and we proved the $\mathcal{A}_1$-continuity of the operator U. □

The following corollary is a direct consequence of Theorem 1.10 and will play a central role in the construction of superoscillations in Theorem 2.5. For the special case of a monome $H(z) = z^p$ this operator is already considered in [34, Theorem 2.3]. A similar operator as (1.29) is already considered in [16, Theorem 2], but defined via infinite products there.

Corollary 1.11. Let $\lambda \in \mathbb{C}$ and $H : \mathbb{C} \to \mathbb{C}$ be an entire function. Then the differential operator

$$U := \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \Big(H\Big(-i\frac{d}{dz} \Big) \Big)^n, \tag{1.29}$$

acts continuously as an operator $U : \mathcal{A}_1(\mathbb{C}) \to \mathcal{A}_1(\mathbb{C})$. Moreover, for every $F \in \mathcal{A}_1(\mathbb{C})$ with power series coefficients bounded as $|f_k| \leq A\frac{B^k}{k!}$ for some $A, B \geq 0$, the image UF admits the exponential bound

$$|(UF)(z)| \leq Ae^{|\lambda| \sum_{l=0}^{\infty} |h_l| B^l} e^{B|z|}, \quad z \in \mathbb{C}, \tag{1.30}$$

where $(h_l)_l$ are the power series coefficients of $H(z) = \sum_{l=0}^{\infty} h_l z^l$.

Proof. This operator U is of the form (1.21) with the entire function $H(\,\cdot\,)$ replaced by $H(-i\,\cdot\,)$, and the constant functions $a_n(z) = \frac{\lambda^n}{n!}$, which obviously satisfy (1.20).

In order to check the estimate (1.30), we note, that in (1.26) we derived the representaton

$$(UF)(z) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{k=0}^{\infty} \Big(\Big(H\Big(-i\frac{d}{dz} \Big) \Big)^n F \Big)_k z^k, \quad z \in \mathbb{C}.$$

Choosing $D = \sum_{l=0}^{\infty} |h_l| B^l$ we can use (1.25) to estimate

$$|(UF)(z)| \leq \sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!} \sum_{k=0}^{\infty} AD^n \frac{B^k}{k!} |z|^k = Ae^{D|\lambda|} e^{B|z|}, \quad z \in \mathbb{C}. \qquad \square$$

2. Classification of superoscillations

As mentioned in the introduction, the existing literature consists of various variants of superoscillating functions, defined in different ways and having different types of oscillatory properties. The aim of this chapter is now to unify these concepts and categorize them into two classes of superoscillating functions.

Beside the difference in their oscillatory behaviour, all superoscillations have in common, that they are entire functions and (linear) combinations of plane waves with frequencies lying within a bounded range. I.e., for some $k_0 > 0$ and some complex valued Borel measure μ on $[-k_0, k_0]$ we consider functions of the form

$$F(z) = \int_{-k_0}^{k_0} e^{i\kappa z} d\mu(\kappa), \quad z \in \mathbb{C}. \tag{2.1}$$

In order to be superoscillatory, we additionally have to ensure that some oscillatory behaviour occurs, which exceeds the intrinsic frequency range $[-k_0, k_0]$. There are now two different ways how this oscillatory behaviour manifests itself.

The first type of superoscillations considers sequences of functions for which the superoscillatory property is realized by the convergence to a plane wave with a frequency outside the expected range.

Definition 2.1 (Type I superoscillations)**.** A sequence of functions $(F_n)_n$, each of the form (2.1) with corresponding measures μ_n but a common maximal frequency $k_0 > 0$, is called a *Type I superoscillating sequence*, if there exists some $k \in \mathbb{R} \setminus [-k_0, k_0]$, such that

$$\lim_{n \to \infty} F_n(z) = e^{ikz}, \quad \text{in } \mathcal{A}_1(\mathbb{C}). \tag{2.2}$$

In contrast, the following Type II superoscillation is a property of a single function only. Here, the superoscillatory behaviour is connected to the number of zeros in a certain interval.

Definition 2.2 (Type II superoscillations)**.** A function F of the form (2.1) is called a *Type II superoscillating function* on the interval $[a, b] \subseteq \mathbb{R}$, if

$$F \text{ has at least } \frac{k_0(b-a)}{\pi} \text{ isolated zeros in the open interval } (a, b). \tag{2.3}$$

The following lemma ensures that the functions (2.1) are indeed elements in the space $\mathcal{A}_1(\mathbb{C})$ of exponentially bounded entire functions (1.1) and hence the convergence (2.2) is well defined.

Lemma 2.3. Let $k_0 > 0$ and μ be a complex Borel measure on $[-k_0, k_0]$. Then the function F in (2.1) is an element in $\mathcal{A}_1(\mathbb{C})$.

Proof. The integrand $e^{i\kappa z}$ in (2.1) is an entire function in z and its derivative can be estimated by

$$\Big|\frac{d}{dz}e^{i\kappa z}\Big| = |\kappa|e^{-\kappa\,\mathrm{Im}(z)} \leq k_0 e^{k_0|z|}, \quad \kappa \in [-k_0, k_0],\, z \in \mathbb{C}. \tag{2.4}$$

Since the interval $[-k_0, k_0]$ is of finite measure, the κ-uniform upper bound (2.4) is integrable and hence ensures the complex differentiability of the function (2.1). Moreover, the exponential boundedness follows from the estimate

$$|F(z)| = \Big|\int_{-k_0}^{k_0} e^{i\kappa z} d\mu(\kappa)\Big| \leq \int_{-k_0}^{k_0} |e^{i\kappa z}| d|\mu|(\kappa) \leq |\mu|([-k_0, k_0])|e^{k_0|z|}, \quad z \in \mathbb{C},$$

where $|\mu|$ is the variation of the complex measure μ. This verifies that $F \in \mathcal{A}_1(\mathbb{C})$. □

The next lemma verifies that the two specific representations (0.1) and (0.6) of superoscillations are indeed covered by the general form (2.1).

Lemma 2.4. Let $k_0 > 0$ and $F : \mathbb{C} \to \mathbb{C}$ be one of the following types of functions:

(i) A linear combination of plane waves

$$F(z) = \sum_{j=0}^{n} C_j e^{ik_j z}, \quad z \in \mathbb{C},$$

with coefficients $C_j \in \mathbb{C}$ and $k_j \in [-k_0, k_0]$.

(ii) An integral of the form

$$F(z) = \int_{\mathbb{R}} C(u) e^{ik(u)z} du, \quad z \in \mathbb{C},$$

for some $C \in L^1(\mathbb{R})$, $k(u) \in [-k_0, k_0]$ for every $u \in \mathbb{R}$.

Then there exists some complex Borel measure μ on $[-k_0, k_0]$, such that F admits the representation (2.1).

Proof.

(i) Choosing the Dirac measure

$$\mu(B) := \sum_{j=0, k_j \in B}^{n} C_j,$$

for any Borel set $B \subseteq [-k_0, k_0]$, then F obviously admits the representation (2.1).

(ii) In the first step we define the complex measure

$$\sigma(A) := \int_A C(u) du,$$

for any Borel set $A \subseteq \mathbb{R}$. With this measure, the function F admits the representation

$$F(z) = \int_{\mathbb{R}} e^{ik(u)z} d\sigma(u), \quad z \in \mathbb{C}.$$

Choosing furthermore

$$\mu(B) := \sigma\big(\{ u \in \mathbb{R} \mid k(u) \in B \} \big)$$

for any Borel set $B \subseteq [-k_0, k_0]$, we can even rewrite F as

$$F(z) = \int_{-k_0}^{k_0} e^{ikz} d\mu(k), \quad z \in \mathbb{C},$$

which is exactly the form (2.1). □

The following sections now collect superoscillations of the existing literature, generalize them if possible, and prove that they fit in either of those two classes. In particular, the Sections 2.1–2.4 treat Type I superoscillating sequences and in Section 2.5 & 2.6 Type II superoscillating functions are considered.

2.1. Constructing Type I superoscillations via generalized Schrödinger equation

In this section we derive two methods, which take a given Type I superoscillating sequence, for example the one in (0.1), and construct a new family of Type I superoscillating sequences out of it. The idea was first considered in [16], revisited in [28] and uses the generalized free Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(t,z) = -H\Big(-i\frac{\partial}{\partial z}\Big)\Psi(t,z), \quad t,z \in \mathbb{C}, \tag{2.5}$$

where for some entire $H(z) = \sum_{n=0}^{\infty} h_n z^n$ the operator $H(-i\frac{d}{dz}) := \sum_{n=0}^{\infty} h_n(-i\frac{d}{dz})^n$ is defined as the corresponding infinite order differential operator acting in $\mathcal{A}_1(\mathbb{C})$. Choosing the initial condition $\Psi(0,z) = e^{ikz}$, the equation (2.5) has the explicit solution

$$\Psi(t,z) = e^{iH(k)t} e^{ikz}, \quad t,z \in \mathbb{C}.$$

Consequently, taking some function F_n of the form (2.1) as initial condition, the solution formally looks like

$$\Psi_n(t,z) = \int_{-k_0}^{k_0} e^{iH(\kappa)t} e^{i\kappa z} d\mu_n(\kappa), \quad t,z \in \mathbb{C}. \tag{2.6}$$

If we now assume that the initial conditions $(F_n)_n$ form a Type I superoscillatory sequence, one may expect, that also at other times $t \in \mathbb{C}$ the sequence $(\Psi_n(t,\,\cdot\,))_n$ is still a Type I superoscillating sequence. This method is precisely specified in Theorem 2.5 (i). It will also turn out in Theorem 2.5 (ii), that at the point $z = 0$, the sequence $(\Psi_n(\,\cdot\,,0))_n$ is Type I superoscillating sequence in the time variable t.

The novelty of Theorem 2.5 compared to the works [16, 28], is firstly, that in (2.5) arbitrary Type I superoscillating functions of the form (2.1) are chosen, while in [16, 28] only the standard example (0.1) was considered. Secondly, it is proven, that the resulting functions $F_n^{(1)}$ and $F_n^{(2)}$ in (2.9) and (2.10) converge in $\mathcal{A}_1(\mathbb{C})$, while only uniform convergence on compact subsets was proven in [16, 28].

Theorem 2.5. Let $(F_n)_n$ be a Type I superoscillating sequence, which satisfy (2.1) in the form

$$F_n(z) = \int_{-k_0}^{k_0} e^{i\kappa z} d\mu_n(\kappa), \quad z \in \mathbb{C}, \tag{2.7}$$

for some $k_0 > 0$ and complex Borel measures μ_n on $[-k_0, k_0]$, and which converge as

$$\lim_{n\to\infty} F_n(z) = e^{ikz} \quad \text{in } \mathcal{A}_1(\mathbb{C}), \tag{2.8}$$

for some $k \in \mathbb{R} \setminus [-k_0, k_0]$.

(i) Then for every entire function $H : \mathbb{C} \to \mathbb{C}$, also the sequence

$$F_n^{(1)}(z) := e^{-H(k)} \int_{-k_0}^{k_0} e^{H(\kappa)} e^{i\kappa z} d\mu_n(\kappa), \quad z \in \mathbb{C},\, n \in \mathbb{N}, \tag{2.9}$$

is a Type I superoscillating sequence, with limit $\lim\limits_{n\to\infty} F_n^{(1)}(z) = e^{ikz}$ in $\mathcal{A}_1(\mathbb{C})$.

(ii) For every entire function $H : \mathbb{C} \to \mathbb{C}$, which satisfies $H(\kappa) \in [-h_0, h_0]$ for every $\kappa \in [-k_0, k_0]$ and $H(k) \in \mathbb{R} \setminus [-h_0, h_0]$, for some $h_0 > 0$, the sequence

$$F_n^{(2)}(z) := \int_{-k_0}^{k_0} e^{iH(\kappa)z} d\mu_n(\kappa), \quad z \in \mathbb{C},\, n \in \mathbb{N}, \tag{2.10}$$

is a Type I superoscillating sequence, with limit $\lim\limits_{n\to\infty} F_n^{(2)}(z) = e^{iH(k)z}$ in $\mathcal{A}_1(\mathbb{C})$.

Proof. For every $t \in \mathbb{C}$ we consider the operator

$$U(t) := \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \Big(H\Big(-i\frac{d}{dz} \Big) \Big)^m, \tag{2.11}$$

which, according to Corollary 1.11, is continuous in $\mathcal{A}_1(\mathbb{C})$. Starting with the power series representation $H(z) = \sum_{l=0}^{\infty} h_l z^l$, it turns out that the operator $H(-i\frac{d}{dz})$ acts on plane waves $e^{i\kappa z}$, as the multiplication

$$H\Big(-i\frac{d}{dz} \Big) e^{i\kappa z} = \sum_{l=0}^{\infty} h_l (-i)^l \frac{d^l}{dz^l} e^{i\kappa z} = \sum_{l=0}^{\infty} h_l \kappa^l e^{i\kappa z} = H(\kappa) e^{i\kappa z}.$$

Consequently, the operator $U(t)$ acts as

$$(U(t)e^{i\kappa\,\cdot})(z) = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} \Big(H\Big(-i\frac{d}{dz} \Big) \Big)^m e^{i\kappa z} = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} (H(\kappa))^m e^{i\kappa z} = e^{iH(\kappa)t} e^{i\kappa z}. \tag{2.12}$$

Since the right hand side of this equation is uniformly bounded as

$$|e^{iH(\kappa)t}e^{i\kappa z}| \leq e^{|H(\kappa)t|}e^{|\kappa||z|} \leq e^{t\sum_{l=0}^\infty |h_l|k_0^l}e^{k_0|z|}, \quad \kappa \in [-k_0, k_0],$$

and since the interval $[-k_0, k_0]$ has finite measure with respect to μ_n, a version of the dominated convergence theorem allows to carry the operator U inside the integral (2.7), namely

$$\begin{aligned}(U(t)F_n)(z) &= \Big(U(t)\int_{-k_0}^{k_0} e^{i\kappa\,\cdot}\,d\mu_n(\kappa)\Big)(z)\\ &= \int_{-k_0}^{k_0}(U(t)e^{i\kappa\,\cdot})(z)d\mu_n(\kappa) = \int_{-k_0}^{k_0} e^{iH(\kappa)t}e^{i\kappa z}d\mu_n(\kappa).\end{aligned} \tag{2.13}$$

(i) In order to prove, that (2.9) is a Type I superoscillating sequence, we choose $t = -i$ in (2.13) and get

$$(U(-i)F_n)(z) = \int_{-k_0}^{k_0} e^{H(\kappa)}e^{i\kappa z}d\mu_n(\kappa) = e^{H(k)}F_n^{(1)}(z), \quad z \in \mathbb{C}.$$

Since $U(-i)$ is continuous in $\mathcal{A}_1(\mathbb{C})$ and $F_n \to e^{ik\,\cdot}$ in $\mathcal{A}_1(\mathbb{C})$, we conclude the convergence

$$\begin{aligned}\lim_{n\to\infty} F_n^{(1)}(z) &= e^{-H(k)}\lim_{n\to\infty}(U(-i)F_n)(z)\\ &= e^{-H(k)}(U(-i)e^{ik\,\cdot})(z) = e^{ikz} \quad \text{in } \mathcal{A}_1(\mathbb{C}).\end{aligned} \tag{2.14}$$

The fact that $F_n^{(1)}$ is of the form (2.1) was already proven in Lemma 2.4 (ii), and hence this proves that $(F_n^{(1)})_n$ is indeed a Type I superoscillating sequence.

(ii) According to the $\mathcal{A}_1$-convergence (2.8) and Lemma 1.5, there exist $A_n, B \geq 0$ with $\lim_{n\to\infty} A_n = 0$, such that we can estimate the power series coefficients of the difference $F_n(z) - e^{ikz} = \sum_{l=0}^\infty (f_{n,l} - \frac{(ik)^l}{l!})z^l$ by

$$\Big|f_{n,l} - \frac{(ik)^l}{l!}\Big| \leq A_n\frac{B^l}{l!}, \quad l, n \in \mathbb{N}_0.$$

From the estimate (1.30) we then conclude

$$\big|\big(U(t)(F_n - e^{ik\,\cdot})\big)(z)\big| \leq A_n e^{D|t|}e^{B|z|},$$

using $D := \sum_{l=0}^\infty |h_l|B^l$. This estimate shows in particular for $z = 0$ the convergence

$$\lim_{n\to\infty}\sup_{t\in\mathbb{C}}\big|\big(U(t)(F_n - e^{ik\,\cdot})\big)(0)\big|e^{-D|t|} \leq \lim_{n\to\infty} A_n = 0.$$

In other words this is nothing else than the $\mathcal{A}_1$-convergence

$$\lim_{n\to\infty}(U(t)F_n)(0) = (U(t)e^{ik\,\cdot})(0),$$

in the variable t. Using the identities (2.12) and (2.13) this convergence can be written as

$$\lim_{n\to\infty} F_n^{(2)}(t) = \lim_{n\to\infty} \int_{-k_0}^{k_0} e^{iH(\kappa)t} d\mu_n(\kappa)$$
$$= \lim_{n\to\infty} (U(t)F_n)(0) = (U(t)e^{ik\,\cdot})(0) = e^{iH(k)t}. \tag{2.15}$$

Since $H(\kappa) \in [-h_0, h_0]$ for every $\kappa \in [-k_0, k_0]$ by assumption, it follows again from Lemma 2.4 (ii), that $F_n^{(2)}$ admits the representation (2.1) for some complex measure μ_n on $[-h_0, h_0]$. Together with the convergence (2.15) to a plane wave with frequency $H(k) \in \mathbb{R}\setminus[-h_0, h_0]$, this proves, that $(F_n^{(2)})_n$ is a Type I superoscillating sequence. □

2.2. Type I Superoscillations as integrals

Another way of constructing superoscillating functions was initiated by M. Berry in [43], where he considered functions of the form

$$F_\delta(x) = \frac{1}{\delta\sqrt{2\pi}} \int_{\mathbb{R}} e^{ik(u)x} e^{-\frac{(u-ia)^2}{2\delta^2}} du, \quad x \in \mathbb{R}.$$

The idea is, that

$$\frac{1}{\delta\sqrt{2\pi}} e^{-\frac{(u-ia)^2}{2\delta^2}} \to \delta(u - ia), \quad \text{as } \delta \to 0^+,$$

approximates the complex delta function and consequently

$$F_\delta(x) \to e^{ik(ia)x}, \quad \text{as } \delta \to 0^+,$$

converges to a plane wave. In the following Theorem 2.6 we want to revisit this idea and prove that under certain assumptions the resulting functions form a Type I superoscillating sequence, see Corollary 2.7.

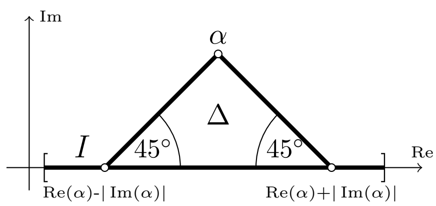
Moreover, we also improve the above construction of [43] in three ways. Firstly, an additional function g is allowed in the integral (2.16). This function does not affect the superoscillatory property of the F_δ's, but allows to modify their shape. Secondly, precise assumptions on the involved functions g and k are given. Thirdly and most importantly, while in [43] mainly the complex saddle point approximation is used to derive properties as the local wavenumber, the convergence in the space $\mathcal{A}_1(\mathbb{C})$ is proven here. In particular, this is necessary for $(F_\delta)_\delta$ to be a Type I superoscillating sequence according to Definition 2.1.

Theorem 2.6. Let $I \subseteq \mathbb{R}$ be some closed but not necessarily bounded interval. Consider continuously differentiable functions $k, g : I \to \mathbb{C}$, which for some $B \geq 0$ admit the bounds

$$\sup_{u\in I} |k(u)| < \infty \quad \text{and} \quad \sup_{u\in I} |g(u)| e^{-B|u|} < \infty.$$

Furthermore, let $\alpha \in \mathbb{C} \setminus \mathbb{R}$ be such that

$$\big[\operatorname{Re}(\alpha) - |\operatorname{Im}(\alpha)|\,,\, \operatorname{Re}(\alpha) + |\operatorname{Im}(\alpha)|\big] \subseteq I,$$



and k, g extend to the closed triangle Δ with the corners α and $\operatorname{Re}(\alpha) \pm |\operatorname{Im}(\alpha)|$, in a way that they are holomorphic on the interior of Δ and continuously differentiable on Δ. Then for every $\delta > 0$, the function

$$F_\delta(z) := \frac{1}{\delta\sqrt{2\pi}} \int_I g(u) e^{ik(u)z} e^{-\frac{(u-\alpha)^2}{2\delta^2}} du, \quad z \in \mathbb{C}, \tag{2.16}$$

satisfies $F_\delta \in \mathcal{A}_1(\mathbb{C})$ and converges as

$$\lim_{\delta \to 0^+} F_\delta(z) = g(\alpha) e^{ik(\alpha)z} \quad \text{in } \mathcal{A}_1(\mathbb{C}). \tag{2.17}$$

Corollary 2.7. Let $I \subseteq \mathbb{R}$ and $k, g : I \to \mathbb{C}$ be as in Theorem 2.6 and additionally satisfy

$$g(\alpha) = 1, \quad k(\alpha) \in \mathbb{R} \setminus [-k_0, k_0] \quad \text{and} \quad k(u) \in [-k_0, k_0] \text{ for every } u \in I,$$

for some $k_0 > 0$. Then $(F_\delta)_\delta$ from (2.17) is a Type I superoscillating sequence.

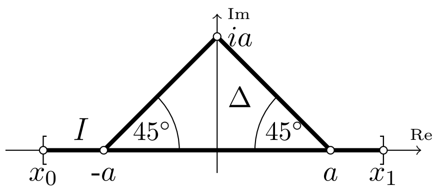
Proof. The convergence

$$\lim_{\delta \to 0^+} F_\delta(z) = e^{ik(\alpha)z} \quad \text{in } \mathcal{A}_1(\mathbb{C}), \tag{2.18}$$

follows from Theorem 2.6 and the assumption $g(\alpha) = 1$. Since $k(u) \in [-k_0, k_0]$ fpr every $u \in I$, the representation (2.1) follows from Lemma 2.4 (ii). Finally, since the frequency $k(\alpha)$ of the limit function in (2.18) lies in $\mathbb{R} \setminus [-k_0, k_0]$, we indeed verified, that $(F_\delta)_\delta$ is a Type I superoscillating sequence. □

Now we come back to the proof of the main result of this section, Theorem 2.6.

Proof of Theorem 2.6. Without loss of generality we choose $\alpha = ia$ for some $a > 0$. This simplifies the geometry to

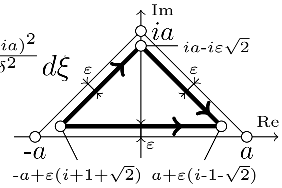


Furthermore, we will use

$$C := \sup_{u \in I} |k(u)|, \quad A := \sup_{u \in I} |g(u)| e^{-B|u|}, \quad x_0 := \inf(I), \quad x_1 := \sup(I).$$

In *Step 1* we will interchange the part $-a \to a$ of the integration path in (2.16) by the triangle path $-a \to ia \to a$. Since the functions g, k are holomorphic in the interior of Δ, we have to scale the triangle by some sufficiently small parameter $\varepsilon > 0$, in order to apply the Cauchy theorem

$$\int_{-a+\varepsilon(i+1+\sqrt{2})}^{a+\varepsilon(i-1-\sqrt{2})} g(\xi)e^{ik(\xi)z}e^{-\frac{(\xi-ia)^2}{2\delta^2}}d\xi = \int_{-a+\varepsilon(i+1+\sqrt{2})}^{ia-i\varepsilon\sqrt{2}} g(\xi)e^{ik(\xi)z}e^{-\frac{(\xi-ia)^2}{2\delta^2}}d\xi + \int_{ia-i\varepsilon\sqrt{2}}^{a+\varepsilon(i-1-\sqrt{2})} g(\xi)e^{ik(\xi)z}e^{-\frac{(\xi-ia)^2}{2\delta^2}}d\xi.$$



Since g and k are continuous on the closed triangle Δ, we are allowed to apply the limit $\varepsilon \to 0^+$ and end up with

$$\int_{-a}^{a} g(\xi)e^{ik(\xi)z}e^{-\frac{(\xi-ia)^2}{2\delta^2}}d\xi = \int_{-a}^{ia} g(\xi)e^{ik(\xi)z}e^{-\frac{(\xi-ia)^2}{2\delta^2}}d\xi + \int_{ia}^{a} g(\xi)e^{ik(\xi)z}e^{-\frac{(\xi-ia)^2}{2\delta^2}}d\xi.$$

Hence we can split up the function F_δ into the four parts

$$\begin{aligned} F_\delta(z) = &\underbrace{\frac{1}{\delta\sqrt{2\pi}}\int_{x_0}^{-a} g(u)e^{ik(u)z}e^{-\frac{(u-ia)^2}{2\delta^2}}du}_{=:F_\delta^{(1)}(z)} + \underbrace{\frac{1}{\delta\sqrt{2\pi}}\int_{-a}^{ia} g(\xi)e^{ik(\xi)z}e^{-\frac{(\xi-ia)^2}{2\delta^2}}d\xi}_{=:F_\delta^{(2)}(z)} \\ &+ \underbrace{\frac{1}{\delta\sqrt{2\pi}}\int_{ia}^{a} g(\xi)e^{ik(\xi)z}e^{-\frac{(\xi-ia)^2}{2\delta^2}}d\xi}_{=:F_\delta^{(3)}(z)} + \underbrace{\frac{1}{\delta\sqrt{2\pi}}\int_{a}^{x_1} g(u)e^{ik(u)z}e^{-\frac{(u-ia)^2}{2\delta^2}}du}_{=:F_\delta^{(4)}(z)}. \end{aligned} \tag{2.19}$$

In *Step 2* we will apply the limit $\delta \to 0^+$ to (2.19). Starting with $F_\delta^{(4)}(z)$, we estimate

$$|F_\delta^{(4)}(z)| \leq \frac{A}{\delta\sqrt{2\pi}}e^{C|z|}e^{\frac{a^2}{2\delta^2}}\int_a^{x_1} e^{Bu}e^{-\frac{u^2}{2\delta^2}}du \leq \frac{A}{2}e^{C|z|}e^{Ba}\Lambda\Big(\frac{a}{\delta\sqrt{2}} - \frac{B\delta}{\sqrt{2}}\Big),$$

where in the second equation the integral identity (A.18) was used. Due to the asymptotics (A.22) and since $a > 0$, this proves the $\mathcal{A}_1$-convergence

$$\sup_{z\in\mathbb{C}} |F_\delta^{(4)}(z)|e^{-C|z|} \leq \frac{A}{2}e^{Ba}\Lambda\Big(\frac{a}{\delta\sqrt{2}} - \frac{B\delta}{\sqrt{2}}\Big) \longrightarrow 0, \quad \text{as } \delta \to 0^+.$$

In the same way we also observe

$$\lim_{\delta\to 0^+}\sup_{z\in\mathbb{C}} |F_\delta^{(1)}(z)|e^{-C|z|} = 0.$$

For the function $F_\delta^{(3)}(z)$ we use $\int_{ia}^{a} e^{-\frac{(\xi-ia)^2}{2\delta^2}}d\xi = \frac{\delta\sqrt{\pi}}{\sqrt{2}}\,\mathrm{erf}\big(\frac{a}{\delta\sqrt{i}}\big)$, to rewrite the difference

$$\begin{aligned} F_\delta^{(3)}(z) - \frac{g(ia)}{2}e^{ik(ia)z} &= \frac{1}{\delta\sqrt{2\pi}}\int_{ia}^{a}\Big(g(\xi)e^{ik(\xi)z} - \frac{g(ia)}{\mathrm{erf}\big(\frac{a}{\delta\sqrt{i}}\big)}e^{ik(ia)z}\Big)e^{-\frac{(\xi-ia)^2}{2\delta^2}}d\xi \\ &= -\frac{1}{2}\int_{ia}^{a}\Big(g(\xi)e^{ik(\xi)z} - \frac{g(ia)}{\mathrm{erf}\big(\frac{a}{\delta\sqrt{i}}\big)}e^{ik(ia)z}\Big)\frac{d}{d\xi}\,\mathrm{erfc}\Big(\frac{\xi-ia}{\delta\sqrt{2}}\Big)d\xi \\ &= -\frac{g(ia)}{2}e^{ik(a)z}\,\mathrm{erfc}\Big(\frac{a}{\delta\sqrt{i}}\Big) + \frac{1}{2}\int_{ia}^{a}\frac{d}{d\xi}\Big(g(\xi)e^{ik(\xi)z}\Big)\,\mathrm{erfc}\Big(\frac{\xi-ia}{\delta\sqrt{2}}\Big)d\xi, \end{aligned}$$

where in the last line we applied integration by parts. Since g, g', k, k' are continuous on the closed triangle Δ, we denote their respective suprema with $\|\cdot\|$. Using this, we can now estimate the difference by

$$\Big|F_\delta^{(3)}(z)-\frac{g(ia)}{2}e^{ik(ia)z}\Big|\leq\Big(\frac{|g(ia)|}{2}\Big|\operatorname{erfc}\Big(\frac{a}{\delta\sqrt{i}}\Big)\Big| + \frac{\|g'\|+\|g\|\|k'\||z|}{\sqrt{2}}\int_0^a\Big|\operatorname{erfc}\Big(\frac{s}{\delta\sqrt{i}}\Big)\Big|ds\Big)e^{\|k\||z|}.$$

Using

$$\Big|\operatorname{erfc}\Big(\frac{s}{\delta\sqrt{i}}\Big)\Big|=\Big|\Lambda\Big(\frac{s}{\delta\sqrt{i}}\Big)\Big|\leq\Lambda\Big(\frac{s}{\delta\sqrt{2}}\Big)\leq\min\Big\{\frac{\delta\sqrt{2}}{s\sqrt{\pi}},\Lambda(0)\Big\}=\min\Big\{\frac{\delta\sqrt{2}}{s\sqrt{\pi}},1\Big\},$$

by (A.22), gives for every $0<\delta\leq\frac{a\sqrt{\pi}}{\sqrt{2}}$ the estimate

$$\int_0^a\Big|\operatorname{erfc}\Big(\frac{s}{\delta\sqrt{i}}\Big)\Big|ds\leq\int_0^{\frac{\delta\sqrt{2}}{\sqrt{\pi}}}1dt+\frac{\delta\sqrt{2}}{\sqrt{\pi}}\int_{\frac{\delta\sqrt{2}}{\sqrt{\pi}}}^a\frac{1}{s}ds=\frac{\delta\sqrt{2}}{\sqrt{\pi}}\Big(1+\ln\Big(\frac{a\sqrt{\pi}}{\delta\sqrt{2}}\Big)\Big),$$

which then leads to the final estimate

$$\Big|F_\delta^{(3)}(z)-\frac{g(ia)}{2}e^{ik(ia)z}\Big|\leq\frac{\delta}{\sqrt{\pi}}\Big(\frac{|g(a)|}{a\sqrt{2}}+\big(\|g'\|+\|g\|\|k'\||z|\big)\Big(1+\ln\Big(\frac{a\sqrt{\pi}}{\delta\sqrt{2}}\Big)\Big)\Big)e^{\|k\||z|}.$$

Since $|z|e^{\|k\||z|}\leq\frac{1}{e\|k\|}e^{2\|k\||z|}$, this estimate proves the convergence

$$\lim_{\delta\to0^+}F_\delta^{(3)}(z)=\frac{g(ia)}{2}e^{ik(ia)z}\quad\text{in }\mathcal{A}_1(\mathbb{C}).$$

For the same reason also

$$\lim_{\delta\to0^+}F_\delta^{(2)}(z)=\frac{g(ia)}{2}e^{ik(ia)z}\quad\text{in }\mathcal{A}_1(\mathbb{C}),$$

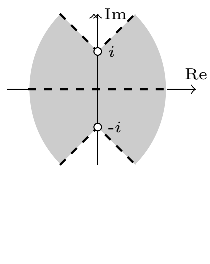
and we proved that all the terms in (2.19) converge in $\mathcal{A}_1(\mathbb{C})$, and consequently also their sum, as it is the statement of the theorem. □

Next we will point out some possible choices of functions k in Corollary 2.7, which lead to Type I superoscillatory sequences $(F_\delta)_\delta$. In particular, we will take the functions k from the original paper [43], which then also shows, that the result of this section indeed cover all the situations investigated there.

Example 2.8.

- We start with the function $k(u)=\frac{1}{1+u^2}$ integrate over $I=\mathbb{R}$ in (2.16). Since k is holomorphic on $\mathbb{C}\setminus\{\pm i\}$, the allowed values $\alpha\in\mathbb{C}\setminus\mathbb{R}$ for which the triangle Δ is contained in $\mathbb{C}\setminus\{\pm i\}$, are characterized by the condition

$$0\neq|\operatorname{Im}(\alpha)|<1+|\operatorname{Re}(\alpha)|.$$

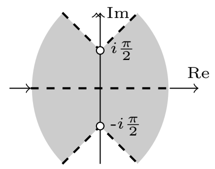


In particular, choosing $g \equiv 1$ and $\alpha = ia$, for some $0 < a < 1$, then $k(u) \in [-1,1]$ for every $u \in \mathbb{R}$ and $k(ia) > 1$. This gives the Type I superoscillatory sequence

$$F_\delta(z) = \frac{1}{\delta\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{iz}{1+u^2}} e^{-\frac{(u-ia)^2}{2\delta^2}} du \overset{\delta\to 0^+}{\longrightarrow} e^{\frac{iz}{1-a^2}} \quad \text{in } \mathcal{A}_1(\mathbb{C}).$$

- The second example in [43] considers $k(u) = \frac{1}{\cosh(u)}$ and again integrates over $I = \mathbb{R}$. Since k is holomorphic on $\mathbb{C} \setminus i\pi(\mathbb{Z} + \frac{1}{2})$, the allowed values $\alpha \in \mathbb{C} \setminus \mathbb{R}$ for which the triangle Δ is contained in $\mathbb{C} \setminus i\pi(\mathbb{Z} + \frac{1}{2})$, are characterized by the condition

$$0 \neq |\operatorname{Im}(\alpha)| < \frac{\pi}{2} + |\operatorname{Re}(\alpha)|.$$



In particular, choosing $g \equiv 1$ and $\alpha = ia$, for some $0 < a < \frac{\pi}{2}$, then $k(u) \in [-1,1]$ for every $u \in \mathbb{R}$ and $k(ia) > 1$. This gives the Type I superoscillatory sequence

$$F_\delta(z) = \frac{1}{\delta\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{iz}{\cosh(u)}} e^{-\frac{(u-ia)^2}{2\delta^2}} du \overset{\delta\to 0^+}{\longrightarrow} e^{\frac{iz}{\cos(a)}} \quad \text{in } \mathcal{A}_1(\mathbb{C}).$$

- Another example is $k(u) = e^{-u^2}$, which is holomorphic on all of $\mathbb{C}$. Hence, Corollary 2.7 is applicable for every $\alpha \in \mathbb{C} \setminus \mathbb{R}$ and in particular, for every $a > 0$, we can choose $\alpha = ia$ and $g \equiv 1$, to get $k(u) \in [-1,1]$ for every $u \in \mathbb{R}$ and $k(ia) > 1$. This gives the Type I superoscillatory sequence

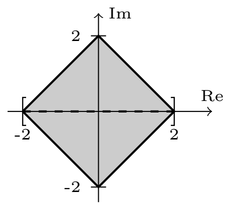
$$F_\delta(z) = \frac{1}{\delta\sqrt{2\pi}} \int_{\mathbb{R}} e^{ie^{-u^2}z} e^{-\frac{(u-ia)^2}{2\delta^2}} du \overset{\delta\to 0^+}{\longrightarrow} e^{ie^{a^2}z} \quad \text{in } \mathcal{A}_1(\mathbb{C}).$$

- Another possible frequency function is $k(u) = \cos(u)$. This function is again entire and Corollary 2.7 is again applicable for every $\alpha \in \mathbb{C} \setminus \mathbb{R}$. In particular for $g \equiv 0$ and for every $a > 0$, we get $k(u) \in [-1,1]$ for every $u \in \mathbb{R}$ and $k(ia) > 1$. This leads to the Type I superoscillatory sequence

$$F_\delta(z) = \frac{1}{\delta\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\cos(u)z} e^{-\frac{(u-ia)^2}{2\delta^2}} du \overset{\delta\to 0^+}{\longrightarrow} e^{i\cosh(a)z} \quad \text{in } \mathcal{A}_1(\mathbb{C}).$$

- One possible frequency function, for which we only integrate along the finite interval $I = [-2,2]$, is $k(u) = 1 - \frac{u^2}{2}$. Here, the values $\alpha \in \mathbb{C} \setminus \mathbb{R}$ are not restricted by the holomorphicity of k, but by the condition $[\operatorname{Re}(\alpha) - |\operatorname{Im}(\alpha)|, \operatorname{Re}(\alpha) + |\operatorname{Im}(\alpha)|] \subseteq I$. Hence the allowed values $\alpha \in \mathbb{C} \setminus \mathbb{R}$ are those who satisfy

$$0 \neq |\operatorname{Im}(\alpha)| \leq |2 \pm \operatorname{Re}(\alpha)|.$$



In particular, choosing $g \equiv 1$ and $\alpha = ia$, for some $0 < a \leq 2$, one gets $k(u) \in [-1,1]$ for every $u \in [-2,2]$ and $k(ia) > 1$. This gives the Type I superoscillatory sequence

$$F_\delta(z) = \frac{1}{\delta\sqrt{2\pi}} \int_{-2}^{2} e^{i(1-\frac{u^2}{2})z} e^{-\frac{(u-ia)^2}{2\delta^2}} du \overset{\delta \to 0^+}{\longrightarrow} e^{i(1+\frac{a^2}{2})z} \quad \text{in } \mathcal{A}_1(\mathbb{C}).$$

2.3. Type I Superoscillating sinc-function

Not a whole class, but one particular superoscillatory function was considered in [46] by M. Berry. For every $a > 1$ he considered the functions

$$F_\delta(z) = \frac{2}{\delta} e^{-\frac{1}{\delta}} \operatorname{sinc}\left(\sqrt{z^2 - \frac{2iaz}{\delta} - \frac{1}{\delta^2}}\right), \quad z \in \mathbb{C},\ \delta > 0, \tag{2.20}$$

using the Sinus cardinalis $\operatorname{sinc}(z) := \frac{\sin(z)}{z}$. Note, that it does not play any role which branch of the complex square root one uses, since the sign, which distinguishes the two branches, gets cancelled out by the symmetry of sinc. The aim of this section is now to verify that (2.20) indeed is a Type I superoscillatory sequence according to Definition 2.1. This will be done in two steps. In Theorem 2.9 we prove the convergence (2.2) and in Theorem 2.10 the integral representation (2.1).

Theorem 2.9. For every $a > 1$, the functions $(F_\delta)_\delta$ from (2.20) are elements in $\mathcal{A}_1(\mathbb{C})$ and converge as

$$\lim_{\delta \to 0^+} F_\delta(z) = e^{iaz} \quad \text{in } \mathcal{A}_1(\mathbb{C}). \tag{2.21}$$

Proof. First of all, although the complex square root in (2.20) is not an entire function, the power series expansion shows, that

$$\operatorname{sinc}(\sqrt{\xi}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \xi^n, \quad \xi \in \mathbb{C}, \tag{2.22}$$

and hence F_δ is again entire. The illustrative reason for this is, that the sign, which distinguishes the two branches, cancels out when $\sqrt{\xi}$ is put as the argument of $\operatorname{sinc}(\sqrt{\xi})$. In order to show that $F_\delta \in \mathcal{A}_1(\mathbb{C})$ and to prove the convergence (2.21), we define

$$R_\delta(z) := \sqrt{z^2 - \frac{2iaz}{\delta} - \frac{1}{\delta^2}}, \quad z \in \mathbb{C}, \tag{2.23}$$

such that we can write $F_\delta(z) = \frac{2}{\delta} e^{-\frac{1}{\delta}} \operatorname{sinc}(R_\delta(z))$. We will now estimate different regions of z separately. In the first case we will estimate in the whole complex plane except regions around the zeros $i\frac{a \pm b}{\delta}$ of R_δ, where $b := \sqrt{a^2 - 1}$, and in the second and third case we consider the neighborhoods of those zeros.

- If $|z - i\frac{a+b}{\delta}| \geq 1$ and $|z - i\frac{a-b}{\delta}| \geq 1$, the square root (2.23) can be estimated as

$$|R_\delta(z)|^2 = \left|z - i\frac{a+b}{\delta}\right| \left|z - i\frac{a-b}{\delta}\right| \geq \left\{ \begin{array}{ll} |z - i\frac{a-b}{\delta}|, & \text{if } \operatorname{Im}(z) \geq \frac{a}{\delta}, \\ |z - i\frac{a+b}{\delta}|, & \text{if } \operatorname{Im}(z) \leq \frac{a}{\delta}, \end{array} \right\} \geq \max\left\{1, \frac{b}{\delta}\right\}. \tag{2.24}$$

If we now define the function $G_\delta(z) := -\frac{e^{-\frac{1}{\delta}-iR_\delta(z)}}{i\delta R_\delta(z)}$, we can estimate

$$|F_\delta(z) - G_\delta(z)| = \frac{2e^{-\frac{1}{\delta}}}{\delta}\left|\operatorname{sinc}(R_\delta(z)) + \frac{e^{-iR_\delta(z)}}{2iR_\delta(z)}\right| = \frac{e^{-\frac{1}{\delta}-\operatorname{Im}(R_\delta(z))}}{\delta|R_\delta(z)|} \le \frac{e^{-\frac{1}{\delta}}}{\delta}, \quad (2.25)$$

where in the last inequality we used (2.24) and the fact that the square root $R_\delta(z)$ is chosen to have nonnegative imaginary part.

Defining also $H_\delta(z) := -\frac{e^{iaz}}{i\delta R_\delta(z)}$, we can further estimate

$$\begin{aligned}|G_\delta(z) - H_\delta(z)| &= \frac{1}{\delta|R_\delta(z)|}\left|e^{-\frac{1}{\delta}-iR_\delta(z)} - e^{iaz}\right| = \frac{e^{-a\operatorname{Im}(z)}}{\delta|R_\delta(z)|}\left|e^{-i(R_\delta(z)+az-\frac{i}{\delta})} - 1\right| \\ &\le \frac{|R_\delta(z) + az - \frac{i}{\delta}|}{\sqrt{b\delta}} e^{a|z| + |\operatorname{Im}(R_\delta(z)+az-\frac{i}{\delta})|}, \end{aligned} \quad (2.26)$$

where in the last inequality we used (2.24) and that $|e^{i\xi} - 1| \le |\xi| e^{|\operatorname{Im}(\xi)|}$ for every $\xi \in \mathbb{C}$. Since we can write

$$\begin{aligned} R_\delta(z) + az - \frac{i}{\delta} &= \frac{1}{\delta}\Big(\sqrt{\delta^2 z^2 - 2ia\delta z - 1} + a\delta z - i\Big) \\ &= \frac{-b^2\delta z^2}{\sqrt{\delta^2 z^2 - 2ia\delta z - 1} - a\delta z + i}, \end{aligned} \quad (2.27)$$

we can use the two estimates (A.1) and (A.8) to further estimate (2.26) as

$$|G_\delta(z) - H_\delta(z)| \le \frac{\sqrt{b\delta}\,|z|^2}{\min\{b, a-b\}} e^{(2a+1)|z|}. \quad (2.28)$$

Thirdly, we estimate

$$\begin{aligned} |H_\delta(z) - e^{iaz}| &= \frac{|1 + i\delta R_\delta(z)|}{\delta|R_\delta(z)|} e^{-a\operatorname{Im}(z)} \le \frac{\sqrt{\delta}}{\sqrt{b}}\Big(\Big|R_\delta(z) + az - \frac{i}{\delta}\Big| + a|z|\Big) e^{a|z|} \\ &\le \Big(\frac{\sqrt{b}\,\delta^{\frac{3}{2}}|z|^2}{\min\{b, a-b\}} + \frac{a\sqrt{\delta}\,|z|}{\sqrt{b}}\Big) e^{a|z|}, \end{aligned} \quad (2.29)$$

where in the first inequality we used (2.24) in the denominator and in the second inequality (2.27) together with (A.1). Combining now (2.25), (2.28) and (2.29) gives for the difference of $F_\delta(z)$ and e^{iaz} the upper bound

$$\begin{aligned} |F_\delta(z) - e^{iaz}| &\le \frac{e^{-\frac{1}{\delta}}}{\delta} + \frac{\sqrt{b\delta}\,|z|^2}{\min\{b, a-b\}} e^{(2a+1)|z|} + \Big(\frac{\sqrt{b}\,\delta^{\frac{3}{2}}|z|^2}{\min\{b, a-b\}} + \frac{a\sqrt{\delta}\,|z|}{\sqrt{b}}\Big) e^{a|z|} \\ &\le \sqrt{\delta}\,\Big(\frac{e^{-\frac{1}{\delta}}}{\delta^{\frac{3}{2}}} + \frac{4\sqrt{b}\,(1+\delta)}{e^2 \min\{b, a-b\}} + \frac{a}{e\sqrt{b}}\Big) e^{2(a+1)|z|}, \end{aligned} \quad (2.30)$$

where in the second inequality we used $|z| \le \frac{1}{e} e^{|z|}$ and $|z|^2 \le \frac{4}{e^2} e^{|z|}$.

- If $|z - i\frac{a+b}{\delta}| \leq 1$ we can estimate

$$|R_\delta(z)|^2 = \Big|z - i\frac{a+b}{\delta}\Big|\,\Big|z - i\frac{a-b}{\delta}\Big| \leq \Big|z - i\frac{a-b}{\delta}\Big| \leq 1 + \frac{2b}{\delta}.$$

With the additional $|\operatorname{sinc}(\xi)| \leq e^{|\xi|}$, $\xi \in \mathbb{C}$, this then leads to

$$|F_\delta(z) - e^{iaz}| \leq \frac{2}{\delta}e^{-\frac{1}{\delta}}e^{|R_\delta(z)|} + e^{-a\operatorname{Im}(z)} \leq \frac{2}{\delta}e^{-\frac{1}{\delta}+\sqrt{1+\frac{2b}{\delta}}} + e^{-a(\frac{a+b}{\delta}-1)}. \tag{2.31}$$

- For $|z - i\frac{a-b}{\delta}| \leq 1$ we similarly get

$$|F_\delta(z) - e^{iaz}| \leq \frac{2}{\delta}e^{-\frac{1}{\delta}+\sqrt{1+\frac{2b}{\delta}}} + e^{-a(\frac{a+b}{\delta}-1)}. \tag{2.32}$$

Combining the three estimates (2.30), (2.31) and (2.32) then proves the $\mathcal{A}_1$-convergence (2.21). ☐

Next, we derive some integral representation for the functions F_δ.

Theorem 2.10. For every $a > 1$, $\delta > 0$, the function F_δ from (2.20) admits the integral representation

$$F_\delta(z) = \frac{1}{\delta}\int_{-1}^{1} e^{\frac{ak-1}{\delta}} J_0\Big(\frac{\sqrt{(a^2-1)(1-k^2)}}{\delta}\Big)e^{ikz}dk, \quad z \in \mathbb{C},$$

where J_0 is the Bessel function of order zero.

Proof. In Lemma A.3 we already calculated the integral

$$\operatorname{sinc}\big(\sqrt{z^2+b^2}\big) = \frac{1}{2}\int_{-1}^{1} e^{ikz} J_0\big(b\sqrt{1-k^2}\big)dk, \quad z \in \mathbb{C}.$$

If we replace $z \to z - \frac{ia}{\delta}$ and set $b = \frac{\sqrt{a^2-1}}{\delta}$ immediately gives the stated integral

$$F_\delta(z) = \frac{2}{\delta}e^{-\frac{1}{\delta}}\operatorname{sinc}\Big(\sqrt{z^2 - \frac{2iaz}{\delta} - \frac{1}{\delta}}\Big) = \frac{1}{\delta}\int_{-1}^{1} e^{ikz}e^{\frac{ak-1}{\delta}} J_0\Big(\frac{\sqrt{(a^2-1)(1-k^2)}}{\delta}\Big)dk. \quad ☐$$

Finally, we will now combine Theorem 2.9 and Theorem 2.10 to show that the functions F_δ in (2.16) indeed form a Type I superoscillating sequence.

Corollary 2.11. For every $a > 1$, the functions $(F_\delta)_\delta$ from (2.20) are a Type I superoscillating sequence.

Proof. The convergence

$$\lim_{\delta\to 0^+} F_\delta(z) = e^{iaz}, \quad \text{in } \mathcal{A}_1(\mathbb{C}),$$

to a plane wave with frequency $a > 1$ is proven in Theorem 2.9. Moreover, the integral representation

$$F_\delta(z) = \int_{-1}^{1} e^{ikz}d\mu_\delta(z), \quad z \in \mathbb{C},$$

is satisfied due to Theorem 2.10 and Lemma 2.4. Hence F_δ is indeed a Type I superoscillating sequence according to Definition 2.1. ☐

2.4. Type I Superoscillating sum of plane waves

The standard example (0.1) of a superoscillating function admits a representation as a linear combination of plane waves

$$F_n(z) = \sum_{j=0}^{n} C_j(n) e^{ik_j(n)z}, \quad z \in \mathbb{C},\, n \in \mathbb{N}_0, \tag{2.33}$$

with specific frequencies $k_j(n) = 1 - \frac{2j}{n}$ and coefficients $C_j(n) = \binom{n}{j}(\frac{1+k}{2})^{n-j}(\frac{1-k}{2})^j$. These frequencies and coefficients are chosen in a way that

$$\lim_{n\to\infty} F_n(z) = e^{ikz}, \quad z \in \mathbb{C}, \tag{2.34}$$

converges to a single plane wave with frequency k, which may be arbitrary large. It was a fundamental problem for a long time how many of those superoscillating functions exist, until in the recent paper [12] the authors managed to find coefficients $C_j(n)$ for any set of given frequencies $k, k_j(n) \in \mathbb{C}$, such that the corresponding sequence (2.33) converges as in (2.34).

While in [12] the convergence (2.34) is understood as uniform on compact subsets of $\mathbb{C}$, we will go one step further in this section and prove the $\mathcal{A}_1$-convergence in Theorem 2.13, under certain additional assumptions on the frequencies $k_j(n)$. As a consequence the resulting functions $(F_n)_n$ form a Type I superoscillating sequence.

The main preparatory result is the following Lemma 2.12, which constructs linear combinations of plane waves with prescribed values of its derivatives at the origin. Many parts of the proof of this theorem are similar to the original paper [12, Theorem 2.1, Theorem 2.2].

Lemma 2.12. Consider pairwise disjoint frequencies $(k_j)_{j=0}^n \subseteq \mathbb{C}$ and arbitrary values $(a_l)_{l=0}^n \subseteq \mathbb{C}$. Then there exist unique coefficients $(C_j)_{j=0}^n \subseteq \mathbb{C}$, determined by the linear system (2.37), such that the function

$$F(z) \coloneqq \sum_{j=0}^{n} C_j e^{ik_j z}, \quad z \in \mathbb{C}, \tag{2.35}$$

has derivatives with values at the origin given by

$$F^{(l)}(0) = a_j, \quad l \in \{0, \dots, n\}. \tag{2.36}$$

Proof. The l-th derivative of the function (2.35) is given by

$$F^{(l)}(z) = \sum_{j=0}^{n} C_j (ik_j)^l e^{ik_j z}, \quad z \in \mathbb{C},\, l \in \{0, \dots, n\}.$$

By the requirement (2.36), the values of these derivatives at $z = 0$ should equal

$$\sum_{j=0}^{n} C_j (ik_j)^l = a_l, \quad l \in \{0, \dots, n\}.$$

This linear system of equations can now also be written in the matrix form

$$\begin{pmatrix} (ik_0)^0 & \dots & (ik_n)^0 \\ \vdots & \ddots & \vdots \\ (ik_0)^n & \dots & (ik_n)^n \end{pmatrix} \begin{pmatrix} C_0 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix}. \tag{2.37}$$

Since the coefficient matrix is a Vandermonde matrix, its determinant is given by

$$\det \begin{pmatrix} (ik_0)^0 & \dots & (ik_n)^0 \\ \vdots & \ddots & \vdots \\ (ik_0)^n & \dots & (ik_n)^n \end{pmatrix} = i^{\frac{n(n+1)}{2}} \prod_{0\leq l<j\leq n} (k_j - k_l), \tag{2.38}$$

and hence nonvanishing due to the pairwise disjoint frequencies $(k_j)_{j=0}^n$. This means, that the system (2.37) is uniquely solvable and leads to the required coefficients $(C_j)_{j=0}^n$. □

Consider now for every $n \in \mathbb{N}$ pairwise disjoint frequencies $(k_j(n))_{j=0}^n \subseteq \mathbb{C}$, and choose the particular values $a_l = (ik)^l$ for the derivatives in (2.36). Then Lemma 2.12 states the existence of functions $F_n(z) = \sum_{j=0}^n C_j(n)e^{ik_j(n)z}$, for which the first n derivatives at $z = 0$ equal

$$F_n^{(l)}(0) = (ik)^l, \quad l \in \{0, \dots, n\}.$$

I.e., the first $n + 1$ terms of the power series expansion of F_n coincide with the one from the plane wave e^{ikz}. Hence it is reasonable to expect some convergence $F_n(z) \to e^{ikz}$, as $n \to \infty$. Indeed, under some additional assumptions on the difference between the frequencies $k_j(n)$, the following Theorem 2.13 proves the $\mathcal{A}_1$-convergence of this sequence.

Theorem 2.13. Let $k_0 > 0$ and consider for every $n \in \mathbb{N}$ pairwise disjoint frequencies $(k_j(n))_{j=0}^n \subseteq [-k_0, k_0]$, such that there exists some $\delta > 0$ with

$$\prod_{l=0, l\neq j}^n |k_l(n) - k_j(n)| \geq \delta^n, \quad n \in \mathbb{N}, \, j \in \{0, \dots, n\}. \tag{2.39}$$

Then, for every $k \in \mathbb{R} \setminus [-k_0, k_0]$, the sequence of functions

$$F_n(z) := \sum_{j=0}^n \bigg(\prod_{l=0, l\neq j}^n \frac{k_l(n) - k}{k_l(n) - k_j(n)} \bigg) e^{ik_j(n)z}, \quad z \in \mathbb{C}, \, n \in \mathbb{N}, \tag{2.40}$$

is a Type I superoscillatory sequence with limit $\lim\limits_{n\to\infty} F_n(z) = e^{ikz}$ in $\mathcal{A}_1(\mathbb{C})$.

Remark 2.14. Note, that if we choose the frequencies $k_j(n) = 1 - \frac{2j}{n}$ as it is done in (0.1), we do not end up with the coefficients $C_j(n)$ from (0.2). This means, that although the coefficients in Lemma 2.12 are uniquely determined by the values of the derivatives at $z = 0$, there are different ways how to linear combine the plane waves $e^{ik_j(n)z}$ and still end up with a sequence convergent to e^{ikz}.

Proof of Theorem 2.13. By Lemma 2.12, the coefficients $C_j(n)$, which ensure that the function $F_n(z) = \sum_{j=0}^{n} C_j(n)e^{ik_j(n)}$ satisfies

$$F_n^{(l)}(0) = (ik)^l, \quad l \in \{0, \dots, n\}, \tag{2.41}$$

are uniquely determined by the linear system (2.37), i.e.

$$\begin{pmatrix} (ik_0(n))^0 & \dots & (ik_n(n))^0 \\ \vdots & \ddots & \vdots \\ (ik_0(n))^n & \dots & (ik_n(n))^n \end{pmatrix} \begin{pmatrix} C_0(n) \\ \vdots \\ C_n(n) \end{pmatrix} = \begin{pmatrix} (ik)^0 \\ \vdots \\ (ik)^n \end{pmatrix}.$$

Note, that all the imaginary units cancel and the equation reduces to

$$\begin{pmatrix} k_0(n)^0 & \dots & k_n(n)^0 \\ \vdots & \ddots & \vdots \\ k_0(n)^n & \dots & k_n(n)^n \end{pmatrix} \begin{pmatrix} C_0(n) \\ \vdots \\ C_n(n) \end{pmatrix} = \begin{pmatrix} k^0 \\ \vdots \\ k^n \end{pmatrix}.$$

Using Cramer's rule, this system admits the solution

$$\begin{aligned} C_j(n) &= \frac{\det \begin{pmatrix} k_0(n)^0 & \dots & k_{j-1}(n)^0 & k^0 & k_{j+1}(n)^0 & \dots & k_n(n)^0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ k_0(n)^n & \dots & k_{j-1}(n)^n & k^n & k_{j+1}(n)^n & \dots & k_n(n)^n \end{pmatrix}}{\det \begin{pmatrix} k_0(n)^0 & \dots & k_n(n)^0 \\ \vdots & \ddots & \vdots \\ k_0(n)^n & \dots & k_n(n)^n \end{pmatrix}} \\ &= \frac{\prod_{0 \le i < j} (k - k_i(n)) \prod_{j < l \le n} (k_l(n) - k)}{\prod_{0 \le i < j} (k_j(n) - k_i(n)) \prod_{j < l \le n} (k_l(n) - k_j(n))} \\ &= \prod_{l=0, l \neq j}^{n} \frac{k_l(n) - k}{k_l(n) - k_j(n)}, \quad j \in \{0, \dots, n\}, \end{aligned}$$

where we used, that both determinants are Vandermonde determinants of the form (2.38). Since these are exactly the coefficients of the functions (2.40), this proves that the functions F_n admit the values (2.41) of their derivatives at $z = 0$.

In order verify the convergence $F_n(z) \to e^{ikz}$, we first note, that the estimate (2.39), together with the explicit form (2.40) implies the following estimate of the functions F_n,

$$\begin{aligned} |F_n(z)| &\le \sum_{j=0}^{n} \Big(\prod_{l=0, l \neq j}^{n} \frac{|k_l(n) - k|}{|k_l(n) - k_j(n)|} \Big) e^{|k_j(n)z|} \\ &\le (k_0 + |k|)^n \sum_{j=0}^{n} \Big(\prod_{l=0, l \neq j}^{n} \frac{1}{|k_l(n) - k_j(n)|} \Big) e^{k_0|z|} \\ &\le (n+1) \Big(\frac{k_0 + |k|}{\delta} \Big)^n e^{k_0|z|}, \quad z \in \mathbb{C}. \end{aligned} \tag{2.42}$$

Next, we write the difference between $F_n(z)$ and e^{ikz} as the Taylor series

$$F_n(z) - e^{ikz} = \sum_{l=0}^{\infty} \frac{F_n^{(l)}(0) - (ik)^l}{l!} z^l = \sum_{l=n+1}^{\infty} \frac{F_n^{(l)}(0) - (ik)^l}{l!} z^l, \quad z \in \mathbb{C}, \tag{2.43}$$

where in the second equality we used the values (2.41) of the derivatives of F_n. For every $\alpha > 1$ and for every $z \in \mathbb{C} \setminus \{0\}$, we now use the Cauchy integral formula, to estimate

$$\begin{aligned}
|F_n^{(l)}(0) - (ik)^l| &= \Big| \frac{l!}{2\pi i} \int_{|\xi| = \alpha|z|} \frac{F_n(\xi) - e^{ik\xi}}{\xi^{l+1}} d\xi \Big| \\
&\leq \frac{l!}{2\pi\alpha^l |z|^l} \int_0^{2\pi} \Big| F_n(\alpha|z|e^{i\varphi}) - e^{ik\alpha|z|e^{i\varphi}} \Big| d\varphi \\
&\leq \frac{l!}{\alpha^l|z|^l} \Big((n+1)\Big(\frac{k_0 + |k|}{\delta}\Big)^n e^{\alpha k_0|z|} + e^{\alpha|k||z|} \Big) \\
&\leq \frac{l!}{\alpha^l|z|^l} \Big((n+1)\Big(\frac{k_0 + |k|}{\delta}\Big)^n + 1 \Big) e^{\alpha|k||z|},
\end{aligned}$$

where in the third line we used the estimate (2.42). Plugging this into the Taylor series (2.43), gives

$$\begin{aligned}
|F_n(z) - e^{ikz}| &\leq \Big((n+1)\Big(\frac{k_0 + |k|}{\delta}\Big)^n + 1 \Big) e^{\alpha|k||z|} \sum_{l=n+1}^{\infty} \frac{1}{\alpha^l} \\
&= \frac{1}{(\alpha - 1)\alpha^n} \Big((n+1)\Big(\frac{k_0 + |k|}{\delta}\Big)^n + 1 \Big) e^{\alpha|k||z|}, \quad z \in \mathbb{C} \setminus \{0\}.
\end{aligned}$$

Due to (2.43), this inequality also holds for $z = 0$. Choosing now $\alpha > \max\{\frac{k_0+|k|}{\delta}, 1\}$ leads to the $\mathcal{A}_1$-convergence

$$\sup_{z \in \mathbb{C}} |F_n(z) - e^{ikz}| e^{-\alpha|k||z|} \leq \frac{1}{(\alpha - 1)\alpha^n} \Big((n+1)\Big(\frac{k_0 + |k|}{\delta}\Big)^n + 1 \Big) \to 0, \quad \text{as } n \to \infty.$$

Finally, the fact that the functions F_n are of the form (2.1) is already shown in Lemma 2.4 (i). This now proves that the sequence $(F_n)_n$ is indeed a Type I superoscillating sequence with respect to Definition 2.1. □

In the following corollary we use the more explicit condition (2.44) on the distance between the frequencies $k_j(n)$, than the one required in (2.39). In particular the frequencies $k_j(n) = 1 - \frac{2j}{n}$ of (0.2) belong to this setting.

Corollary 2.15. Let $k_0 > 0$ and consider for every $n \in \mathbb{N}$ pairwise disjoint frequencies $(k_j(n))_{j=0}^n \subseteq [-k_0, k_0]$, such that there exists some $\delta > 0$ with

$$|k_l(n) - k_j(n)| \geq \frac{\delta}{n}, \quad n \in \mathbb{N},\ l \neq j \in \{0, \dots, n\}. \tag{2.44}$$

Then, for every $k \in \mathbb{R} \setminus [-k_0, k_0]$, the sequence of functions

$$F_n(z) := \sum_{j=0}^{n} \Big(\prod_{l=0, l \neq j}^{n} \frac{k_l(n) - k}{k_l(n) - k_j(n)} \Big) e^{ik_j(n)z}, \quad z \in \mathbb{C},\ n \in \mathbb{N},$$

is a Type I superoscillatory sequence with limit $\lim_{n \to \infty} F_n(z) = e^{ikz}$ in $\mathcal{A}_1(\mathbb{C})$.

Proof. We will prove that the condition (2.44) implies the condition (2.39). Without loss of generality we will assume that

$$k_0(n) < k_1(n) < \cdots < k_n(n), \quad n \in \mathbb{N},$$

are in ascending order. Then the condition (2.44) implies the condition

$$|k_j(n) - k_l(n)| \geq \frac{\delta|j-l|}{n}, \quad j \neq l \in \{0, \ldots, n\}.$$

Then the estimate (2.39) is satisfied by

$$\prod_{l=0, l\neq j}^{n} |k_l(n) - k_j(n)| \geq \prod_{l=0, l\neq j}^{n} \frac{\delta|j-l|}{n} \geq \frac{\delta^n j!(n-j)!}{n^n} = \frac{\delta^n n!}{n^n \binom{n}{j}} \geq \left(\frac{\delta}{2e}\right)^n,$$

where in the last inequality we used $\binom{n}{j} \leq 2^n$ as well as $\frac{n^n}{n!} \leq e^n$, which is a consequence of the Stirling formula. □

2.5. Energy optimized Type II superoscillating functions

While the Sections 2.1–2.4 all consider Type I superoscillating sequences, this section will be the first dealing with Type II superoscillating functions.

The problem of constructing bandlimited functions of minimal energy (L^2-norm), whose graph passes through a prescribed set of points, was first addressed in 1965 by L. Levi [97]. Similar methods were later on used by A. Kempf, P. Ferreira and collaborators. In [79, 81, 83, 84, 85, 92, 95, 102] the authors construct L^2-functions with compactly supported Fourier transform, which change their sign arbitrarily often in an arbitrary small interval and are optimal with respect to energy minimization. In [78, 80] it is even shown, that the energy expense to create such superoscillatory behaviour grows exponentially with the number of oscillations and polynomially with the inverse bandwidth. Also additional constraints, as matching derivatives, were allowed in [57, 84]. Furthermore, a slightly different kind of optimization problem was considered in [91], where not the energy of the whole function is minimized, but the ratio of the energy inside the superoscillatory region and the energy of the whole wave.

The following Theorem 2.16 revisits the works mentioned above and in particular [84], but also generalizes previous results in the sense that prescribed derivatives of higher order are allowed.

Theorem 2.16. Let $k_0 > 0$, $a_n^{(j)} \in \mathbb{C}$, $x_n^{(j)} \in \mathbb{R}$ for $j \in \{0, \ldots, J\}$, $n \in \{1, \ldots, N_j\}$. Furthermore, for every $j \in \{1, \ldots, J\}$ let $(x_n^{(j)})_{n=1}^{N_j}$ be pairwise disjoint. Then one function $F \in L^2(\mathbb{R})$ satisfying

(i) $\operatorname{supp}(\mathcal{F}[F]) \subseteq [-k_0, k_0]$,

(ii) $F^{(j)}(x_n^{(j)}) = a_n^{(j)}, \quad j \in \{0, \ldots, J\}, n \in \{1, \ldots, N_j\}$, and having minimal L^2-norm, is given by

$$F(x) = \sum_{j=0}^{J} \sum_{n=1}^{N_j} c_n^{(j)} \operatorname{sinc}^{(j)} \big(k_0(x_n^{(j)} - x)\big), \tag{2.45}$$

where the coefficients $c_n^{(j)}$ are the unique solution of the linear system (2.49).

Note, that since the Fourier transform $\mathcal{F}[F]$ is compactly supported, it follows from the Paley-Wiener theorem that the function F is infinitely often differentiable. This in particular means, that the requirements on the derivative in (ii) are well defined.

The following corollary is a special case of Theorem 2.16 with $J = 0$, i.e., only the amplitudes of the function are prescribed.

Corollary 2.17. Let $k_0 > 0$, $(a_n)_{n=1}^N \subseteq \mathbb{C}$ and pairwise disjoint points $(x_n)_{n=1}^N \subseteq \mathbb{R}$. Then one function $F \in L^2(\mathbb{R})$ satisfying

(i) $\operatorname{supp}(\mathcal{F}[F]) \subseteq [-k_0, k_0]$,

(ii) $F(x_n) = a_n, \quad n \in \{1, \dots, N\}$.

and having minimal L^2-norm, is given by

$$F(x) = \sum_{n=1}^{N} c_n \operatorname{sinc} \big(k_0(x_n - x)\big), \tag{2.46}$$

where the coefficients c_n are the unique solution of the linear system $S\mathbf{c} = \mathbf{a}$ with the matrix entries

$$S_{m,n} = \operatorname{sinc} \big(k_0(x_m - x_n)\big).$$

Remark 2.18. In order to use Corollary 2.17 for the construction of superoscillations, one can choose points $(x_n)_{n=1}^N \in [a, b]$ in ascending order and amplitudes $(a_n)_{n=1}^N$ with alternating sign. For example

$$x_n = a + \frac{n-1}{N-1}(b-a) \quad \text{and} \quad a_n = (-1)^{n-1}. \tag{2.47}$$

This way, the resulting function F from (2.46) changes its sign at least $N - 1$ times and consequently has at least $N - 1$ zeros in the open interval (a, b). If we choose $N - 1 \geq \frac{k_0(b-a)}{\pi}$, the assumption (2.3) is satisfied. Moreover, due to the requirement (i) in Corollary 2.17, the function F admits the representation

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-k_0}^{k_0} \mathcal{F}[F](k) e^{ikx} dk, \quad x \in \mathbb{R}.$$

Hence by Lemma 2.4 (ii) it is of the form (2.1), which then makes the function F a Type II superoscillating function.

Additionally, prescribing also values of the derivatives, as it is done in Theorem 2.16, allows to control the shape of the function, see Figure 2.5.

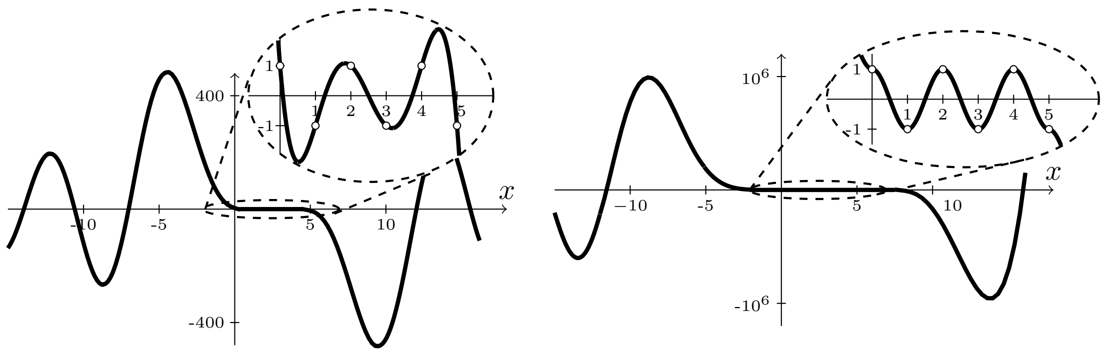


Figure 2.1.: In this example we construct a function with maximum frequency $k_0 = 1$ and want to prescribe the values in the points $x_n = n-1$, $n \in \{1, \ldots, 6\}$. In the left picture the function (2.45) is plotted when only the values $F(x_n) = (-1)^{n+1}$ are fixed. While the function oscillates with a high frequency in the interval $[0, 5]$, the shape of the function is by no means a plane wave. In the right picture, we additionally fix the values $F'(x_n) = 0$ of the derivatives, with the result, that the shape of f, inside the superoscillatory region $[0, 5]$, is modified and now approximates a plane wave. However, the cost for adapting the shape lies in the outer region, which is of magnitudes larger that in the left picture.

Remark 2.19. The optimality result (2.46) can also be interpreted in another way. If we only consider normalized functions $F \in L^2(\mathbb{R})$ with $\|F\|_{L^2(\mathbb{R})} = 1$, and allow the amplitudes $(a_n)_{n=1}^N$ being scaled to $(\mu a_n)_{n=1}^N$, then the function (2.46), divided by its normalization, allow the largest value of μ and consequently has the biggest amplitude of superoscillations. A similar problem in maximizing the amplitude of a bandlimited signal with normalized energy, was already considered in [96].

Proof of Theorem 2.16. We want to minimize the following problem:

$$\begin{aligned} &\min \|F\|^2_{L^2(\mathbb{R})}, \quad \text{under the restrictions} \\ &\operatorname{supp}(\mathcal{F}[F]) \subseteq [-k_0, k_0] \quad \text{and} \quad F^{(j)}(x_n^{(j)}) = a_n^{(j)}, \quad j \in \{0, \ldots, J\}, n \in \{1, \ldots, N_j\}. \end{aligned}$$

We can translate this problem into Fourier space, i.e., we look for $G \in L^2([-k_0, k_0])$, which is the representative for $G = \mathcal{F}[F]$, satisfying the minimization problem

$$\min \|G\|_{L^2([-k_0,k_0])}, \quad \text{under the restrictions} \tag{2.48a}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-k_0}^{k_0} (i\xi)^j e^{i\xi x_n^{(j)}} G(\xi) d\xi = a_n^{(j)}, \quad j \in \{0, \ldots, J\}, n \in \{1, \ldots, N_j\}. \tag{2.48b}$$

The Lagrangian of this minimization problem is given by

$$L = |G(\xi)|^2 + \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{J} \sum_{n=1}^{N_j} \Big(\lambda_n^{(j)} \operatorname{Re}\big((i\xi)^j e^{i\xi x_n^{(j)}} G(\xi)\big) + \mu_n^{(j)} \operatorname{Im}\big((i\xi)^j e^{i\xi x_n^{(j)}} G(\xi)\big)\Big),$$

with Lagrange multipliers $\lambda_n^{(j)}$, $\mu_n^{(j)} \in \mathbb{R}$. Differentiating with respect to $\operatorname{Re}(G)$ and

$\mathrm{Im}(G)$ then gives the two equations

$$2\,\mathrm{Re}(G(\xi)) + \frac{1}{\sqrt{2\pi}}\sum_{j=0}^{J}\sum_{n=1}^{N_j}\Big(\lambda_n^{(j)}\,\mathrm{Re}\left((i\xi)^j e^{i\xi x_n^{(j)}}\right) + \mu_n^{(j)}\,\mathrm{Im}\left((i\xi)^j e^{i\xi x_n^{(j)}}\right)\Big) = 0$$

$$2\,\mathrm{Im}(G(\xi)) + \frac{1}{\sqrt{2\pi}}\sum_{j=0}^{J}\sum_{n=1}^{N_j}\Big(-\lambda_n^{(j)}\,\mathrm{Im}\left((i\xi)^j e^{i\xi x_n^{(j)}}\right) + \mu_n^{(j)}\,\mathrm{Re}\left((i\xi)^j e^{i\xi x_n^{(j)}}\right)\Big) = 0,$$

which can be written as one complex valued equation

$$G(\xi) = \frac{\sqrt{\pi}}{\sqrt{2}}\sum_{j=0}^{J}\sum_{n=1}^{N_j}\frac{c_n^{(j)}}{k_0^{j+1}}(-i\xi)^j e^{-i\xi x_n^{(j)}},$$

where we defined $c_n^{(j)} = -\frac{k_0^{j+1}}{\pi}(\lambda_n^{(j)} + i\mu_n^{(j)})$. Multiplying $\frac{(i\xi)^l}{\sqrt{2\pi}\,k_0^l}e^{i\xi x_m^{(l)}}$, for $l \in \{0,\dots,J\}$, $m \in \{1,\dots,N_l\}$ and integrating over $\xi \in [-k_0, k_0]$, gives

$$\frac{a_m^{(l)}}{k_0^l} = \frac{1}{\sqrt{2\pi}\,k_0^l}\int_{-k_0}^{k_0}(i\xi)^l e^{i\xi x_m^{(l)}} g(\xi)d\xi = \sum_{j=0}^{J}\sum_{n=1}^{N_j}\frac{(-1)^j c_n^{(j)}}{2k_0^{j+l+1}}\int_{-k_0}^{k_0}(i\xi)^{j+l}e^{i\xi(x_m^{(l)}-x_n^{(j)})}d\xi,$$

where in the first equation we used the conditions (2.48b). This is a linear system of equations of the form

$$S\mathbf{c} = \mathbf{a}, \tag{2.49}$$

with the vectors

$$\mathbf{a} := \left(\frac{a_1^{(0)}}{k_0^0},\dots,\frac{a_{N_1}^{(0)}}{k_0^0},\dots,\frac{a_1^{(J)}}{k_0^J},\dots,\frac{a_{N_J}^{(J)}}{k_0^J}\right)^\top,$$
$$\mathbf{c} := \left(c_1^{(0)},\dots,c_{N_1}^{(0)},\dots,c_1^{(J)},\dots,c_{N_J}^{(J)}\right)^\top,$$

and the matrix

$$S := \begin{pmatrix} S_{1,1}^{(0,0)} & \dots & S_{1,N_1}^{(0,0)} & \dots & S_{1,1}^{(0,J)} & \dots & S_{1,N_J}^{(0,J)} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ S_{N_1,1}^{(0,0)} & \dots & S_{N_1,N_1}^{(0,0)} & \dots & S_{N_1,1}^{(0,J)} & \dots & S_{N_1,N_J}^{(0,J)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ S_{1,1}^{(J,0)} & \dots & S_{1,N_1}^{(J,0)} & \dots & S_{1,1}^{(J,J)} & \dots & S_{1,N_J}^{(J,J)} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ S_{N_J,1}^{(J,0)} & \dots & S_{N_J,N_1}^{(J,0)} & \dots & S_{N_J,1}^{(J,J)} & \dots & S_{N_J,N_J}^{(J,J)} \end{pmatrix} \tag{2.50}$$

$$S_{m,n}^{(l,j)} = \frac{(-1)^j}{2k_0^{j+l+1}}\int_{-k_0}^{k_0}(i\xi)^{j+l}e^{i\xi(x_m^{(l)}-x_n^{(j)})}d\xi = (-1)^j\,\mathrm{sinc}^{(j+l)}\left(k_0(x_m^{(l)} - x_n^{(j)})\right).$$

In order to show that the matrix S is invertible, we show that it is positive definite. To do so, let $\mathbf{v} \in \mathbb{C}^{N_1+\cdots+N_J}$ and consider the inner product

$$\begin{aligned}\langle S\mathbf{v}, \mathbf{v}\rangle &= \sum_{l=0}^{J}\sum_{m=1}^{N_l}\sum_{j=0}^{J}\sum_{n=1}^{N_j} \frac{(-1)^j}{2k_0^{j+l+1}} \int_{-k_0}^{k_0} (i\xi)^{j+l} e^{i\xi(x_m^{(l)}-x_n^{(j)})} d\xi v_n^{(j)} \overline{v_m^{(l)}} \\ &= \frac{1}{2k_0}\int_{-k_0}^{k_0} \sum_{l=0}^{J}\sum_{m=1}^{N_l}\sum_{j=0}^{J}\sum_{n=1}^{N_j} \overline{\frac{(-i\xi)^l}{k_0^l} e^{-i\xi x_m^{(l)}} v_m^{(l)}} \frac{(-i\xi)^j}{k_0^j} e^{-i\xi x_n^{(j)}} v_n^{(j)} d\xi \\ &= \frac{1}{2k_0}\int_{-k_0}^{k_0} \left| \sum_{j=0}^{J}\sum_{n=1}^{N_j} \frac{(-i\xi)^j}{k_0^j} e^{-i\xi x_n^{(j)}} v_n^{(j)} \right|^2 d\xi \geq 0.\end{aligned}$$

Moreover, since for every $j \in \{0, \ldots, J\}$ the $(x_n^{(j)})_{n=1}^{N_j}$ are pairwise disjoint, all the functions $(i\xi)^j e^{-i\xi x_n^{(j)}}$ are linear independent and it is obvious that this integral vanishes if and only if the vector $\mathbf{v} = 0$ vanishes. This shows that the matrix S is positive definite and hence invertible. The unique solution $\mathbf{c}$ of (2.49) now determines the optimal solution of the minimization problem in Fourier space (2.48). Inverse Fourier transforming then also gives the optimal solution in real space

$$\begin{aligned}F(x) = F^{-1}[G](x) &= \sum_{j=0}^{J}\sum_{n=1}^{N_j} \frac{c_n^{(j)}}{2k_0^{j+1}} \int_{-k_0}^{k_0} (i\xi)^j e^{i\xi(x_n^{(j)}-x)} d\xi \\ &= \sum_{j=0}^{J}\sum_{n=1}^{N_j} c_n^{(j)} \operatorname{sinc}^{(j)}\big(k_0(x_n^{(j)} - x)\big),\end{aligned}$$

which is exactly the stated representation of the minimizer (2.45) □

Although the result (2.45) gives an explicit solution of the minimization problem, it is shown in [80, 82] that the actual calculation of the coefficients $c_n^{(j)}$ is problematic since the corresponding coefficient matrix S in (2.50) is highly ill conditioned as the points $x_n^{(j)}$ are located close to each other. Moreover, the sensitivity of the amplitudes $a_n^{(j)}$ with respect to errors of the coefficients $c_n^{(j)}$ is investigated in [85]. Improvements of this issue by using the technique of oversampled signal reconstruction are done in [81], by shifting the interpolation points in [82], or by constructing only approximations of superoscillating functions in [95].

2.6. Type II superoscillating product of shifted functions

While solving the linear system (2.49) is very difficult from a computational point of view, we want to present a very efficient way of constructing Type II superoscillating functions in this last section. The idea was given by A. Kempf in [63] and is to multiply many, slightly shifted versions of a functions of low bandwidth. For example

$$F(x) = \prod_{j=1}^{n} \sin\Big(\frac{x - \varepsilon_j}{n}\Big),$$

with $\varepsilon_j > 0$ being small displacements which determine the final spacing between the zeros of F. One can also consider

$$F(x) = \prod_{j=1}^{n} \operatorname{sinc}\Big(\frac{x-\varepsilon_j}{n}\Big),$$

for an L^2-superoscillating function.

The following theorem generalizes this idea and proves that this method indeed yields Type II superoscillating functions.

Theorem 2.20. Let $k_0 > 0$ and $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and $f \not\equiv 0$. Moreover, f admits one of the following representations:

(i) A linear combination of plane waves

$$f(x) = \sum_{l=0}^{n} C_l e^{ik_l x}, \quad x \in \mathbb{R}, \tag{2.51}$$

with coefficients $C_l \in \mathbb{C}$ and frequencies $k_l \in [-k_0, k_0]$.

(ii) $f \in L^2(\mathbb{R})$ with compactly supported Fourier transform $\operatorname{supp}(\mathcal{F}[f]) \subseteq [-k_0, k_0]$.

Then for any interval $[a, b] \subseteq \mathbb{R}$, $n \geq \frac{k_0(b-a)}{\pi}$ and pairwise disjoint points $(\varepsilon_j)_{j=1}^n \subseteq (a, b)$, the product of shifted functions

$$F(x) = \prod_{j=1}^{n} f\Big(\frac{x-\varepsilon_j}{n}\Big), \quad x \in \mathbb{R}, \tag{2.52}$$

is a Type II superoscillating function.

Proof.

(i) If we assume f to be of the form (2.51), we can rewrite the product as

$$\begin{aligned} F(x) &= \prod_{j=1}^{n} \sum_{l=0}^{n} C_l e^{\frac{i}{n}k_l(x-\varepsilon_j)} = \sum_{l_1,\dots,l_n=0}^{n} \prod_{j=1}^{n} \Big(C_{l_j} e^{\frac{i}{n}k_{l_j}(x-\varepsilon_j)} \Big) \\ &= \sum_{l_1,\dots,l_n=0}^{n} \Big(\prod_{j=1}^{n} C_{l_j} e^{-\frac{i}{n}k_{l_j}\varepsilon_j} \Big) e^{\frac{i}{n}\sum\limits_{j=1}^{n} k_{l_j} x}, \quad x \in \mathbb{R}. \end{aligned}$$

Hence F is a linear combination of plane waves with frequencies bounded as

$$\frac{1}{n}\Big|\sum_{j=1}^{n} k_{l_j}\Big| \leq \frac{1}{n}\sum_{j=1}^{n} k_0 = k_0, \quad l_1, \dots, l_n \in \{0, \dots, n\},$$

and consequently of the form (2.1) due to Lemma 2.4 (i).

We also see immediately, that $F(\varepsilon_j) = 0$ for every $j \in \{1, \dots, n\}$, which means, that F has at least n zeros in the open interval (a, b). Since F obviously extends to an entire function, these zeros are also isolated. The assumption $n \geq \frac{k_0(b-a)}{\pi}$ then indeed ensures that F is a Type II superoscillating function according to Definition 2.2.

(ii) Considering now the second type of functions, we note, that the Fourier transform of each individual factor is given by

$$\mathcal{F}\Big[f\Big(\frac{\cdot-\varepsilon_j}{n}\Big)\Big](k)=e^{-i\frac{\varepsilon_j}{n}k}\mathcal{F}\Big[f\Big(\frac{\cdot}{n}\Big)\Big](k)=ne^{-i\frac{\varepsilon_j}{n}k}\mathcal{F}[f](nk),\quad k\in\mathbb{R}.$$

Since by assumption $\operatorname{supp}(\mathcal{F}[f])\subseteq[-k_0,k_0]$, we have

$$\operatorname{supp}\Big(\mathcal{F}\Big[f\Big(\frac{\cdot-\varepsilon_j}{n}\Big)\Big]\Big)\subseteq\Big[-\frac{k_0}{n},\frac{k_0}{n}\Big],\quad j\in\{1,\dots,n\}.$$

By the convolution theorem about the Fourier transform of products we know, that the Fourier transform of F is included in the sum of the individual supports

$$\operatorname{supp}(\mathcal{F}[F])\subseteq\sum_{j=1}^{n}\Big[-\frac{k_0}{n},\frac{k_0}{n}\Big]=[-k_0,k_0].$$

Hence we can represent the function F via its inverse Fourier transform

$$F(x)=\int_{-k_0}^{k_0}\mathcal{F}[F](k)e^{ikx}dk,\quad x\in\mathbb{R}.$$

Lemma 2.4 (ii) then ensures the representation (2.1).

Since again $F(\varepsilon_j)=0$ for every $j\in\{1,\dots,n\}$, the function F admits at least $n\geq\frac{k_0(b-a)}{\pi}$ zeros inside the interval (a,b). The compactly supported Fourier transform of the function f in particular implies, that it extends to an entire function by the Paley-Wiener theorem. Hence the zeros are isolated and we verified that also in this case the function (2.52) is a Type II superoscillating function. □

3. Fresnel type integrals

In this chapter we develop the so called Fresnel integral technique, which is a method to make sense of integrals of the form

$$\int_{\mathbb{R}} e^{iy^2} f(y)dy, \tag{3.1}$$

in particular in situations where the function f itself is not integrable. The basic idea is to use the Cauchy theorem to rotate the domain of integration into the complex plane and consequently make the oscillating prefactor e^{iy^2} a Gaussian e^{-y^2} whose decay at infinity ensures integrability.

The results of this chapter, in particular Corollary 3.2 with the choice $a = 1$ and $x = 0$, will show that for holomorphic functions f which are exponentially bounded, one can insert the Gaussian $e^{-\varepsilon(y-y_0)^2}$ and view the integral (3.1) as the limit

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} e^{-\varepsilon(y-y_0)^2} e^{iy^2} f(y)dy.$$

Under some slightly stronger growth assumptions, which in particular implies that f is bounded on the real line, such a regularization is not necessary and one can regard the integral (3.1) as the limit

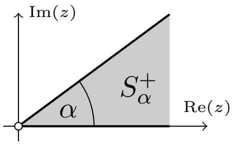
$$\lim_{R_1,R_2\to\infty} \int_{-R_1}^{R_2} e^{iy^2} f(y)dy.$$

However, under both assumptions one has the absolutely convergent representation

$$e^{i\alpha} \int_{\mathbb{R}} e^{i(ye^{i\alpha})^2} f(ye^{i\alpha})dy,$$

for some $\alpha > 0$. Note, that the subsequent Fresnel type integral technique is in two ways an improvement of the version in [7]. The first improvement lies in the fact, that we allow an exponential growth of order $p \in (0, 2)$ in (3.3) and (3.6), while in [7] only $p = 1$ was considered. The second improvement lies roughly speaking in the fact, that in [7] the function f had to be holomorphic in a neighborhood of the closed cone S_α (3.20), but here it is enough for f to be holomorphic in the interior of S_α with a continuous extension to the boundary.

Proposition 3.1 (Fresnel type integral)**.** Let $a > 0$, $x \in \mathbb{R}$ and consider for $\alpha \in (0, \frac{\pi}{2})$ the sector



$$S_\alpha^+ := \{\, z \in \mathbb{C} \setminus \{0\} \mid \operatorname{Arg}(z) \in [0, \alpha] \,\}, \tag{3.2}$$

and a continuous function $f : S_\alpha^+ \to \mathbb{C}$ which is holomorphic on the interior of S_α^+. Then the following assertions hold.

(i) If f satisfies the estimate

$$|f(z)| \leq Ae^{B|z|^p}, \quad z \in S_\alpha^+, \tag{3.3}$$

for some and $A, B \geq 0$ and $p \in (0,2)$, then for every $y_0 \in \mathbb{R}$

$$\lim_{\varepsilon \to 0^+} \int_0^\infty e^{-\varepsilon(y-y_0)^2} e^{ia(y-x)^2} f(y) dy = e^{i\alpha} \int_0^\infty e^{ia(ye^{i\alpha}-x)^2} f(ye^{i\alpha}) dy, \tag{3.4}$$

where both integrands are absolutely integrable. Moreover, for $0 < \varepsilon < \frac{2a}{\tan(\alpha)}$ moreover get

$$\int_0^\infty e^{-\varepsilon(y-y_0)^2} e^{ia(y-x)^2} f(y) dy = e^{i\alpha} \int_0^\infty e^{-\varepsilon(ye^{i\alpha}-y_0)^2} e^{ia(ye^{i\alpha}-x)^2} f(ye^{i\alpha}) dy. \tag{3.5}$$

(ii) If f satisfies the estimate

$$|f(z)| \leq Ae^{B|\operatorname{Im}(z)|^p}, \quad z \in S_\alpha^+, \tag{3.6}$$

for some $A, B \geq 0$ and $p \in (0,2)$, then

$$\lim_{R \to \infty} \int_0^R e^{ia(y-x)^2} f(y) dy = e^{i\alpha} \int_0^\infty e^{ia(ye^{i\alpha}-x)^2} f(ye^{i\alpha}) dy, \tag{3.7}$$

where the integrand on the right hand side is absolutely integrable, and also the integrand on the left hand side is absolutely integrable for every $R > 0$.

Proof. Since the calculation is the same, we will for simplicity only consider $x = 0$, $a = 1$ and $y_0 = 0$.

We start by proving (i). For any η in the interior of S_α^+ and with $|\eta| \leq 1$, we define the shifted function

$$f_\eta(z) := f(z + \eta), \quad z \in S_\alpha^+ - \eta. \tag{3.8}$$

Then f_η is holomorphic on the interior of $S_\alpha^+ - \eta$ and admits the exponential bound

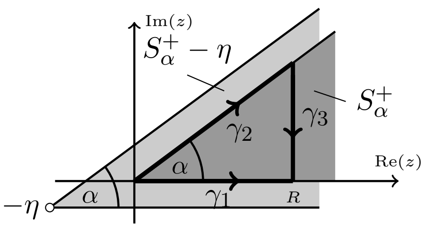
$$|f_\eta(z)| \leq Ae^{B|z+\eta|^p} \leq Ae^{B2^p(|z|^p+|\eta|^p)} \leq Ae^{B2^p(|z|^p+1)} = \widetilde{A}e^{\widetilde{B}|z|^p}, \quad z \in S_\alpha^+ - \eta, \tag{3.9}$$

by (3.3) and using the new constants $\widetilde{A} := Ae^{B2^p}$ and $\widetilde{B} = B2^p$. Fixing $R > 0$, we then consider the integration path

$$\gamma_1 := \{\, y \mid 0 \leq y \leq R \,\},$$

$$\gamma_2 := \left\{\, ye^{i\alpha} \;\middle|\; 0 \leq y \leq \frac{R}{\cos(\alpha)} \,\right\},$$

$$\gamma_3 := \{\, R + iy \mid R\tan(\alpha) \geq y \geq 0 \,\}.$$



Since the integration paths γ_1, γ_2, γ_3 lies inside the interior of $S_\alpha^+ - \eta$, where f_η is holomorphic, Cauchy's theorem yields for every $\varepsilon > 0$

$$\int_{\gamma_1} e^{(i-\varepsilon)z^2} f_\eta(z) dz = \int_{\gamma_2} e^{(i-\varepsilon)z^2} f_\eta(z) dz + \int_{\gamma_3} e^{(i-\varepsilon)z^2} f_\eta(z) dz. \tag{3.10}$$

Using the exponential bound (3.9), we can estimate the integral along γ_3 as

$$\begin{aligned}\Big|\int_{\gamma_3} e^{(i-\varepsilon)z^2} f_\eta(z)dz\Big| &\le \widetilde{A}e^{-\varepsilon R^2}\int_0^{R\tan(\alpha)} e^{\varepsilon y^2-2Ry+\widetilde{B}|R+iy|^p}dy\\ &\le \widetilde{A}e^{-\varepsilon R^2+\frac{\widetilde{B}R^p}{\cos^p(\alpha)}}\int_0^{R\tan(\alpha)} e^{-y(2R-\varepsilon y)}dy\\ &\le \widetilde{A}R\tan(\alpha)e^{-\varepsilon R^2+\frac{\widetilde{B}R^p}{\cos^p(\alpha)}},\end{aligned}$$

where in the last line we restricted $\varepsilon \le \frac{2}{\tan(\alpha)}$ to conclude $\varepsilon y \le 2R$. This estimate proves the convergence

$$\lim_{R\to\infty}\int_{\gamma_3} e^{(i-\varepsilon)z^2} f_\eta(z)dz = 0,$$

and consequently, in the limit $R \to \infty$, the integrals (3.10) become

$$\int_0^\infty e^{(i-\varepsilon)y^2} f_\eta(y)dy = e^{i\alpha}\int_0^\infty e^{(i-\varepsilon)(ye^{i\alpha})^2} f_\eta(ye^{i\alpha})dy. \tag{3.11}$$

Here both integrals are absolutely convergent, the left hand side because of the factor $e^{-\varepsilon y^2}$ and the right hand side due to the estimate

$$\begin{aligned}\big|e^{(i-\varepsilon)(ye^{i\alpha})^2} f_\eta(ye^{i\alpha})\big| &\le \widetilde{A}e^{-(\sin(2\alpha)+\varepsilon\cos(2\alpha))y^2+\widetilde{B}y^p},\\ &\le \widetilde{A}e^{-(\frac{2}{\tan(\alpha)}-\varepsilon)\sin^2(\alpha)y^2+\widetilde{B}y^p},\end{aligned} \tag{3.12}$$

which is integrable for every $\varepsilon < \frac{2}{\tan(\alpha)}$. Moreover, since the upper bound (3.12) is η-independent, we can apply the dominated convergence theorem to both sides of (3.11) and obtain

$$\int_0^\infty e^{(i-\varepsilon)y^2} f(y)dy = e^{i\alpha}\int_0^\infty e^{(i-\varepsilon)(ye^{i\alpha})^2} f(ye^{i\alpha})dy, \tag{3.13}$$

which is exactly the identity (3.5). Finally, we want to apply the limit $\varepsilon \to 0^+$ to this equation. By the estimate (3.12) for f instead of f_η, i.e., formally putting $\eta = 0$, the integrand on the right hand side of (3.13) is bounded by some majorant which decreases as $\varepsilon \to 0^+$. The dominated convergence theorem then yields the stated limit (3.4).

For the proof of (ii) we again consider the function f_η from (3.8) with $|\eta| \le 1$, which in this case satisfies the estimate

$$|f_\eta(z)| \le Ae^{B|\operatorname{Im}(z+\eta)|^p} \le Ae^{B2^p(|\operatorname{Im}(z)|^p+|\operatorname{Im}(\eta)|^p)} \le \widetilde{A}e^{\widetilde{B}|\operatorname{Im}(z)|^p}, \quad z \in S_\alpha^+ - \eta, \tag{3.14}$$

using the constants $\widetilde{A} := Ae^{B2^p}$ and $\widetilde{B} := B2^p$. In the same way as we derived (3.10), we also get

$$\int_{\gamma_1} e^{iz^2} f_\eta(z)dz = \int_{\gamma_2} e^{iz^2} f_\eta(z)dz + \int_{\gamma_3} e^{iz^2} f_\eta(z)dz. \tag{3.15}$$

With the exponential bound (3.14) we can estimate for every $R > \big(\frac{\widetilde{B}\tan^{p-1}(\alpha)}{2}\big)^{2-p}$ the integral along γ_3 as

$$\begin{aligned}\Big|\int_{\gamma_3} e^{iz^2} f_\eta(z)dz\Big| &\leq \widetilde{A}\int_0^{R\tan(\alpha)} e^{-y(2R-\widetilde{B}y^{p-1})}dy \\ &\leq \widetilde{A}\int_0^{\infty} e^{-y(2R-\widetilde{B}R^{p-1}\tan^{p-1}(\alpha))}dy \\ &= \frac{\widetilde{A}}{2R-\widetilde{B}R^{p-1}\tan^{p-1}(\alpha)}. \end{aligned} \tag{3.16}$$

Hence also in this case the integral over γ_3 vanishes in the limit $R \to \infty$ and (3.15) becomes

$$\lim_{R\to\infty}\int_0^R e^{iy^2} f_\eta(y)dy = e^{i\alpha}\int_0^\infty e^{i(ye^{i\alpha})^2} f_\eta(ye^{i\alpha})dy, \tag{3.17}$$

where the integrand on the right hand side is absolutely integrable, since the inequality (3.14) gives the estimate

$$\Big|e^{i(ye^{i\alpha})^2} f_\eta(ye^{i\alpha})\Big| \leq \widetilde{A}e^{-y^2\sin(2\alpha)+\widetilde{B}y^p\sin^p(\alpha)}. \tag{3.18}$$

This estimate also gives an η-uniform majorant which allows to carry the limit $\eta \to 0$ inside the integral on the right hand side of (3.17), i.e.,

$$\lim_{\eta\to 0}\lim_{R\to\infty}\int_0^R e^{iy^2} f_\eta(y)dy = e^{i\alpha}\int_0^\infty e^{i(ye^{i\alpha})^2} f(ye^{i\alpha})dy. \tag{3.19}$$

On the left hand side we still have to interchange the two limits, which means we have to show that (3.17) converges uniformly in η. To do so, we use the Cauchy theorem (3.15) to estimate

$$\begin{aligned}&\Big|\int_0^R e^{iy^2} f_\eta(y)dy - e^{i\alpha}\int_0^\infty e^{i(ye^{i\alpha})^2} f_\eta(ye^{i\alpha})dy\Big| \\ &\qquad\leq \Big|\int_{\gamma_3} e^{iz^2} f_\eta(z)dz\Big| + \Big|\int_{\frac{R}{\cos(\alpha)}}^\infty e^{i(ye^{i\alpha})^2} f_\eta(ye^{i\alpha})dy\Big|.\end{aligned}$$

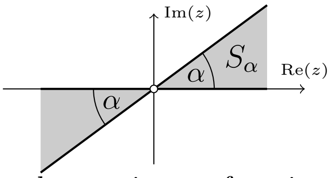
The integral along γ_3 converges uniformly due to (3.16), but also the second integral can be estimated by the η-independent upper bound (3.18). Hence the convergence (3.17) is indeed uniform in η and we are allowed to interchange the limits in (3.19) and get

$$e^{i\alpha}\int_0^\infty e^{i(ye^{i\alpha})^2} f(ye^{i\alpha})dy = \lim_{R\to\infty}\lim_{\eta\to 0}\int_0^R e^{iy^2} f_\eta(y)dy = \lim_{R\to\infty}\int_0^R e^{iy^2} f(y)dy,$$

where in the second equality we were allowed to carry the limit inside the integral due to the η-independent upper bound (3.18). This finally proves the second assertion (3.7). □

The Fresnel integral technique of Proposition 3.1 can also be applied on the negative semi axis, which leads to the following corollary.

Corollary 3.2. Let $a > 0$, $x \in \mathbb{R}$ and consider for some $\alpha \in (0, \frac{\pi}{2})$ the double sector



$$S_\alpha := \{ z \in \mathbb{C} \setminus \{0\} \mid \operatorname{Arg}(z) \in [0, \alpha] \cup [\pi, \pi + \alpha] \}, \tag{3.20}$$

and a continuous function $f : S_\alpha \to \mathbb{C}$ which is holomorphic on the interior of S_α. Then the following assertions hold.

(i) If f satisfies the estimate

$$|f(z)| \leq A e^{B|z|^p}, \quad z \in S_\alpha, \tag{3.21}$$

for some $A, B \geq 0$ and $p \in (0, 2)$, then for every $y_0 \in \mathbb{R}$

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} e^{-\varepsilon (y - y_0)^2} e^{ia(y-x)^2} f(y) dy = e^{i\alpha} \int_{\mathbb{R}} e^{ia(ye^{i\alpha} - x)^2} f(ye^{i\alpha}) dy, \tag{3.22}$$

where both integrands are absolutely integrable. Moreover, for $0 < \varepsilon < \frac{2a}{\tan(\alpha)}$ we even get

$$\int_{\mathbb{R}} e^{-\varepsilon (y - y_0)^2} e^{ia(y-x)^2} f(y) dy = e^{i\alpha} \int_{\mathbb{R}} e^{-\varepsilon (ye^{i\alpha} - y_0)^2} e^{ia(ye^{i\alpha} - x)^2} f(ye^{i\alpha}) dy. \tag{3.23}$$

(ii) If f satisfies the estimate

$$|f(z)| \leq A e^{B|\operatorname{Im}(z)|^p}, \quad z \in S_\alpha, \tag{3.24}$$

for some $A, B \geq 0$ and $p \in (0, 2)$, then

$$\lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} e^{ia(y-x)^2} f(y) dy = e^{i\alpha} \int_{\mathbb{R}} e^{ia(ye^{i\alpha} - x)^2} f(ye^{i\alpha}) dy, \tag{3.25}$$

where the integrand on the right hand side is absolutely integrable, and also the integrand on the left hand side is absolutely integrable for every $R_1, R_2 > 0$.

4. Schrödinger equation on $\mathbb{R}$

The central topic of this chapter is the investigation of the Cauchy problem (4.3) of the time dependent Schrödinger equation. In particular we consider the Green's function approach, which means to write the solution Ψ as an integral of the form (4.4), where G is the so called Green's function. The main result of this chapter will then be Theorem 4.6, which puts the integral (4.4) into a mathematical rigorous framework using Fresnel integrals (3.22) and gives a continuous dependency between the initial condition F and the solution Ψ. In the application on superoscillations in Chapter 6, we then put some superoscillating function as initial condition and conclude from this continuity property, that also the solution attains some superoscillatory behaviour at later times $t > 0$.

In the history of superoscillations, continuous dependency problems of this kind were treated using infinite order differential operators of the form (1.17), see for example [4, 19, 20, 22, 28, 33, 35, 36, 66, 67]. However, in our proof we avoid this detour and follow a direct path by using estimates of the integral representation (4.4). At this point we also want to mention the recent work [101], where the authors give sufficient conditions on the moments $\int_{\mathbb{R}} y^m G(t,x,y)dy$ of the Green's function to prove some continuous dependency of solution and initial condition of the Schrödinger equation.

We start to specify in detail in which sense we want to understand the Cauchy problem (4.3). It will be convenient to view the solution (and its derivatives) in the context of absolutely continuous functions. The linear space of absolutely continuous functions on some open interval $I \subseteq \mathbb{R}$ will be denoted by $\mathrm{AC}(I)$. Recall, that a function $f : I \to \mathbb{C}$ is said to be *absolutely continuous*, if there exists some $g \in L^1_{\mathrm{loc}}(I)$, such that

$$f(y) - f(x) = \int_x^y g(s)ds, \quad x, y \in I. \tag{4.1}$$

Also observe, that $f \in \mathrm{AC}(I)$ is differentiable almost everywhere and its derivative f' coincides with g in (4.1) almost everywhere. For $T \in (0, \infty]$ we shall now work with the space

$$\mathrm{AC}_{1,2}((0,T)\times\mathbb{R}) := \left\{ \Psi : (0,T)\times\mathbb{R} \to \mathbb{C} \;\middle|\; \begin{array}{l} \Psi(\,\cdot\,,x) \in \mathrm{AC}((0,T)),\ x \in \mathbb{R} \\ \Psi(t,\,\cdot\,), \frac{\partial}{\partial x}\Psi(t,\,\cdot\,) \in \mathrm{AC}(\mathbb{R}),\ t \in (0,T) \end{array} \right\}. \tag{4.2}$$

Let $V : (0,T)\times\mathbb{R} \to \mathbb{C}$ be some potential and $F : \mathbb{R} \to \mathbb{C}$ some initial condition. We call a function $\Psi \in \mathrm{AC}_{1,2}((0,T)\times\mathbb{R})$ a *solution* of the time dependent Schrödinger equation, if it satisfies

$$i\frac{\partial}{\partial t}\Psi(t,x) = \Big(-\frac{\partial^2}{\partial x^2} + V(t,x)\Big)\Psi(t,x), \qquad \text{f.a.e. } t \in (0,T),\ x \in \mathbb{R}, \tag{4.3a}$$

$$\lim_{t\to 0^+}\Psi(t,x) = F(x), \qquad x \in \mathbb{R}. \tag{4.3b}$$

As already mentioned, the starting point will always be the corresponding *Green's function* $G : (0,T) \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, which only depends on the potential V and is independent of the initial condition F. With this Green's function the solution Ψ then admits the (formal) representation

$$\Psi(t,x) = \int_{\mathbb{R}} G(t,x,y)F(y)dy, \quad t \in (0,T),\ x \in \mathbb{R}. \tag{4.4}$$

Since we want to apply this theory to superoscillations, one of the important tasks of this chapter is to make sense of this integral for exponential initial values of the form (0.1), or more general for exponentially bounded initial conditions $F \in \mathcal{A}_p(\mathbb{C})$ from (1.1). It turns out, that the integrand in (4.4) is not in $L^1(\mathbb{R})$ and hence the integral does not exist in the Lebesgue sense. Instead, the Fresnel integral technique of Chapter 3 will be used for a rigorous interpretation of the integral. Roughly speaking, the integration path is rotated into the complex plane, where the integrand becomes absolutely integrable and the wave function becomes

$$\Psi(t,x) = e^{i\alpha} \int_{\mathbb{R}} G(t,x,ye^{i\alpha})F(ye^{i\alpha})dy, \tag{4.5}$$

for some $\alpha \in (0, \frac{\pi}{2})$. Equivalently, (4.4) can also be interpreted as the regularized integral

$$\Psi(t,x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} e^{-\varepsilon y^2} G(t,x,y)F(y)dy,$$

and under certain stronger assumptions, see Remark 4.7, even as the improper Riemann integral

$$\Psi(t,x) = \lim_{R_1,R_2 \to \infty} \int_{-R_1}^{R_2} G(t,x,y)F(y)dy.$$

Next we collect a set of assumptions on the Green's function, which ensure that the wave function (4.4) is well defined (in one of the just mentioned equivalent senses) and a solution of the Cauchy problem (4.3). The precise formulation of this statement, and also the set of allowed initial conditions, is given in Theorem 4.6.

Assumption 4.1. Let $T \in (0,\infty]$ and $G : (0,T) \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. For some $\alpha \in (0, \frac{\pi}{2})$ let S_α be the double sector (3.20), and suppose that G admits a continuation to a function $G : (0,T) \times \mathbb{R} \times S_\alpha \to \mathbb{C}$, such that for every fixed $t \in (0,T)$, $x \in \mathbb{R}$ the mapping $G(t,x,\,\cdot\,)$ is continuous on S_α and holomorphic on the interior of S_α. Moreover, it will be assumed that G satisfies the following properties (i)–(iii).

(i) For every fixed $z \in S_\alpha$, the function $G(\,\cdot\,,\,\cdot\,,z) \in \mathrm{AC}_{1,2}((0,T) \times \mathbb{R})$ is a solution of the time dependent Schrödinger equation

$$i\frac{\partial}{\partial t}G(t,x,z) = \Big(-\frac{\partial^2}{\partial x^2} + V(t,x)\Big)G(t,x,z), \quad \text{f.a.e. } t \in (0,T),\ x \in \mathbb{R}, \tag{4.6}$$

with $V : (0,T) \times \mathbb{R} \to \mathbb{C}$ the considered potential.

(ii) For every $x \in \mathbb{R}$ there exists some $x_0 > |x|$, such that for every $F \in \mathcal{H}(\mathbb{C})$ we have the initial condition

$$\lim_{t \to 0^+} \int_{-x_0}^{x_0} G(t,x,y)F(y)dy = F(x). \tag{4.7}$$

(iii) There exists $a \in \mathrm{AC}((0,T))$ with $a(t) > 0$ and $\lim_{t\to 0^+} a(t) = \infty$, such that the function $\widetilde{G}$ in the decomposition

$$G(t,x,z) = e^{ia(t)(z-x)^2}\widetilde{G}(t,x,z), \quad t \in (0,T),\ x \in \mathbb{R},\ z \in S_\alpha, \tag{4.8}$$

is for every $t \in (0,T)$, $x \in \mathbb{R}$ exponentially bounded as

$$|\widetilde{G}(t,x,z)| \leq A_0(t,x)e^{B_0(t,x)|z|^q}, \quad z \in S_\alpha, \tag{4.9a}$$

$$\Big|\frac{\partial}{\partial x}\widetilde{G}(t,x,z)\Big|, \Big|\frac{\partial^2}{\partial x^2}\widetilde{G}(t,x,z)\Big|, \Big|\frac{\partial}{\partial t}\widetilde{G}(t,x,z)\Big| \leq A_1(t,x)e^{B_1(t,x)|z|^q}, \quad z \in S_\alpha. \tag{4.9b}$$

Here $q \in (0,2)$ and $A_0, A_1, B_0, B_1 : (0,T) \times \mathbb{R} \to [0,\infty)$ are continuous and for every $x \in \mathbb{R}$

$$\frac{A_0(\,\cdot\,,x)}{\sqrt{a(\,\cdot\,)}} \text{ and } B_0(\,\cdot\,,x) \text{ are bounded as } t \to 0^+. \tag{4.10}$$

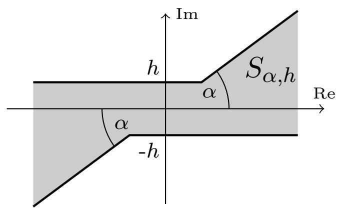
We briefly comment on some of the conditions in Assumption 4.1 and also refer the reader to Section 7.1–7.4 for explicit examples of Green's functions.

Remark 4.2. The holomorphic extension of G in the z-variable is needed to apply the Fresnel integral technique of Corollary 3.2. The crucial assumption is the decomposition (4.8), where the exponential $e^{ia(t)(z-x)^2}$ (quadratic in z) is separated from the remainder $\widetilde{G}$, which admits the exponential growth (4.9a) of order $q < 2$. The rotation of the integration path in (3.22) from the real line into the complex plane, turns the factor $e^{ia(t)(z-x)^2}$ into a Gaussian, which then dominates the exponential growth of $\widetilde{G}$ in the integral (4.5).

Remark 4.3. Note, that the Equation (4.8) of Assumption 4.1 introduces the reduced Green's function $\widetilde{G}(t,x,z)$. From a practical point of view it is often easier to differentiate $\widetilde{G}$ instead of G. Hence we can translate the Schrödinger equation (4.6) or (4.13) for G into an equivalent differential equation for $\widetilde{G}$, namely

$$i\frac{\partial}{\partial t}\widetilde{G} = \Big(-\frac{\partial^2}{\partial x^2} + 4ia(z-x)\frac{\partial}{\partial x} + (4a^2 + a')(z-x)^2 - 2ia + V\Big)\widetilde{G}. \tag{4.11}$$

Since in practical applications the initial condition (4.7) is often hard to verify, the following Assumption 4.4 gives an opportunity to replace it by the simple limit (4.14). Roughly speaking, the limit (4.14) is one way how the Green's function can approach $\delta(x-y)$ as $t \to 0^+$, while (4.7) allows any approximation. However, in order to use this simplification it is necessary for the Green's function to be holomorphic (and satisfy (4.9a)) not only on S_α but also in a neighborhood of the real line. More precisely, for $\alpha \in (0, \frac{\pi}{2})$ and $h > 0$ we define the domain



$$S_{\alpha,h} := S_\alpha \cup \big(\mathbb{R} + i[-h,h]\big), \tag{4.12}$$

Assumption 4.4. Let $T \in (0,\infty]$ and $G : (0,T) \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. For some $\alpha \in (0, \frac{\pi}{2})$, $h > 0$ suppose that G admits a continuation to a function $G : (0,T) \times \mathbb{R} \times S_{\alpha,h} \to \mathbb{C}$, such that for every fixed $t \in (0,T)$, $x \in \mathbb{R}$ the mapping $G(t,x,\,\cdot\,)$ is continuous on $S_{\alpha,h}$ and holomorphic on the interior of S_α. Moreover, it will be assumed that G satisfies the following properties (i)–(iii).

(i) For every fixed $z \in S_\alpha$, the function $G(\,\cdot\,,\,\cdot\,,z) \in \mathrm{AC}_{1,2}((0,T) \times \mathbb{R})$ is a solution of the time dependent Schrödinger equation

$$i\frac{\partial}{\partial t}G(t,x,z) = \Big(-\frac{\partial^2}{\partial x^2} + V(t,x)\Big)G(t,x,z), \quad \text{f.a.e. } t \in (0,T),\ x \in \mathbb{R}, \tag{4.13}$$

with $V : (0,T) \times \mathbb{R} \to \mathbb{C}$ the considered potential.

(ii) With the function $a(t)$ from (4.15), the Green's function admits the limit

$$\lim_{t\to 0^+} \frac{G(t,x,x)}{\sqrt{a(t)}} = \frac{1}{\sqrt{i\pi}}, \quad x \in \mathbb{R}. \tag{4.14}$$

(iii) There exists $a \in \mathrm{AC}((0,T))$ with $a(t) > 0$ and $\lim_{t\to 0^+} a(t) = \infty$, such that the function $\widetilde{G}$ in the decomposition

$$G(t,x,z) = e^{ia(t)(z-x)^2}\widetilde{G}(t,x,z), \quad t \in (0,T),\ x \in \mathbb{R},\ z \in S_{\alpha,h}, \tag{4.15}$$

is for every $t \in (0,T)$, $x \in \mathbb{R}$ exponentially bounded as

$$|\widetilde{G}(t,x,z)| \leq A_0(t,x)e^{B_0(t,x)|z|^q}, \quad z \in S_{\alpha,h} \tag{4.16a}$$

$$\Big|\frac{\partial}{\partial x}\widetilde{G}(t,x,z)\Big|,\ \Big|\frac{\partial^2}{\partial x^2}\widetilde{G}(t,x,z)\Big|,\ \Big|\frac{\partial}{\partial t}\widetilde{G}(t,x,z)\Big| \leq A_1(t,x)e^{B_1(t,x)|z|^q}, \quad z \in S_\alpha. \tag{4.16b}$$

Here $q \in (0,2)$ and $A_0, A_1, B_0, B_1 : (0,T) \times \mathbb{R} \to [0,\infty)$ are continuous and for every $x \in \mathbb{R}$

$$\frac{A_0(\,\cdot\,,x)}{\sqrt{a(\,\cdot\,)}} \text{ and } B_0(\,\cdot\,,x) \text{ are bounded as } t \to 0^+. \tag{4.17}$$

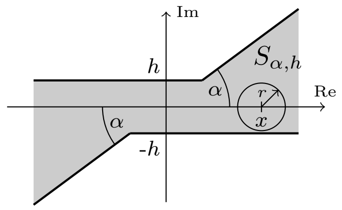
The following lemma proves that the Assumption 4.4 is indeed stronger than the Assumption 4.1.

Lemma 4.5. If $G : (0,T) \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ satisfies Assumption 4.4, it also satisfies Assumption 4.1.

Proof. The only thing to check is the initial condition (4.7). For fixed $x \in \mathbb{R}$, we first generalize (4.14) in the sense, that for any $z(t) \in S_{\alpha,h}$, with $z(t) \overset{t\to 0^+}{\longrightarrow} x$, we have

$$\lim_{t\to 0^+} \frac{\widetilde{G}(t,x,z(t))}{\sqrt{a(t)}} = \frac{1}{\sqrt{i\pi}}. \tag{4.18}$$

Note, that any closed ball of radius $0 < r < h$ around x is contained in the interior of $S_{\alpha,h}$, where $\widetilde{G}(t,x,\,\cdot\,)$ is holomorphic.



Hence we are allowed to apply the Cauchy integral formula to write

$$\begin{aligned}\widetilde{G}(t,x,z(t)) - \widetilde{G}(t,x,x) &= \frac{1}{2\pi i}\int_{|z-x|=r}\Big(\frac{\widetilde{G}(t,x,z)}{z-z(t)} - \frac{\widetilde{G}(t,x,z)}{z-x}\Big)dz \\ &= \frac{z(t)-x}{2\pi i}\int_{|z-x|=r}\frac{\widetilde{G}(t,x,z)}{(z-z(t))(z-x)}dz \\ &= \frac{z(t)-x}{2\pi}\int_0^{2\pi}\frac{\widetilde{G}(t,x,x+re^{i\theta})}{x+re^{i\theta}-z(t)}d\theta.\end{aligned}$$

Using (4.9a), we can further estimate the integrand to get

$$\begin{aligned}\big|\widetilde{G}(t,x,z(t)) - \widetilde{G}(t,x,x)\big| &\le \frac{A_0(t,x)|z(t)-x|}{2\pi}\int_0^{2\pi}\frac{e^{B_0(t,x)|x+re^{i\theta}|^q}}{|x+re^{i\theta}-z(t)|}d\theta \\ &\le \frac{A_0(t,x)|z(t)-x|}{r-|z(t)-x|}e^{B_0(t,x)(|x|+r)^q}.\end{aligned}$$

Since $\frac{A_0(t,x)}{\sqrt{a(t)}}$ and $B_0(t,x)$ are bounded as $t \to 0^+$ and $\lim_{t\to 0^+} z(t) = x$, it follows, that

$$\lim_{t\to 0^+}\frac{|\widetilde{G}(t,x,z(t)) - \widetilde{G}(t,x,x)|}{\sqrt{a(t)}} = 0.$$

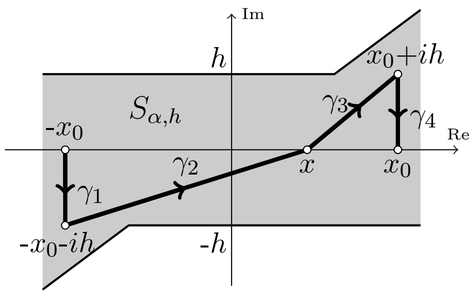
With (4.14) and the decomposition (4.8), we then obtain the limit (4.18), namely

$$\lim_{t\to 0^+}\frac{\widetilde{G}(t,x,z(t))}{\sqrt{a(t)}} = \lim_{t\to 0^+}\frac{\widetilde{G}(t,x,x)}{\sqrt{a(t)}} = \lim_{t\to 0^+}\frac{G(t,x,x)}{\sqrt{a(t)}} = \frac{1}{\sqrt{i\pi}}.$$

For the actual proof of the initial condition (4.7), we choose any $x_0 > |x|$ and use the Cauchy theorem to change the integration path $[-x_0, x_0]$ to

$$\begin{aligned}\int_{-x_0}^{x_0} G(t,x,y)F(y)dy = &\int_{\gamma_1} G(t,x,z)F(z)dz + \int_{\gamma_2} G(t,x,z)F(z)dz \\ &+ \int_{\gamma_3} G(t,x,z)F(z)dz + \int_{\gamma_4} G(t,x,z)F(z)dz,\end{aligned}$$

$$\begin{aligned}\gamma_1 &= \{\, -x_0 - is \mid s \in [0,h] \,\}, \\ \gamma_2 &= \{\, -x_0 - ih + s(x + x_0 + ih) \mid s \in [0,1] \,\}, \\ \gamma_3 &= \{\, x + s(x_0 - x + ih) \mid s \in [0,1] \,\}, \\ \gamma_4 &= \{\, x_0 + i(h-s) \mid s \in [0,h] \,\}.\end{aligned}$$



Since F is holomorphic, it is in particular bounded on $[-x_0, x_0] + i[-h, h]$, i.e.

$$|F(z)| \leq A_F, \quad z \in [-x_0, x_0] + i[-h, h].$$

Using this, together with the estimate (4.16a), the integral along γ_4 can be estimated by

$$\begin{aligned}
\Big| \int_{\gamma_4} G(t,x,z)F(z)dz \Big| &= \Big| \int_0^h G(t,x,x_0 + i(h-s))F(x_0 + i(h-s))ds \Big| \\
&\leq A_F A_0(t,x) \int_0^h e^{-2a(t)(x_0-x)(h-s)} e^{B_0(t,x)|x_0+i(h-s)|^q} ds \\
&\leq A_F A_0(t,x) e^{B_0(t,x)|x_0+ih|^q} \int_0^h e^{-2a(t)(x_0-x)s} ds \\
&\leq \frac{A_F A_0(t,x)}{2a(t)(x_0-x)} e^{B_0(t,x)|x_0+ih|^q}.
\end{aligned}$$

Since $\frac{A_0(\,\cdot\,,x)}{\sqrt{a(\,\cdot\,)}}$ and $B_0(\,\cdot\,,x)$ are bounded as $t \to 0^+$ and $a(t) \overset{t\to 0^+}{\longrightarrow} \infty$, this inequality proves the convergence

$$\lim_{t\to 0^+} \int_{\gamma_4} G(t,x,z)F(z)dz = 0. \tag{4.19}$$

In the same way one proves that also

$$\lim_{t\to 0^+} \int_{\gamma_1} G(t,x,z)F(z)dz = 0. \tag{4.20}$$

The integral along γ_3 can be written as

$$\begin{aligned}
&\int_{\gamma_3} G(t,x,z)F(z)dz \\
&\quad = (x_0 - x + ih) \int_0^1 G\big(t,x,x + s(x_0 - x + ih)\big) F\big(x + s(x_0 - x + ih)\big) ds \\
&\quad = \frac{x_0 - x + ih}{\sqrt{a(t)}} \int_0^{\sqrt{a(t)}} G\Big(t,x,x + \frac{s(x_0 - x + ih)}{\sqrt{a(t)}}\Big) F\Big(x + \frac{s(x_0 - x + ih)}{\sqrt{a(t)}}\Big) ds.
\end{aligned}$$

Since the integrand is bounded as

$$\begin{aligned}
&\Big| \frac{1}{\sqrt{a(t)}} G\Big(t,x,x + \frac{s(x_0 - x + ih)}{\sqrt{a(t)}}\Big) F\Big(x + \frac{s(x_0 - x + ih)}{\sqrt{a(t)}}\Big) \Big| \\
&\qquad \leq \frac{A_F A_0(t,x)}{\sqrt{a(t)}} e^{-2h(x_0-x)s^2} e^{B_0(t,x)\left|x + \frac{s(x_0-x+ih)}{\sqrt{a(t)}}\right|^q} \\
&\qquad \leq \frac{A_F A_0(t,x)}{\sqrt{a(t)}} e^{-2h(x_0-x)s^2} e^{B_0(t,x)(|x| + s|x_0-x+ih|)^q},
\end{aligned}$$

where we chose $t > 0$ small enough, such that $a(t) \geq 1$. Since $\frac{A_0(t,x)}{\sqrt{a(t)}}$ and $B_0(t,x)$ are bounded as $t \to 0^+$, this upper bound can be made t independent and integrable. Hence we are allowed to apply the dominated convergence theorem and carry the limit inside the integral

$$\lim_{t\to 0^+}\int_{\gamma_3} G(t,x,z)F(z)dz = (x_0-x+ih)\int_0^\infty \lim_{t\to 0^+}\frac{G\Big(t,x,x+\frac{s(x_0-x+ih)}{\sqrt{a(t)}}\Big)}{\sqrt{a(t)}}F(x)ds.$$

$$= \frac{F(x)(x_0-x+ih)}{\sqrt{i\pi}}\int_0^\infty e^{is^2(x_0-x+ih)^2}ds = \frac{F(x)}{2}. \qquad (4.21)$$

where in the first line we used the continuity of the function F and in the second line the convergence (4.18) from the first part of the proof. In the same way we also get

$$\lim_{t\to 0^+}\int_{\gamma_2} G(t,x,z)F(z)dz = \frac{F(x)}{2}. \qquad (4.22)$$

Combining now the limits (4.19), (4.20), (4.21) and (4.22), we conclude the initial value (4.7) and consequently proved that G indeed satisfies the Assumption 4.1. □

The following Theorem 4.6 is the main result of this chapter. It will be shown that under Assumption 4.1 or Assumption 4.4 the integral (4.4) is meaningful as a Fresnel type integral and that the resulting function Ψ is a solution of the time dependent Schrödinger equation (4.3). We also show that the solution continuously depends on the initial condition. Note, that the assumed $\mathcal{A}_p$-convergence on the initial conditions is stronger than the resulting uniform convergence on compact sets (4.25) of the solutions. However, this $\mathcal{A}_p$-convergence is justified, since $\mathcal{A}_1(\mathbb{C})$ is the natural space in which superoscillations are normally treated, see Definition 2.1 and $\mathcal{A}_q(\mathbb{C})$ is also a suitable space for the applications on the supershift, see Theorem 6.5 and Theorem 6.7.

Theorem 4.6. Let $G:(0,T)\times\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ be as in Assumption 4.1 or Assumption 4.4. Then for every $F\in\mathcal{A}_p(\mathbb{C})$, $p\in(0,2)$, the wave function

$$\Psi(t,x) := \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} e^{-\varepsilon y^2}G(t,x,y)F(y)dy, \quad t\in(0,T),\ x\in\mathbb{R}, \qquad (4.23)$$

exists and $\Psi\in \mathrm{AC}_{1,2}((0,T)\times\mathbb{R})$ is a solution of the Cauchy problem

$$i\frac{\partial}{\partial t}\Psi(t,x) = \Big(-\frac{\partial^2}{\partial x^2}+V(t,x)\Big)\Psi(t,x), \qquad \text{f.a.e. } t\in(0,T),\ x\in\mathbb{R}, \qquad (4.24a)$$

$$\lim_{t\to 0^+}\Psi(t,x) = F(x), \qquad x\in\mathbb{R}. \qquad (4.24b)$$

Moreover, for any sequence of initial conditions $(F_n)_n\subseteq\mathcal{A}_p(\mathbb{C})$ which converge as $F_n \overset{n\to\infty}{\longrightarrow} F$ in $\mathcal{A}_p(\mathbb{C})$, also the corresponding solutions (4.23) converge as

$$\lim_{n\to\infty}\Psi(t,x;F_n) = \Psi(t,x;F), \qquad (4.25)$$

for fixed $t\in(0,T)$ and uniformly on compact subsets of $\mathbb{R}$.

Note, that for convenience we used the notation $\Psi(t,x;F)$ to emphasize the initial condition.

Remark 4.7. If one replaces the growth condition (4.9a) in Assumption 4.1 by the stronger condition

$$|\widetilde{G}(t,x,z)| \leq A_0(t,x)e^{B_0(t,x)|\operatorname{Im}(z)|^q}, \quad t \in (0,T),\, x \in \mathbb{R},\, z \in S_\alpha, \tag{4.26}$$

for some $q \in (0,2)$, and also choose some initial condition $F \in \mathcal{A}_p(\mathbb{C})$, $p \in (0,2)$ which is bounded as

$$|F(z)| \leq Ae^{B|\operatorname{Im}(z)|^p}, \quad z \in S_\alpha, \tag{4.27}$$

then it follows from Corollary 3.2 (ii), that the wave function (4.23) can be written in the equivalent form

$$\Psi(t,x) = \lim_{R_1,R_2\to\infty} \int_{-R_1}^{R_2} G(t,x,y)F(y)dy, \quad t \in (0,T),\, x \in \mathbb{R}.$$

We point out that the stronger growth condition (4.26) on the Green's function is for $q = 1$ satisfied in all the examples of Chapter 7. However, the growth condition (4.27) for the initial condition F is rather restrictive and it is desirable to allow also initial conditions that may be unbounded on the real line. See for example the type of initial conditions which arise naturally for the supershift property in Theorem 6.5.

Proof of Theorem 4.6. Since Assumption 4.4 is stronger than Assumption 4.1 due to Lemma 4.5, it is sufficient to prove the theorem for Green's functions satisfying Assumption 4.1. First we note, that due to $F \in \mathcal{A}_p(\mathbb{C})$, there exists $A, B \geq 0$ such that

$$|F(z)| \leq Ae^{B|z|^p}, \quad z \in S_\alpha. \tag{4.28}$$

Step 1. In the first step we want to apply Corollary 3.2 (i), to show, that the expression (4.23) for the wave function is meaningful and give a representation using Fresnel type integrals. For this, we fix $t \in (0,T)$, $x \in \mathbb{R}$ and use the estimates (4.9a) and (4.28) to get

$$\begin{aligned} |\widetilde{G}(t,x,z)F(z)| &\leq AA_0(t,x)e^{B_0(t,x)|z|^q+B|z|^p} \\ &\leq AA_0(t,x)e^{(B+B_0(t,x))(1+|z|)^{\max\{p,q\}}} \\ &\leq AA_0(t,x)e^{(B+B_0(t,x))2^{\max\{p,q\}}(1+|z|^{\max\{p,q\}})} \\ &= \widetilde{A}_0(t,x)e^{\widetilde{B}_0(t,x)|z|^{\widetilde{p}}}, \quad z \in S_\alpha, \end{aligned} \tag{4.29}$$

where we introduced the new coefficients

$$\widetilde{p} \coloneqq \max\{p,q\}, \quad \widetilde{B}_0(t,x) \coloneqq (B+B_0(t,x))2^{\tilde{p}}, \quad \widetilde{A}_0(t,x) \coloneqq AA_0(t,x)e^{\widetilde{B}_0(t,x)}, \tag{4.30}$$

Hence, due to the decomposition (4.8), the assumptions of Corollary 3.2 are satisfied, which means, that the wave function (4.23) exists and admits the absolute integrable representation

$$\begin{aligned} \Psi(t,x) &= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} e^{-\varepsilon y^2} e^{ia(t)(y-x)^2}\widetilde{G}(t,x,y)F(y)dy \\ &= e^{i\alpha}\int_{\mathbb{R}} e^{ia(t)(ye^{i\alpha}-x)^2}\widetilde{G}(t,x,ye^{i\alpha})F(ye^{i\alpha})dy \\ &= e^{i\alpha}\int_{\mathbb{R}} G(t,x,ye^{i\alpha})F(ye^{i\alpha})dy. \end{aligned} \tag{4.31}$$

Step 2. We show that the function Ψ in (4.23), is a solution of the Schrödinger equation (4.24a). Roughly speaking, since G is already a solution of (4.6) by Assumption 4.1 (i), it is sufficient to carry the derivatives inside the integral (4.31).

For the time derivative we note, that $G(\,\cdot\,,x,z) \in \mathrm{AC}((0,T))$ for every $x \in \mathbb{R}$, $z \in S_\alpha$ by Assumption 4.1 (i). Hence, for any $t_0 \in (0,T)$ we have

$$G(t,x,z) = G(t_0,x,z) + \int_{t_0}^{t} \frac{\partial}{\partial \tau} G(\tau,x,z)d\tau, \quad t \in (0,T),\ x \in \mathbb{R},\ z \in S_\alpha,$$

which leads to the following integral representation of the wave function (4.31)

$$\Psi(t,x) = \Psi(t_0,x) + e^{i\alpha} \int_{\mathbb{R}} \int_{t_0}^{t} \frac{\partial}{\partial \tau} G(\tau,x,ye^{i\alpha})d\tau F(ye^{i\alpha})dy. \tag{4.32}$$

Using the decomposition (4.8), we can write the derivative as

$$\frac{\partial}{\partial \tau} G(\tau,x,ye^{i\alpha}) = \Big(ia'(\tau)(ye^{i\alpha}-x)^2 \widetilde{G}(\tau,x,ye^{i\alpha}) + \frac{\partial}{\partial \tau}\widetilde{G}(\tau,x,ye^{i\alpha})\Big) e^{ia(\tau)(ye^{i\alpha}-x)^2}.$$

Using the estimate (4.29) and a similar estimate for $\frac{\partial}{\partial \tau}\widetilde{G}(t,x,z)F(z)$ with the coefficients

$$\widetilde{B}_1(t,x) := (B + B_1(t,x))2^{\tilde{p}} \quad \text{and} \quad \widetilde{A}_1(t,x) := AA_1(t,x)e^{\widetilde{B}_1(t,x)},$$

we get

$$\begin{aligned}
&\Big|\frac{\partial}{\partial \tau} G(\tau,x,ye^{i\alpha})F(ye^{i\alpha})\Big| \\
&\quad = \Big|ia'(\tau)(ye^{i\alpha}-x)^2 \widetilde{G}(\tau,x,ye^{i\alpha}) + \frac{\partial}{\partial \tau}\widetilde{G}(\tau,x,ye^{i\alpha})\Big| \Big|e^{ia(\tau)(ye^{i\alpha}-x)^2} F(ye^{i\alpha})\Big| \\
&\quad \le \Big(|a'(\tau)||ye^{i\alpha}-x|^2 \widetilde{A}_0(\tau,x) + \widetilde{A}_1(\tau,x)\Big) e^{-a(\tau)\sin(2\alpha)y^2} e^{\widetilde{B}_0(\tau,x)|y|^{\tilde{p}} + 2a(\tau)\sin(\alpha)|xy|}.
\end{aligned}$$

Since $\widetilde{A}_0$, $\widetilde{A}_1$, $\widetilde{B}_0$, $\widetilde{B}_1$ and a are assumed to be continuous and $a' \in L^1_{\mathrm{loc}}((0,T))$ due to the absolute continuity of a, the right hand side of this estimate is integrable on $[t_0,t]$. Additionally, the factor $e^{-a(\tau)\sin(2\alpha)y^2}$ implies integrability with respect to $y \in \mathbb{R}$. Hence we observe absolute integrability on $[t_0,t] \times \mathbb{R}$ and are allowed to interchange the order of integration in (4.32) by the Fubini theorem. I.e., we obtain

$$\Psi(t,x) = \Psi(t_0,x) + e^{i\alpha} \int_{t_0}^{t} \int_{\mathbb{R}} \frac{\partial}{\partial \tau} G(\tau,x,ye^{i\alpha})F(ye^{i\alpha})dyd\tau.$$

In particular, this shows $\Psi(\,\cdot\,,x) \in \mathrm{AC}((0,T))$, the t-derivative exists almost everywhere and is given by

$$\frac{\partial}{\partial t}\Psi(t,x) = e^{i\alpha} \int_{\mathbb{R}} \frac{\partial}{\partial t} G(t,x,ye^{i\alpha})F(ye^{i\alpha})dy.$$

Using the same argument, also $\Psi(t,\,\cdot\,)$ and $\frac{\partial}{\partial x}\Psi(t,\,\cdot\,)$ are absolutely continuous on $\mathbb{R}$, with spatial derivatives almost everywhere given by

$$\frac{\partial}{\partial x}\Psi(t,x) = e^{i\alpha} \int_{\mathbb{R}} \frac{\partial}{\partial x} G(t,x,ye^{i\alpha})F(ye^{i\alpha})dy, \tag{4.33}$$

$$\frac{\partial^2}{\partial x^2}\Psi(t,x) = e^{i\alpha} \int_{\mathbb{R}} \frac{\partial^2}{\partial x^2} G(t,x,ye^{i\alpha})F(ye^{i\alpha})dy.$$

This means $\Psi \in \mathrm{AC}_{1,2}((0,T)\times\mathbb{R})$ and from (4.6) we conclude, that the Schrödinger equation (4.24a) is satisfied for almost every $t\in(0,T)$, $x\in\mathbb{R}$.

In *Step 3* we verify the initial condition (4.24b). To do so, we fix $x\in\mathbb{R}$, let $x_0>|x|$ be from Assumption 4.1 (ii) and split up the integral (4.23) as

$$\Psi(t,x) = \underbrace{\lim_{\varepsilon\to 0^+}\int_{-\infty}^{-x_0} e^{-\varepsilon y^2} G(t,x,y)F(y)dy}_{=:\Psi_1(t,x)} + \underbrace{\lim_{\varepsilon\to 0^+}\int_{-x_0}^{x_0} e^{-\varepsilon y^2} G(t,x,y)F(y)dy}_{=:\Psi_2(t,x)}$$
$$+ \underbrace{\lim_{\varepsilon\to 0^+}\int_{x_0}^{\infty} e^{-\varepsilon y^2} G(t,x,y)F(y)dy}_{=:\Psi_3(t,x)}.$$

We will now investigate all three integrals separately. Starting with Ψ_3, we can, similar as for (4.31), use Proposition 3.1 for the shifted integrand $z\mapsto G(t,x,x_0+z)F(x_0+z)$ to write the integral as

$$\begin{aligned}\Psi_3(t,x) &= \lim_{\varepsilon\to 0^+}\int_0^{\infty} e^{-\varepsilon(x_0+y)^2} G(t,x,x_0+y)F(x_0+y)dy\\ &= e^{i\alpha}\int_0^{\infty} G(t,x,x_0+ye^{i\alpha})F(x_0+ye^{i\alpha})dy.\end{aligned}$$

Using the inequality (4.29), we can now estimate this integral as

$$\begin{aligned}|\Psi_3(t,x)| &\le \widetilde{A}_0(t,x)\int_0^{\infty} e^{-a(t)\sin(2\alpha)y^2-2a(t)\sin(\alpha)(x_0-x)y+\widetilde{B}_0(t,x)|x_0+ye^{i\alpha}|^{\widetilde{p}}}dy\\ &\le \widetilde{A}_0(t,x)\int_0^{\infty} e^{-a(t)\sin(2\alpha)y^2-2a(t)\sin(\alpha)(x_0-x)y+\widetilde{B}_0(t,x)(x_0+y)^{\widetilde{p}}}dy\\ &= \frac{\widetilde{A}_0(t,x)}{\sqrt{a(t)}}\int_0^{\infty} e^{-\sin(2\alpha)y^2-2\sqrt{a(t)}\,\sin(\alpha)(x_0-x)y+\widetilde{B}_0(t,x)\left(x_0+\frac{y}{\sqrt{a(t)}}\right)^{\widetilde{p}}}dy.\end{aligned}$$

According to (4.10) and (4.30) we know that $\frac{\widetilde{A}_0}{\sqrt{a}}$ and $\widetilde{B}_0$ remain finite in the limit $t\to 0^+$, and also that $\lim_{t\to 0^+} a(t)=\infty$. Therefore, since $x_0>x$, the integrand vanishes in the limit $t\to 0^+$ and so does the whole function

$$\lim_{t\to 0^+}\Psi_3(t,x)=0. \tag{4.34}$$

In the same way we also get

$$\lim_{t\to 0^+}\Psi_1(t,x)=0. \tag{4.35}$$

For the function $\Psi_2(t,x)$ we first note, that due to the dominated convergence theorem we are allowed to carry the limit $\varepsilon\to 0^+$ inside the integral and get

$$\Psi_2(t,x)=\int_{-x_0}^{x_0} G(t,x,y)F(y)dy.$$

Since $F\in\mathcal{A}_p(\mathbb{C})$ is an entire function, the initial value

$$\lim_{t\to 0^+}\Psi_2(t,x)=F(x) \tag{4.36}$$

follows from assumption (4.7). Combining now (4.34), (4.35) and (4.36) gives the initial value (4.24b) of $\Psi(t,x)$ and hence finishes Step 3 of the proof.

Step 4. It remains to check the continuous dependency (4.25) of the wave function on the initial condition. According to the $\mathcal{A}_p$-convergence (1.2) of the initial conditions, we define the coefficients

$$A_n \coloneqq \sup_{z\in\mathbb{C}} |F(z)-F_n(z)|e^{-C|z|^p}, \quad n\in\mathbb{N},$$

for which obviously

$$\lim_{n\to\infty} A_n = 0 \quad \text{and} \quad |F(z)-F_n(z)| \le A_n e^{C|z|^p}, \quad z\in\mathbb{C}, \tag{4.37}$$

holds true. Let $x_0>0$ be arbitrary and for every $t\in(0,T)$, $x\in[-x_0,x_0]$ we split up the Ψ-integral (4.23) into

$$\Psi(t,x;F) = \Psi_1(t,x;F)+\Psi_2(t,x;F)+\Psi_3(t,x;F),$$

with

$$\begin{aligned}
\Psi_1(t,x;F) &\coloneqq \lim_{\varepsilon\to 0^+}\int_{-\infty}^{-x_0} e^{-\varepsilon y^2}G(t,x,y)F(y)dy,\\
\Psi_2(t,x;F) &\coloneqq \lim_{\varepsilon\to 0^+}\int_{-x_0}^{x_0} e^{-\varepsilon y^2}G(t,x,y)F(y)dy,\\
\Psi_3(t,x;F) &\coloneqq \lim_{\varepsilon\to 0^+}\int_{x_0}^{\infty} e^{-\varepsilon y^2}G(t,x,y)F(y)dy.
\end{aligned}$$

We will now prove the convergence (4.25) for the three parts of the wave function separately. Starting with Ψ_3, we use Proposition 3.1 with the shifted integrand $z\mapsto G(t,x,x_0+z)F(x_0+z)$, to write the integral as

$$\Psi_3(t,x;F) = e^{i\alpha}\int_0^\infty G(t,x,x_0+ye^{i\alpha})F(x_0+ye^{i\alpha})dy,$$

and estimate the difference by

$$\begin{aligned}
&|\Psi_3(t,x;F)-\Psi_3(t,x;F_n)|\\
&\quad = \left|e^{i\alpha}\int_0^\infty G(t,x,x_0+ye^{i\alpha})\big(F(x_0+ye^{i\alpha})-F_n(x_0+ye^{i\alpha})\big)dy\right|\\
&\quad \le A_nA_0(t,x)\int_0^\infty e^{-a(t)\sin(2\alpha)y^2-2a(t)(x_0-x)y\sin(\alpha)+B_0(t,x)|x_0+ye^{i\alpha}|^q+C|x_0+ye^{i\alpha}|^p}dy\\
&\quad \le \frac{A_nA_0(t,x)}{\sqrt{a(t)}}\int_0^\infty e^{-\sin(2\alpha)y^2-2\sqrt{a(t)}\,(x_0-x)y\sin(\alpha)+B_0(t,x)\left(x_0+\frac{y}{\sqrt{a(t)}}\right)^q+C\left(x_0+\frac{y}{\sqrt{a(t)}}\right)^p}dy,
\end{aligned}$$

Since A_0 and B_0 are continuous by Assumption 4.1 (iii), they are in particular bounded on the compact set $[-x_0,x_0]$ and therefore can be made x-independent. Also, the right hand side converges to zero as $A_n \overset{n\to\infty}{\longrightarrow} 0$ by (4.37), which proves that

$$\lim_{n\to\infty}\Psi_3(t,x;F_n) = \Psi_3(t,x;F)$$

for every fixed $t \in (0,T)$ and uniformly on $[-x_0, x_0]$. Following the same arguments, one obtains the same convergence for Ψ_1, namely

$$\lim_{n\to\infty} \Psi_1(t,x;F_n) = \Psi_1(t,x;F),$$

for every fixed $t \in (0,T)$ and uniformly on $[-x_0, x_0]$. Finally, the difference between the Ψ_2-functions can be estimated by

$$\begin{aligned}|\Psi_2(t,x;F) - \Psi_2(t,x;F_n)| &= \lim_{\varepsilon\to 0^+} \left| \int_{-x_0}^{x_0} e^{-\varepsilon y^2} G(t,x,y)(F(y) - F_n(y))dy \right| \\ &\leq A_n A_0(t,x) \int_{-x_0}^{x_0} e^{B_0(t,x)|y|^q + C|y|^p} dy \\ &\leq 2A_n A_0(t,x) x_0 e^{B_0(t,x)x_0^q + Cx_0^p}.\end{aligned}$$

Also here, since A_0 and B_0 are continuous by Assumption 4.1 (ii), they in particular are bounded on the compact set $[-x_0, x_0]$ and the right hand side can be made independent of x. Since also $A_n \stackrel{n\to\infty}{\longrightarrow} 0$ by (4.37), this proves that

$$\lim_{n\to\infty} \Psi_2(t,x;F_n) = \Psi_2(t,x;F)$$

for every fixed $t > 0$ and uniformly on $[x_0, x_0]$. Since $x_0 > 0$ was arbitrary, the uniform convergence holds on any compact subset $K \subseteq \mathbb{R}$, which verifies the convergence (4.25) and hence finishes the proof. □

5. Schrödinger equation on $\mathbb{R} \setminus \{0\}$

In comparison to the previous Chapter 4, where we considered the Schrödinger equation on the whole real line, we remove the point $x = 0$ now. This means, that we consider the Schrödinger equation on $\mathbb{R} \setminus \{0\}$, and allow boundary or transmission conditions at $x = 0$. The latter also implies distributional potentials as the Dirac δ- or δ'-potential, since they are mathematically implemented via transmission conditions at the point of interaction. Similar as in (4.2), it is convenient to view derivatives in the context of absolutely continuous functions. In particular we define

$$\mathrm{AC}(\dot{\mathbb{R}}) \coloneqq \Big\{ f : \dot{\mathbb{R}} \to \mathbb{C} \ \Big| \ f|_{(0,\infty)} \in \mathrm{AC}((0,\infty)) \text{ and } f|_{(-\infty,0)} \in \mathrm{AC}((-\infty,0) \Big\}$$

to work for $T \in (0, \infty]$ with the space

$$\mathrm{AC}_{1,2}((0,T) \times \dot{\mathbb{R}}) \coloneqq \left\{ \Psi : (0,T) \times \dot{\mathbb{R}} \to \mathbb{C} \ \middle| \ \begin{array}{l} \Psi(\,\cdot\,, x) \in \mathrm{AC}((0,T)),\ x \in \dot{\mathbb{R}} \\ \Psi(t,\,\cdot\,), \frac{\partial}{\partial x}\Psi(t,\,\cdot\,) \in \mathrm{AC}(\dot{\mathbb{R}}),\ t \in (0,T) \end{array} \right\}, \tag{5.1}$$

where for a shorter notation we introduced the notion $\dot{\mathbb{R}} \coloneqq \mathbb{R} \setminus \{0\}$.

Let $V : (0,T) \times \dot{\mathbb{R}} \to \mathbb{C}$ be some potential, $M, N \in \mathbb{C}^{2\times 2}$ matrices describing the transmission condition and $F : \dot{\mathbb{R}} \to \mathbb{C}$ some initial condition. We call a function $\Psi \in \mathrm{AC}_{1,2}((0,T) \times \dot{\mathbb{R}})$ a *solution* of the time dependent Schrödinger equation, if it satisfies

$$i\frac{\partial}{\partial t}\Psi(t,x) = \Big(-\frac{\partial^2}{\partial x^2} + V(t,x) \Big)\Psi(t,x), \qquad \text{f.a.e. } t \in (0,T),\ x \in \dot{\mathbb{R}}, \tag{5.2a}$$

$$M \begin{pmatrix} \Psi(t,0^+) \\ \Psi(t,0^-) \end{pmatrix} = N \begin{pmatrix} \frac{\partial}{\partial x}\Psi(t,0^+) \\ -\frac{\partial}{\partial x}\Psi(t,0^-) \end{pmatrix}, \qquad t \in (0,T), \tag{5.2b}$$

$$\lim_{t \to 0^+} \Psi(t,x) = F(x), \qquad x \in \dot{\mathbb{R}}. \tag{5.2c}$$

The corresponding *Green's function* is a function $G : (0,T) \times \dot{\mathbb{R}} \times \dot{\mathbb{R}} \to \mathbb{C}$, which depends on the potential V and the matrices M, N, but not on the initial condition F, such that the solution Ψ admits the (formal) representation

$$\Psi(t,x) = \int_{\mathbb{R}} G(t,x,y)F(y)dy, \quad t \in (0,T),\ x \in \dot{\mathbb{R}}. \tag{5.3}$$

In Assumption 4.1 and Assumption 4.4 we already introduced a set of properties for the Green's function on $\mathbb{R}$. The following adapted Assumption 5.1 and Assumption 5.2 for the Green's function on $\dot{\mathbb{R}}$ are similar, the main difference only lies in the additional transmission condition (5.5).

Assumption 5.1. Let $T \in (0, \infty]$ and $G : (0, T) \times \dot{\mathbb{R}} \times \dot{\mathbb{R}} \to \mathbb{C}$. For some $\alpha \in (0, \frac{\pi}{2})$ let S_α be the double sector (3.20), and suppose that G admits a continuation to a function $G : (0, T) \times \dot{\mathbb{R}} \times S_\alpha \to \mathbb{C}$, such that for every fixed $t \in (0, T)$, $x \in \dot{\mathbb{R}}$ the mapping $G(t, x, \cdot\,)$ is continuous on S_α and holomorphic on the interior of S_α. Moreover, it will be assumed that G satisfies the following properties (i)–(iii).

(i) For every fixed $z \in S_\alpha$, the function $G(\,\cdot\,, \cdot\,, z) \in \mathrm{AC}_{1,2}((0, T) \times \dot{\mathbb{R}})$ is a solution of the time dependent Schrödinger equation

$$i\frac{\partial}{\partial t}G(t, x, z) = \Big(- \frac{\partial^2}{\partial x^2} + V(t, x)\Big)G(t, x, z), \quad \text{f.a.e. } t \in (0, T),\ x \in \dot{\mathbb{R}}, \tag{5.4}$$

with $V : (0, T) \times \dot{\mathbb{R}} \to \mathbb{C}$ the considered potential. Moreover, for every $y \in \dot{\mathbb{R}}$ the Green's function satisfies the transmission condition

$$M \begin{pmatrix} G(t, 0^+, y) \\ G(t, 0^-, y) \end{pmatrix} = N \begin{pmatrix} \frac{\partial}{\partial x}G(t, 0^+, y) \\ -\frac{\partial}{\partial x}G(t, 0^-, y) \end{pmatrix}, \quad t \in (0, T), \tag{5.5}$$

with matrices $M, N \in \mathbb{C}^{2\times 2}$.

(ii) For every $x \in \dot{\mathbb{R}}$ there exists some $x_0 > |x|$, such that for every $F \in \mathcal{H}(\mathbb{C})$ we have the initial condition

$$\lim_{t\to 0^+} \int_{-x_0}^{x_0} G(t, x, y)F(y)dy = F(x). \tag{5.6}$$

(iii) There exists $a \in \mathrm{AC}((0, T))$ with $a(t) > 0$ and $\lim_{t\to 0^+} a(t) = \infty$, such that the function $\widetilde{G}$ in the decomposition

$$G(t, x, z) = e^{ia(t)(z-x)^2}\widetilde{G}(t, x, z), \quad t \in (0, T),\ x \in \dot{\mathbb{R}},\ z \in S_\alpha, \tag{5.7}$$

is for every $t \in (0, T)$, $x \in \dot{\mathbb{R}}$ exponentially bounded as

$$\big|\widetilde{G}(t, x, z)\big| \le A_0(t, x)e^{B_0(t,x)|z|^q}, \quad z \in S_\alpha, \tag{5.8a}$$

$$\Big|\frac{\partial}{\partial x}\widetilde{G}(t, x, z)\Big| \le A_1(t, x)e^{B_1(t,x)|z|^q}, \quad z \in S_\alpha, \tag{5.8b}$$

$$\Big|\frac{\partial^2}{\partial x^2}\widetilde{G}(t, x, z)\Big|, \Big|\frac{\partial}{\partial t}\widetilde{G}(t, x, z)\Big| \le A_2(t, x)e^{B_2(t,x)|z|^q}, \quad z \in S_\alpha. \tag{5.8c}$$

Here $q \in (0, 2)$ and $A_0, A_1, A_2, B_0, B_1, B_2 : (0, T) \times \dot{\mathbb{R}} \to [0, \infty)$ are continuous and for every $x \in \dot{\mathbb{R}}$

$$\frac{A_0(\,\cdot\,, x)}{\sqrt{a(t)}} \text{ and } B_0(\,\cdot\,, x) \text{ are bounded as } t \to 0^+, \tag{5.9}$$

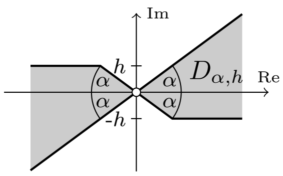
and for every $t \in (0, T)$:

- If $M = N = 0$, no further assumptions.
- If $M \neq 0$, $N = 0$, then $A_0(t, \cdot\,)$, $B_0(t, \cdot\,)$ are bounded as $x \to 0^\pm$.
- If $N \neq 0$, then $A_0(t, \cdot\,)$, $A_1(t, \cdot\,)$, $B_0(t, \cdot\,)$, $B_1(t, \cdot\,)$ are bounded as $x \to 0^\pm$.

Note, that the Remarks 4.2 & 4.3 are still valid for this Assumption 5.1. Moreover, see the Sections 7.5 & 7.6 for explicit examples of Green's functions which satisfy Assumption 5.1.

Similar as we replaced the initial condition (4.7) by the simple limit (4.14) in Assumption 4.4, we can also replace (5.6) by the upcoming limit (5.13) for the Schrödinger equation on $\dot{\mathbb{R}}$. However, in order to use this simplification it is necessary for the Green's function to be holomorphic (and satisfy (5.8a)) not only on S_α but also in a neighborhood of $\dot{\mathbb{R}}$. More precisely, for $\alpha \in (0, \frac{\pi}{2})$ and $h > 0$ on the domain

$$\begin{aligned} D_{\alpha,h} \coloneqq &\{ z \in \mathbb{C} \setminus \{0\} \mid \operatorname{Arg}(z) \in [-\alpha, \alpha] \text{ and } \operatorname{Im}(z) \geq -h \} \\ &\cup \{ z \in \mathbb{C} \setminus \{0\} \mid \operatorname{Arg}(z) \in [\pi - \alpha, \pi + \alpha] \text{ and } \operatorname{Im}(z) \leq h \} . \end{aligned} \tag{5.10}$$



Assumption 5.2. Let $T \in (0, \infty]$ and $G : (0,T) \times \dot{\mathbb{R}} \times \dot{\mathbb{R}} \to \mathbb{C}$. For some $\alpha \in (0, \frac{\pi}{2})$, $h > 0$ suppose that G admits a continuation to a function $G : (0,T) \times \dot{\mathbb{R}} \times D_{\alpha,h} \to \mathbb{C}$, such that for every fixed $t \in (0,T)$, $x \in \dot{\mathbb{R}}$ the mapping $G(t, x, \cdot\,)$ is continuous on $D_{\alpha,h}$ and holomorphic on the interior of $D_{\alpha,h}$. Moreover, it will be assumed that G satisfies the following properties (i)–(iii).

(i) For every fixed $z \in S_\alpha$, the function $G(\,\cdot\,, \cdot\,, z) \in \mathrm{AC}_{1,2}((0,T) \times \dot{\mathbb{R}})$ is a solution of the time dependent Schrödinger equation

$$i\frac{\partial}{\partial t} G(t, x, z) = \Big(- \frac{\partial^2}{\partial x^2} + V(t, x)\Big) G(t, x, z), \quad \text{f.a.e. } t \in (0,T),\ x \in \dot{\mathbb{R}}, \tag{5.11}$$

with $V : (0,T) \times \dot{\mathbb{R}} \to \mathbb{C}$ the considered potential. Moreover, for every $y \in \dot{\mathbb{R}}$ the Green's function satisfies the transmission condition

$$M \begin{pmatrix} G(t, 0^+, y) \\ G(t, 0^-, y) \end{pmatrix} = N \begin{pmatrix} \frac{\partial}{\partial x} G(t, 0^+, y) \\ -\frac{\partial}{\partial x} G(t, 0^-, y) \end{pmatrix}, \quad t \in (0,T), \tag{5.12}$$

with matrices $M, N \in \mathbb{C}^{2\times 2}$.

(ii) With the function $a(t)$ from (5.14), the Green's function admits the limit

$$\lim_{t\to 0^+} \frac{G(t, x, x)}{\sqrt{a(t)}} = \frac{1}{\sqrt{i\pi}}, \quad x \in \dot{\mathbb{R}}. \tag{5.13}$$

(iii) There exists $a \in \mathrm{AC}((0,T))$ with $a(t) > 0$ and $\lim_{t\to 0^+} a(t) = \infty$, such that the function $\widetilde{G}$ in the decomposition

$$G(t, x, z) = e^{ia(t)(z-x)^2} \widetilde{G}(t, x, z), \quad t \in (0,T),\ x \in \dot{\mathbb{R}},\ z \in D_{\alpha,h}, \tag{5.14}$$

is for every $t \in (0,T)$, $x \in \dot{\mathbb{R}}$ exponentially bounded as

$$\left|\widetilde{G}(t,x,z)\right| \le A_0(t,x)e^{B_0(t,x)|z|^q}, \qquad z \in D_{\alpha,h}, \tag{5.15a}$$

$$\left|\frac{\partial}{\partial x}\widetilde{G}(t,x,z)\right| \le A_1(t,x)e^{B_1(t,x)|z|^q}, \qquad z \in S_\alpha, \tag{5.15b}$$

$$\left|\frac{\partial^2}{\partial x^2}\widetilde{G}(t,x,z)\right|, \left|\frac{\partial}{\partial t}\widetilde{G}(t,x,z)\right| \le A_2(t,x)e^{B_2(t,x)|z|^q}, \qquad z \in S_\alpha. \tag{5.15c}$$

Here $q \in (0,2)$ and $A_0, A_1, A_2, B_0, B_1, B_2 : (0,T) \times \dot{\mathbb{R}} \to [0,\infty)$ are continuous and for every $x \in \dot{\mathbb{R}}$

$$\frac{A_0(\,\cdot\,,x)}{\sqrt{a(t)}} \text{ and } B_0(\,\cdot\,,x) \text{ are bounded as } t \to 0^+, \tag{5.16}$$

and for every $t \in (0,T)$:

- If $M = N = 0$, no further assumptions.
- If $M \neq 0$, $N = 0$, then $A_0(t,\,\cdot\,)$, $B_0(t,\,\cdot\,)$ are bounded as $x \to 0^\pm$.
- If $N \neq 0$, then $A_0(t,\,\cdot\,)$, $A_1(t,\,\cdot\,)$, $B_0(t,\,\cdot\,)$, $B_1(t,\,\cdot\,)$ are bounded as $x \to 0^\pm$.

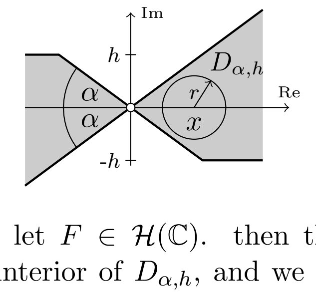
The following lemma proves that the Assumption 5.2 is indeed stronger than the Assumption 5.1.

Lemma 5.3. If $G : (0,T) \times \dot{\mathbb{R}} \times \dot{\mathbb{R}} \to \mathbb{C}$ satisfies Assumption 5.2, it also satisfies Assumption 5.1.

Proof. The only thing to check is the initial condition (5.6). Since the calculation is principally the same for $x < 0$, we only consider $x > 0$ here. First of all, we generalize (5.13) in the sense that for any $z(t) \in D_{\alpha,h}$ with $\lim_{t \to 0^+} z(t) = x$ we have

$$\lim_{t \to 0^+} \frac{\widetilde{G}(t,x,z(t))}{\sqrt{a(t)}} = \frac{1}{\sqrt{i\pi}}. \tag{5.17}$$

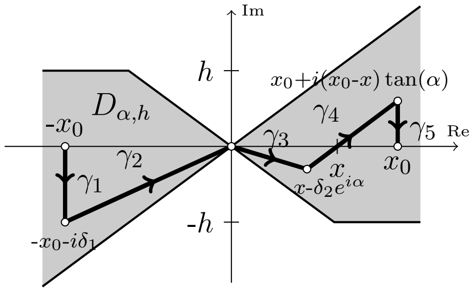
In order to prove this, we follow the same steps as in the proof of (4.18) with the only difference, that we consider a closed ball $B_r(x)$ with radius $0 < r < \min\{h, x\sin(\alpha)\}$ around x. This ball is then obviously included in the interior of $D_{\alpha,h}$.



Choose now any $x_0 > x$ and let $F \in \mathcal{H}(\mathbb{C})$. then there exist $\delta_1, \delta_2 > 0$ such that $-x_0 - i\delta, x - \delta_2 e^{i\alpha}$ lie in the interior of $D_{\alpha,h}$, and we can use the Cauchy theorem to change the integration path $[-x_0, x_0]$ in (5.6) to

$$\int_{-x_0}^{x_0} G(t,x,y)F(y)dy = \int_{\gamma_1} G(t,x,z)F(z)dz + \int_{\gamma_2} G(t,x,z)F(z)dz + \int_{\gamma_3} G(t,x,z)F(z)dz + \int_{\gamma_4} G(t,x,z)F(z)dz + \int_{\gamma_5} G(t,x,z)F(z)dz,$$

$$\gamma_1 = \{\, -x_0 - is \mid 0 \le s \le \delta_1 \,\},$$
$$\gamma_2 = \{\, s(x_0 + i\delta_1) \mid -1 \le s \le 0 \,\},$$
$$\gamma_3 = \Big\{\, s(x - \delta_2 e^{i\alpha} \;\Big|\; 0 \le s \le 1 \,\Big\},$$
$$\gamma_4 = \left\{\, x + se^{i\alpha} \;\middle|\; -\delta \le s \le \frac{x_0 - x}{\cos(\alpha)} \,\right\},$$
$$\gamma_5 = \{\, x_0 + is \mid (x_0 - x)\tan(\alpha) \ge s \ge 0 \,\}.$$



Since F is holomorphic, it is in particular bounded on $\{\, z \in D_{\alpha,h} \mid |\operatorname{Re}(z)| \le x_0 \,\}$, i.e.

$$|F(z)| \le A_F, \quad z \in D_{\alpha,h} \text{ with } |\operatorname{Re}(z)| \le x_0.$$

Using this, together with the estimate (5.15a), the integral along γ_1 can be estimated by

$$\begin{aligned}\Big|\int_{\gamma_1} G(t,x,z)F(z)dz\Big| &= \Big|\int_0^{\delta_1} G(t,x,-x_0-is)F(-x_0-is)ds\Big| \\ &\le A_F A_0(t,x)\int_0^{\delta_1} e^{-2a(t)(x_0+x)s} e^{B_0(t,x)|x_0+is|^q} ds \\ &\le A_F A_0(t,x) e^{B_0(t,x)|x_0+i\delta_1|^q}\int_0^{\delta_1} e^{-2a(t)(x_0+x)s} ds \\ &\le \frac{A_F A_0(t,x)}{2a(t)(x_0+x)} e^{B_0(t,x)|x_0+i\delta_1|^q}.\end{aligned}$$

Since $\frac{A_0(\,\cdot\,,x)}{\sqrt{a(\,\cdot\,)}}$ and $B_0(\,\cdot\,,x)$ are bounded as $t \to 0^+$ and $a(t) \overset{t\to 0^+}{\longrightarrow} \infty$, this inequality proves the convergence

$$\lim_{t\to 0^+}\int_{\gamma_1} G(t,x,z)F(z)dz = 0. \tag{5.18}$$

In the same way one proves that also

$$\lim_{t\to 0^+}\int_{\gamma_2} G(t,x,z)F(z)dz = \lim_{t\to 0^+}\int_{\gamma_3} G(t,x,z)F(z)dz = \lim_{t\to 0^+}\int_{\gamma_5} G(t,x,z)F(z)dz = 0. \tag{5.19}$$

The integral along γ_4 can be written as

$$\begin{aligned}\int_{\gamma_4} G(t,x,z)F(z)dz &= e^{i\alpha}\int_{-\delta_2}^{\frac{x_0-x}{\cos(\alpha)}} G(t,x,x+se^{i\alpha})F(x+se^{i\alpha})ds \\ &= \frac{e^{i\alpha}}{\sqrt{a(t)}}\int_{-\delta_2\sqrt{a(t)}}^{\frac{x_0-x}{\cos(\alpha)}\sqrt{a(t)}} G\Big(t,x,x+\frac{se^{i\alpha}}{\sqrt{a(t)}}\Big)F\Big(x+\frac{se^{i\alpha}}{\sqrt{a(t)}}\Big)ds.\end{aligned}$$

Since the integrand is bounded as

$$\begin{aligned}\Big|\frac{1}{\sqrt{a(t)}}G\Big(t,x,x+\frac{se^{i\alpha}}{\sqrt{a(t)}}\Big)F\Big(x+\frac{se^{i\alpha}}{\sqrt{a(t)}}\Big)\Big| &\le \frac{A_F A_0(t,x)}{\sqrt{a(t)}} e^{-s^2\sin(2\alpha)} e^{B_0(t,x)\big|x+\frac{se^{i\alpha}}{\sqrt{a(t)}}\big|^q} \\ &\le \frac{A_F A_0(t,x)}{\sqrt{a(t)}} e^{-s^2\sin(2\alpha)} e^{B_0(t,x)(x+|s|)^q},\end{aligned}$$

where in the second inequality we chose $t>0$ small enough, such that $a(t)\geq 1$. Since $\frac{A_0(t,x)}{\sqrt{a(t)}}$ and $B_0(t,x)$ are bounded as $t\to 0^+$, the right hand side can be made t-independent. Hence we are allowed to apply the dominated convergence theorem and carry the limit inside the integral

$$\begin{aligned}\lim_{t\to 0^+}\int_{\gamma_4} G(t,x,z)F(z)dz &= e^{i\alpha}\int_{-\infty}^{\infty} e^{is^2e^{2i\alpha}}\lim_{t\to 0^+}\frac{\widetilde{G}\Big(t,x,x+\frac{se^{i\alpha}}{\sqrt{a(t)}}\Big)}{\sqrt{a(t)}}F\Big(x+\frac{se^{i\alpha}}{\sqrt{a(t)}}\Big)ds\\ &= \frac{e^{i\alpha}F(x)}{\sqrt{i\pi}}\int_{-\infty}^{\infty} e^{is^2e^{2i\alpha}}ds = F(x),\end{aligned} \tag{5.20}$$

where in the second line we used the continuity of the function F as well as the convergence (5.17) from the first part of the proof. Combining now the limits (5.18), (5.19) and (5.20), we conclude the initial value (5.6) and consequently proved that G indeed satisfies the Assumption 5.1. □

In the following Threorem 5.4 a similar result as Theorem 4.6 is given for the Schrödinger equation on $\dot{\mathbb{R}}$ and hence for the Green's function satisfying Assumption 5.1. In particular, transmission conditions at $x=0^{\pm}$ are included.

Theorem 5.4. Let $G:(0,T)\times\dot{\mathbb{R}}\times\dot{\mathbb{R}}\to\mathbb{C}$ be as in Assumption 5.1 of Assumption 5.2. Then for every $F\in\mathcal{A}_p(\mathbb{C})$, $p\in(0,2)$, the wave function

$$\Psi(t,x)\coloneqq\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} e^{-\varepsilon y^2}G(t,x,y)F(y)dy,\quad t\in(0,T),\ x\in\dot{\mathbb{R}}, \tag{5.21}$$

exists and $\Psi\in\mathrm{AC}_{1,2}((0,T)\times\dot{\mathbb{R}})$ is a solution of the Cauchy problem

$$i\frac{\partial}{\partial t}\Psi(t,x)=\Big(-\frac{\partial^2}{\partial x^2}+V(t,x)\Big)\Psi(t,x),\qquad \text{f.a.e. } t\in(0,T),\ x\in\dot{\mathbb{R}}, \tag{5.22a}$$

$$M\begin{pmatrix}\Psi(t,0^+)\\ \Psi(t,0^-)\end{pmatrix}=N\begin{pmatrix}\frac{\partial}{\partial x}\Psi(t,0^+)\\ -\frac{\partial}{\partial x}\Psi(t,0^-)\end{pmatrix},\qquad t\in(0,T), \tag{5.22b}$$

$$\lim_{t\to 0^+}\Psi(t,x)=F(x),\qquad x\in\dot{\mathbb{R}}. \tag{5.22c}$$

Moreover, for any sequence of initial conditions $(F_n)_n\subseteq\mathcal{A}_p(\mathbb{C})$ which converge as $F_n\overset{n\to\infty}{\longrightarrow}F$ in $\mathcal{A}_p(\mathbb{C})$, also the corresponding solutions (5.21) converge as

$$\lim_{n\to\infty}\Psi(t,x;F_n)=\Psi(t,x;F), \tag{5.23}$$

for fixed $t\in(0,T)$ and uniformly on compact subsets of $\dot{\mathbb{R}}$.

Note, that for convenience we used the notation $\Psi(t,x;F)$ to emphasize the initial condition.

Remark 5.5. If we replace the growth condition (5.8a) in Assumption 5.1 by the stronger condition

$$|\widetilde{G}(t,x,z)|\leq A_0(t,x)e^{B_0(t,x)|\operatorname{Im}(z)|^q},\quad t\in(0,T),\ x\in\dot{\mathbb{R}},\ z\in S_\alpha, \tag{5.24}$$

for some $q \in (0,2)$, and also choose some initial condition $F \in \mathcal{A}_p(\mathbb{C})$, $p \in (0,2)$, which is bounded as

$$|F(z)| \leq Ae^{B|\operatorname{Im}(z)|^p}, \quad z \in S_\alpha, \tag{5.25}$$

then it follows from Corollary 3.2 (ii), that the wave function (5.21) can be written in the equivalent form

$$\Psi(t,x) = \lim_{R_1,R_2\to\infty} \int_{-R_1}^{R_2} G(t,x,y)F(y)dy, \quad t \in (0,T),\, x \in \dot{\mathbb{R}}.$$

We point out that the stronger growth condition (5.24) on the Green's function is for $q = 1$ satisfied in all examples in Section 7.5 & 7.6. However, the growth condition (5.25) for the initial condition F is rather restrictive and it is desirable to allow also initial conditions that may be unbounded on the real line. See for example the type of initial conditions which arise naturally for the supershift property in Theorem 6.5.

Proof of Theorem 5.4. The proof of the existence of the wave function (5.21) and the fact that $\Psi \in \mathrm{AC}_{1,2}((0,T) \times \dot{\mathbb{R}})$ is a solution of (5.22a) and (5.22c) is the same as in the proof of Theorem 4.6. Also the continuous dependency result (5.23) can be proven in the same way.

It remains to verify the transmission condition (5.22b). If $M = N = 0$, the transmission condition (5.22b) is trivially satisfied. In the case $M \neq 0$ and $N = 0$, we can estimate the integrand of the integral (4.31) as

$$\big|G(t,x,ye^{i\alpha})F(ye^{i\alpha})\big| \leq AA_0(t,x)e^{-a(t)\sin(2\alpha)y^2+2a(t)\sin(\alpha)|xy|+B_0(t,x)|y|^p+B|y|^q},$$

using the decomposition (5.7) as well as the estimate (5.8a). Since by Assumption 5.1 (iii) the coefficients $A_0(t,\,\cdot\,)$ and $B_0(t,\,\cdot\,)$ are bounded in the limit $x \to 0^\pm$, the right hand side can be replaced by some integrable and x-independent majorant, at least in a neighborhood of $x = 0$. Hence we can apply the dominated convergence theorem to (4.31), which gives the boundary value

$$\Psi(t,0^\pm) = e^{i\alpha} \int_{\mathbb{R}} G(t,0^\pm,ye^{i\alpha})F(ye^{i\alpha})dy, \quad t \in (0,T). \tag{5.26}$$

Since moreover the estimate (5.8a) in the limit $x \to 0^\pm$ shows that

$$|\widetilde{G}(t,0^\pm,z)| \leq A_0(t)e^{B_0(t)|z|^p}, \quad t \in (0,T),\, z \in S_\alpha,$$

where $A_0(t)$ and $B_0(t)$ are upper bounds of $A_0(t,\,\cdot\,)$ and $B_0(t,\,\cdot\,)$ in the limit $x \to 0^\pm$, we can apply Corollary 3.2 to rewrite the integral (5.26) into the form

$$\Psi(t,0^\pm) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} e^{-\varepsilon y^2} G(t,0^\pm,y)F(y)dy, \quad t \in (0,T). \tag{5.27}$$

Since the Green's function satisfies the transmission condition (5.5) with $N = 0$, the same equation carries over to the wave function Ψ and we end up with the stated (5.22b).

In the situation $N \neq 0$, also the coefficients $A_1(t,\,\cdot\,)$ and $B_1(t,\,\cdot\,)$ are bounded as $x \to 0^\pm$ by Assumption 5.1 (iii). In the same way as we derived (4.33) in the proof of Theorem 4.6, we also get the Fresnel type integral representation

$$\frac{\partial}{\partial x}\Psi(t,x) = e^{i\alpha} \int_{\mathbb{R}} \frac{\partial}{\partial x} G(t,x,ye^{i\alpha})F(ye^{i\alpha})dy, \quad t \in (0,T),\, x \in \dot{\mathbb{R}}, \tag{5.28}$$

of the spatial derivative. Using the decomposition (5.7) as well as the estimate (5.8a) and (5.8b) to estimate the integrand as

$$\begin{aligned}
&\Big|\frac{\partial}{\partial x} G(t,x,ye^{i\alpha})F(ye^{i\alpha})\Big| \\
&\qquad = \Big|2ia(t)(x-ye^{i\alpha})\widetilde{G}(t,x,ye^{i\alpha}) + \frac{\partial}{\partial x}\widetilde{G}(t,x,ye^{i\alpha})\Big|\Big|e^{ia(t)(ye^{i\alpha}-x)^2}F(ye^{i\alpha})\Big| \\
&\qquad \le A\Big(2A_0(t,x)a(t)|x-ye^{i\alpha}|e^{B_0(t,x)|y|^p} \\
&\qquad\qquad + A_1(t,x)e^{B_1(t,x)|y|^p}\Big)e^{B|y|^q}e^{2a(t)\sin(\alpha)|xy|}e^{-a(t)\sin(2\alpha)y^2}.
\end{aligned}$$

Since, by assumption the coefficients A_0, A_1, B_0, B_1 are bounded as $x \to 0^{\pm}$, the right hand side can be replaced by some integrable and x-independent majorant, at least in a neighborhood of $x = 0$. Hence we can apply the dominated convergence theorem to (5.28), which gives the boundary value

$$\frac{\partial}{\partial x}\Psi(t,0^{\pm}) = e^{i\alpha}\int_{\mathbb{R}} \frac{\partial}{\partial x}G(t,0^{\pm},ye^{i\alpha})F(ye^{i\alpha})dy, \quad t \in (0,T).$$

In the same way as above, Corollary 3.2 then transforms this integral back into the form

$$\frac{\partial}{\partial x}\Psi(t,0^{\pm}) = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} e^{-\varepsilon y^2}\frac{\partial}{\partial x}G(t,0^{\pm},y)F(y)dy, \quad t \in (0,T). \tag{5.29}$$

Since we already know by assumption that G satisfies the transmission condition (5.5), the integral representations (5.27) and (5.29) show, that it carries over to Ψ and gives (5.22b). □

6. Stability of superoscillations and supershifts

It is a question almost as old as superoscillations itself: What happens to a superoscillating function as it evolves in time, when interacting with some quantum mechanical system? Or in other words: Does a frequency shift of the initial condition at $t = 0$ survives the time evolution and leads to a similar shift for the solution at times $t > 0$? As one can imagine, this time persistence of superoscillations is of crucial importance in applications, since one has to guarantee that the superoscillating wave is not right away destroyed by some potential or perturbation. Since, as mentioned in the introduction, superoscillations always appear with an exponentially small amplitude, one could think of them being quickly destroyed by the exponentially large amplitudes of the outside regions. However, it was shown in [51], that at least in the potential free case, superoscillations survive for an unexpected amount of time. An experimental verification of this fact is for example done in [3, 27].

The particular case of the quantum mechanical evolution problem with respect to the one dimensional Schrödinger equation will be the main topic of this chapter. This means, for some potential $V(t,x)$ we consider the Cauchy problem

$$i\frac{\partial}{\partial t}\Psi(t,x) = \Big(-\frac{\partial^2}{\partial x^2} + V(t,x)\Big)\Psi(t,x), \tag{6.1a}$$

$$\Psi(0,x) = F(x), \tag{6.1b}$$

and ask, what happens to the wave function $\Psi(t,x)$ if we put some superoscillatory function F as its initial condition. Will the solution $\Psi(t,x)$ still be superoscillating at times $t > 0$? The first mathematical rigorous treatment of this question was the paper [15], where for the free particle the authors proved a superoscillatory behaviour for any time $t > 0$. Subsequent works then also consider nonconstant potentials as the harmonic oscillator in [22, 33, 35, 55, 56, 68], the electric field in [19, 22, 35, 55], the magnetic field in [22, 66], the centrifugal potential in [22, 33, 67, 68], the step potential in [20] and distributional potentials as δ and δ' in [5, 6, 36]. But also the time persistence with respect to other evolution equations as the Klein-Gordon equation in [21, 88] or the Dirac equation in [72, 88] were considered.

While all the previous works only consider specific potentials and investigate them one by one, the main novelty of the unified approach of this thesis is, that it is based only on certain regularity and growth conditions of the corresponding Green's function but avoids its explicit form, see Assumption 4.1 and Assumption 5.1. It will turn out that all the above mentioned potentials are covered by our unified approach, as verified in Chapter 7.

In order to motivate the structure of this chapter and in particular the time persistence result of Theorem 6.5, we start in Chapter 2 where we introduced two different types of superoscillating functions. The Type II superoscillations of Definition 2.2, which are characterized by their number of sign changes, will not be treated here. We will only consider the Type I superoscillations of Definition 2.1, where the convergence

$$\lim_{n\to\infty} F_n(z) = e^{ikz} \quad \text{in } \mathcal{A}_1(\mathbb{C}),$$

plays a central role. This limit perfectly harmonizes with the continuous dependency results of Theorem 4.6 and Theorem 5.4 and already indicates some convergence of the form

$$\lim_{n\to\infty} \Psi(t,x;F_n) = \Psi(t,x;e^{ik\,\cdot\,}),$$

and hence some superoscillatory property of the wave function. However, although the limit function $\Psi(t,x;e^{ik\,\cdot\,})$ may admit some oscillatory behaviour, it is by no means expected (and also not true) that it is again a plane wave $e^{ik(t)x}$ with some frequency $k(t)$. Hence, the sequence $\Psi(t,x;F_n)$ is no longer superoscillating in the sense that it converges to some plane wave with high frequency. A second issue is, that the desired $\mathcal{A}_1$-convergence in the variable x may in general fail since some arbitrary potential, having for example discontinuities, leads to a wave function which is no longer holomorphic although the initial value was. To overcome this dilemma, the notion of *supershift* was introduced in [68]. The upcoming definition of supershift mainly means, that we forget about the oscillatory behaviour of the plane waves e^{ikz} and replace them by some arbitrary functions $\varphi_k(z)$. The only property we stick to is the convergence (6.4). Whether the functions φ_k admit any kind of oscillatory behaviour has to be investigated independently and is not part of the general theory in this chapter.

Definition 6.1 (Supershift)**.** Let $\mathcal{O},\mathcal{U} \subseteq \mathbb{C}$ with $\mathcal{U} \subsetneq \mathcal{O}$ and X be a metric space. Consider a family

$$\varphi_k : X \to \mathbb{C}, \quad k \in \mathcal{O}, \tag{6.2}$$

of complex valued functions, such that for every $s \in X$ the mapping $k \mapsto \varphi_k(s)$ is bounded on $\mathcal{U}$. We say that a sequence of functions $(\Phi_n)_n$ of the form

$$\Phi_n(s) = \int_{\mathcal{U}} \varphi_\kappa(s) d\mu_n(\kappa), \quad s \in X, \tag{6.3}$$

with complex Borel measures μ_n on $\mathcal{U}$, admits a *supershift*, if there exists some $k \in \mathcal{O}\setminus\mathcal{U}$, such that

$$\lim_{n\to\infty} \Phi_n(s) = \varphi_k(s), \quad s \in X, \tag{6.4}$$

converges uniformly on compact subsets of X.

Remark 6.2. Note, that the integral (6.3) is well defined since any complex measure μ_n is in particular finite and the integrand $\kappa \mapsto \varphi_\kappa(s)$ is bounded by assumption.

Remark 6.3. If the sequence $(\Phi_n)_n$ in (6.3) admits a supershift, then the values of φ_k for some $k \in \mathcal{O} \setminus \mathcal{U}$, outside the smaller set $\mathcal{U}$, can be calculated by only using values φ_k at the points κ inside $\mathcal{U}$. Hence, informally speaking, when considering the mapping $k \mapsto \varphi_k$, there is a breeze of analyticity in the air, see also Theorem 6.7 and Corollary 6.8.

Since we introduced the supershift property with the aim to generalize superoscillations, the following example shows, that the superoscillating sequence (0.1) indeed admits the supershift property.

Example 6.4. Let $X = \mathbb{C}$ and consider for every $k \in \mathcal{O} = \mathbb{R}$ the exponentials

$$\varphi_k(z) = e^{ikz}, \quad z \in \mathbb{C}.$$

Then the functions F_n from Definition 2.1 can be written as

$$F_n(z) = \int_{-k_0}^{k_0} \varphi_\kappa(z) d\mu_n(\kappa), \quad z \in \mathbb{C},$$

due to the representation (2.1). Moreover, by (2.2) it follows that

$$\lim_{n\to\infty} F_n(z) = \varphi_k(z) \quad \text{in } \mathcal{A}_1(\mathbb{C}),$$

for some $k \in \mathbb{R} \setminus [-k_0, k_0]$. Since this $\mathcal{A}_1$-convergence in particular implies the uniform convergence on compact subsets of $\mathbb{C}$, the superoscillating sequence $(F_n)_n$ indeed admits a supershift according to Definition 6.1.

The main result of this chapter is the following Theorem 6.5 on the supershift property of the solution of the Schrödinger equation, which can be viewed as a corollary of the continuous dependence results of Theorem 4.6 and Theorem 5.4. Roughly speaking, we consider a family of initial conditions that admits a supershift (with respect to a slightly stronger form of convergence as in Definition 6.1) and conclude that the corresponding solutions of the Schrödinger equation admit a similar type of supershift, see also Remark 6.6.

Theorem 6.5. Let the function G be as in Assumption 4.1 (or Assumption 5.1) and $\mathcal{O}, \mathcal{U} \subseteq \mathbb{C}$ with $\mathcal{U} \subsetneq \mathcal{O}$. Moreover, for some $p \in (0, 2)$ let $\varphi_k \in \mathcal{A}_p(\mathbb{C})$, $k \in \mathcal{O}$, be a family of functions, such that the required $\mathcal{A}_p$-boundedness is satisfied by

$$|\varphi_k(z)| \leq A(k) e^{B(k)|z|^p}, \quad z \in \mathbb{C}, \tag{6.5}$$

for some $A(k), B(k)$ which are bounded on $\mathcal{U}$. For complex Borel measures μ_n, the functions

$$F_n(z) = \int_{\mathcal{U}} \varphi_\kappa(z) d\mu_n(\kappa), \quad z \in \mathbb{C}, \tag{6.6}$$

are elements in $\mathcal{A}_p(\mathbb{C})$ and if they converge as

$$\lim_{n\to\infty} F_n = \varphi_k \quad \text{in } \mathcal{A}_p(\mathbb{C}), \tag{6.7}$$

to some φ_k with $k \in \mathcal{O} \setminus \mathcal{U}$, then also the sequence of solutions of the Cauchy problem (4.24) (or (5.22)) converges as

$$\lim_{n\to\infty} \Psi(t, x; F_n) = \lim_{n\to\infty} \int_{\mathcal{U}} \Psi(t, x; \varphi_\kappa) d\mu_n(\kappa) = \Psi(t, x; \varphi_k), \tag{6.8}$$

for every $t \in (0, T)$ and uniformly on compact subsets of $\mathbb{R}$ (or $\dot{\mathbb{R}}$).

Proof. In the first step we want to show that F_n is an entire function. To do so, we choose any closed triangle $\Delta\subseteq\mathbb{C}$ and consider the path integral

$$\int_\Delta F_n(z)dz=\int_\Delta\int_{\mathcal{U}}\varphi_\kappa(z)d\mu_n(\kappa)dz.$$

Due to (6.5) and the uniform bounds $A(\kappa)\leq A_\infty$ and $B(\kappa)\leq B_\infty$ for every $\kappa\in\mathcal{U}$, the integrand can be estimated by

$$|\varphi_\kappa(z)|\leq A_\infty e^{B_\infty|z|^p},\quad \kappa\in\mathcal{U},\ z\in\Delta.$$

Hence the double integral over $\Delta\times\mathcal{U}$ is absolute convergent and we are allowed to interchange the order of integration, which leaves us with

$$\int_\Delta F_n(z)dz=\int_{\mathcal{U}}\int_\Delta\varphi_\kappa(z)dzd\mu_n(\kappa)=\int_{\mathcal{U}}0\,d\mu_n(\kappa)=0,$$

where the integral $\int_\Delta\varphi_\kappa(z)dz$ vanishes due to the holomorphicity of φ_κ. By the Theorem of Morera this implies that F_n is an entire function. The fact that it is also exponentially bounded follows from the simple estimate

$$|F_n(z)|\leq\int_{\mathcal{U}}|\varphi_\kappa(z)|d|\mu_n|(\kappa)\leq A_\infty|\mu_n|(\mathcal{U})e^{B_\infty|z|^p},\quad z\in\mathbb{C},$$

where $|\mu_n|$ is the variation of the complex measure μ_n. Hence we verified $F_n\in\mathcal{A}_p(\mathbb{C})$.

The fact, that the convergence (6.7) leads to the convergence (6.8), was already proven in Theorem 4.6 and Theorem 5.4. It is left to show that it is possible to write

$$\Psi(t,x;F_n)=\int_{\mathcal{U}}\Psi(t,x;\varphi_\kappa)d\mu_n(\kappa).$$

Using the representation (4.31) of the solution gives

$$\begin{aligned}\Psi(t,x;F_n)&=e^{i\alpha}\int_{\mathbb{R}}G(t,x,ye^{i\alpha})F_n(ye^{i\alpha})dy\\&=e^{i\alpha}\int_{\mathbb{R}}G(t,x,ye^{i\alpha})\int_{\mathcal{U}}\varphi_\kappa(ye^{i\alpha})d\mu_n(\kappa)dy\\&=e^{i\alpha}\int_{\mathcal{U}}\int_{\mathbb{R}}G(t,x,ye^{i\alpha})\varphi_\kappa(ye^{i\alpha})dyd\mu_n(\kappa)\\&=\int_{\mathcal{U}}\Psi(t,x;\varphi_\kappa)d\mu_n(\kappa),\end{aligned}$$

where we were allowed to interchange the order of integration because of the estimate

$$|G(t,x,ye^{i\alpha})\varphi_\kappa(ye^{i\alpha})|\leq A_\infty A_0(t,x)e^{-a(t)\sin(2\alpha)y^2+2a(t)\sin(\alpha)|xy|+B_0(t,x)|y|^q+B_\infty|y|^p}.$$

Since the complex measure μ_n is finite, and due to the decaying Gausssian $e^{-a(t)\sin(2\alpha)y^2}$, the double is absolute convergent and we are allowed to apply the Fubini theorem. □

Remark 6.6. Since the convergence (6.7) implies uniform convergence on all compact subsets of $\mathbb{R}$, it is clear, that the initial conditions $(F_n)_n$ in (6.6) admit the supershift property of Definition 6.1 with respect to the metric space $X = \mathbb{R}$. Furthermore, at any time $t \in (0,T)$, the convergence (6.8) shows that the sequence $(\Psi(t,x;F_n))_n$ again admits a supershift with the functions (6.2) chosen as $\phi_k(x) := \Psi(t,x;\varphi_k)$.

In the next result we continue the theme of Theorem 6.5 and return to the analyticity issue of Remark 6.3. In fact, the following Theorem 6.7 shows, that analyticity in the variable k of the initial condition implies analyticity in the k-variable in the wave function.

Theorem 6.7. Let G be as in Assumption 4.1 (or Assumption 5.1) and $\Psi(t,x;F)$ the solution of the corresponding Cauchy problem (4.24) (or (5.22)). For some open set $\Omega \subseteq \mathbb{C}$ and $p \in (0,2)$, we consider a family of functions $\varphi_k \in \mathcal{A}_p(\mathbb{C})$, $k \in \Omega$, such that

$$|\varphi_k(z)| \leq A(k) e^{B(k)|z|^p}, \quad z \in \mathbb{C}, \tag{6.9}$$

is satisfied for some $A(k), B(k) \geq 0$, locally bounded. If for every $z \in \mathbb{C}$ the mapping

$$\Omega \ni k \mapsto \varphi_k(z)$$

is holomorphic, then for every fixed $t \in (0,T)$, $x \in \mathbb{R}$ (or $\dot{\mathbb{R}}$), the mapping

$$\Omega \ni k \mapsto \Psi(t,x;\varphi_k)$$

is holomorphic as well.

Proof. Fix $t \in (0,T)$, $x \in \mathbb{R}$ (or $\dot{\mathbb{R}}$). Then for any closed triangle $\Delta \subseteq \Omega$, we have the path integral

$$\int_\Delta \Psi(t,x;\varphi_k)dk = \int_\Delta e^{i\alpha} \int_\mathbb{R} G(t,x,ye^{i\alpha})\varphi_k(ye^{i\alpha})dydk, \tag{6.10}$$

due to the representation (4.31) of the wave function. Here $\alpha \in (0,\frac{\pi}{2})$ is the angle of the double sector S_α in Assumption 4.1 (or Assumption 5.1). In order to interchange the order of integration, we have to prove absolute integrability of the double integral. Firstly, the estimate

$$\big|G(t,x,ye^{i\alpha})\varphi_k(ye^{i\alpha})\big| \leq A(k)A_0(t,x)e^{-a(t)\sin(2\alpha)y^2+2a(t)\sin(\alpha)|xy|+B_0(t,x)|y|^q+B(k)|y|^p}, \tag{6.11}$$

follows from (4.9a) (or (5.8a)) as well as (6.9), and shows that the y-integral is absolutely convergent. Moreover, the coefficients $A(k), B(k)$ are assumed to be bounded on the compact triangle Δ. This means, that the right hand side of (6.11) can be replaced by some k-independent and y-integrable upper bound. Hence, the double integral (6.10) is absolutely convergent and we are allowed to interchange the order of integration and get

$$\int_\Delta \Psi(t,x;\varphi_k)dk = e^{i\alpha}\int_\mathbb{R} G(t,x,ye^{i\alpha})\int_\Delta \varphi_k(ye^{i\alpha})dkdy.$$

Since the mapping $\Omega \ni k \mapsto \varphi_k(ye^{i\alpha})$ is holomorphic the path integral along Δ vanishes and we get

$$\int_\Delta \Psi(t,x;\varphi_k)d\kappa = 0.$$

Due to the Theorem of Morera this implies the analyticity of $\Omega \ni k \mapsto \Psi(t,x;\varphi_k)$. □

In order to appreciate our main results, the following corollary shows how the above Theorem 6.5 and Theorem 6.7 combine in the special case of exponentials $\varphi_k(z) = e^{ikz}$, i.e. in the case of a Type I superoscillating sequence of Definition 2.1.

Corollary 6.8. Let G be as in Assumption 4.1 (or Assumption 5.1) and $(F_n)_n$ a Type I superoscillating sequence. Then the sequence of solutions of the Cauchy problem (4.24) (or (5.22)) converges as

$$\lim_{n\to\infty} \Psi(t,x;F_n) = \lim_{n\to\infty} \int_{-k_0}^{k_0} \Psi(t,x;e^{i\kappa\,\cdot\,})d\mu_n(\kappa) = \Psi(t,x;e^{ik\,\cdot\,}), \tag{6.12}$$

for every $t \in (0,T)$ and uniformly on compact subsets of $\mathbb{R}$ (or $\dot{\mathbb{R}}$). That is, the sequence $\Psi(t,x;F_n)$ admits a supershift. Moreover, for every fixed $t \in (0,T)$, $x \in \mathbb{R}$ (or $x \in \dot{\mathbb{R}}$), the mapping

$$\mathbb{C} \ni \kappa \mapsto \Psi(t,x;e^{i\kappa\,\cdot\,}) \quad \text{is analytic.}$$

Proof. Since the functions $\varphi_\kappa(z) = e^{i\kappa z}$ satisfy the estimate

$$|\varphi_\kappa(z)| \leq e^{|\kappa||z|}, \quad \kappa, z \in \mathbb{C},$$

the assumptions of Theorem 6.5 and Theorem 6.7 are satisfied and the statement of the corollary follows. □

7. Examples of Green's functions

In this chapter we apply the theory of the previous Chapters 4, 5 and 6 to some specific potentials and transmission conditions, where the corresponding Green's function is known explicitly. We start by the simplest case, the free particle $V = 0$ in Section 7.1, and proceed with the electric field $V \sim x$ in Section 7.2, the harmonic oscillator $V \sim x^2$ in Section 7.3, the Pöschl Teller potential $V \sim \frac{1}{\cosh^2(x)}$ in Section 7.4, which are all applications of the Schrödinger equation on $\mathbb{R}$, i.e. Section 4. As examples for the Schrödinger equation on $\dot{\mathbb{R}} := \mathbb{R} \setminus \{0\}$ of Section 5, we consider the centrifugal potential $V \sim \frac{1}{x^2}$ in Section 7.5 as well as arbitrary point interactions in Section 7.6. The main task in the above mentioned examples is to verify either Assumption 4.1 or Assumption 4.4 is satisfied for the potentials on $\mathbb{R}$ and either Assumption 5.1 or Assumption 5.2 is satisfied for the potentials on $\dot{\mathbb{R}}$. Hence in all the cases the corresponding Theorem 4.6 or Theorem 5.4 is applicable and in particular the time persistence of the supershift property in Theorem 6.5 holds true.

While the forthcoming examples of Section 7.1–7.4 were already considered in the paper [7], the centrifugal potential of Section 7.5, in particular in the attractive case $\lambda < 0$, is a potential for which the time persistence of supershifts is proven for the first time. For arbitrary point interactions this result is already known, see [6], but Section 7.6 at least shows that also these distributional potentials fit into the general framework of the unified approach.

7.1. Free particle

The free particle $V(t, x) = 0$ is the easiest example and also the one which was investigated first in the history of time evolution of superoscillations, see [15]. The corresponding Green's function is given by

$$G(t, x, y) = \frac{1}{2\sqrt{i\pi t}} e^{-\frac{(y-x)^2}{4it}}, \quad t > 0,\ x, y \in \mathbb{R}. \tag{7.1}$$

Theorem 7.1. The Green's function (7.1) satisfies Assumption 4.4 for the potential $V(t, x) = 0$.

Proof. It is obvious by simply replacing $y \to z$ in (7.1), that $G(t, x, \cdot)$ extends to an entire function. Moreover, the decomposition (4.15) is satisfied using $a(t) = \frac{1}{4t}$ and

$$\widetilde{G}(t, x, z) = \frac{1}{2\sqrt{i\pi t}}, \quad t > 0,\ x \in \mathbb{R},\ z \in \mathbb{C}.$$

Next we verify the properties (i)–(iii) of Assumption 4.4.

(i) In order to see, that (7.1) is a solution of the time dependent Schrödinger equation (4.13), or equivalently that $\widetilde{G}$ is a solution of (4.11), we explicitly calculate its derivatives, which are

$$\frac{\partial}{\partial x}\widetilde{G}(t,x,z)=0, \qquad \frac{\partial^2}{\partial x^2}\widetilde{G}(t,x,z)=0 \quad \text{and} \quad \frac{\partial}{\partial t}\widetilde{G}(t,x,z)=\frac{-1}{4\sqrt{i\pi}\,t^{\frac{3}{2}}}. \tag{7.2}$$

Hence, $\widetilde{G}(t,x,z)$ satisfies

$$i\frac{\partial}{\partial t}\widetilde{G}(t,x,z)=\Big(-\frac{\partial^2}{\partial x^2}+\frac{z-x}{it}\frac{\partial}{\partial x}-\frac{i}{2t}\Big)\widetilde{G}(t,x,z),$$

which is exactly (4.11) with $a(t)=\frac{1}{4t}$ and $V(t,x)=0$.

(ii) For any $x\in\mathbb{R}$, we trivially obtain the limit (4.14) as

$$\lim_{t\to 0^+}\frac{G(t,x,x)}{\sqrt{a(t)}}=\lim_{t\to 0^+}\frac{1}{\sqrt{i\pi}}=\frac{1}{\sqrt{i\pi}}. \tag{7.3}$$

(iii) The absolute value of $\widetilde{G}$ and its derivatives are

$$\begin{aligned} \big|\widetilde{G}(t,x,z)\big| &= \frac{1}{2\sqrt{\pi t}},\\ \Big|\frac{\partial}{\partial x}\widetilde{G}(t,x,z)\Big| = \Big|\frac{\partial^2}{\partial x^2}\widetilde{G}(t,x,z)\Big| &= 0,\\ \Big|\frac{\partial}{\partial t}\widetilde{G}(t,x,z)\Big| &= \frac{1}{4\sqrt{\pi}\,t^{\frac{3}{2}}}. \end{aligned} \tag{7.4}$$

This shows, that the exponential bound (4.16a) is satisfied with $A_0(t,x)=\frac{1}{2\sqrt{\pi t}}$ and $B_0(t,x)=0$, for which

$$\frac{A_0(t,x)}{\sqrt{a(t)}}=\frac{1}{\sqrt{\pi}} \quad \text{and} \quad B_0(t,x)=0,$$

are obviously bounded as $t\to 0^+$, as claimed in (4.17). By the explicit form (7.2) of the derivatives we immediately see, that also the exponential bounds (4.16b) are satisfied.

Hence the Green's function indeed satisfies Assumption 4.4 and Theorem 7.1 is proven. □

7.2. Time dependent electric field

In this section we consider $V(t,x)=\lambda(t)x$, for some continuous $\lambda:[0,\infty)\to\mathbb{R}$. For the special case of a constant strength $\lambda(t)=\lambda$, this type of potential was already investigated with respect to the time persistence of superoscillations in [19, Theorem 3.6]. Even for some generalized Schrödinger equation, i.e. $-\frac{\partial}{\partial x^2}$ replaced by $f(-i\frac{\partial}{\partial x})$

where f is a polynomial of even degree, the electric field potential was investigated in [22]. The Green's function, associated with electric field potential, is given by

$$G(t,x,y) = \frac{1}{2\sqrt{i\pi t}} e^{i\beta(t)+it\alpha'(t)x+i\alpha(t)y-\frac{(y-x)^2}{4it}}, \quad t>0,\ x,y\in\mathbb{R}, \tag{7.5}$$

where the parameters $\alpha, \beta : (0,\infty) \to \mathbb{C}$ are the solutions of the ordinary differential equations

$$t\alpha''(t) + 2\alpha'(t) = -\lambda(t) \quad \text{and} \quad \beta'(t) = -t^2\alpha'(t)^2, \quad t>0, \tag{7.6}$$

with initial conditions

$$\lim_{t\to 0^+} \alpha(t) = \lim_{t\to 0^+} \beta(t) = \lim_{t\to 0^+} t\alpha'(t) = 0. \tag{7.7}$$

For the particular special case of a time independent electric field $\lambda(t) = \lambda$ with constant strength $\lambda \in \mathbb{R}$, the solutions of the initial value problems (7.6) and (7.7) are in this case explicitly given by

$$\alpha(t) = -\frac{\lambda t}{2} \quad \text{and} \quad \beta(t) = -\frac{\lambda^2 t^3}{12}.$$

Consequently, the Green's function (7.5) becomes

$$G(t,x,y) = \frac{1}{2\sqrt{i\pi t}} e^{-\frac{i\lambda^2 t^3}{12} - \frac{i\lambda t(x+y)}{2} - \frac{(y-x)^2}{4it}}, \quad t>0,\ x,y\in\mathbb{R}.$$

Note, that this Green's function coincides with the ones the authors used in [35, 101].

Theorem 7.2. The Green's function (7.5) satisfies Assumption 4.4 for the potential $V(t,x) = \lambda(t)x$.

Proof. It is obvious by simply replacing $y \to z$ in (7.5), that $G(t,x,\cdot\,)$ extends to an entire function. Moreover, the decomposition (4.15) is satisfied using $a(t) = \frac{1}{4t}$ and

$$\widetilde{G}(t,x,z) = \frac{1}{2\sqrt{i\pi t}} e^{i\beta(t)+it\alpha'(t)x+i\alpha(t)z}, \quad t>0,\ x\in\mathbb{R},\ z\in\mathbb{C}.$$

Next we verify the properties (i)–(iii) of Assumption 4.4.

(i) In order to see, that (7.5) is a solution of the time dependent Schrödinger equation (4.13), or equivalently that $\widetilde{G}$ is a solution of (4.11), we explicitly calculate its derivatives, which are

$$\begin{aligned}
\frac{\partial}{\partial x}\widetilde{G}(t,x,z) &= it\alpha'(t)\widetilde{G}(t,x,z), \\
\frac{\partial^2}{\partial x^2}\widetilde{G}(t,x,z) &= -t^2\alpha'(t)^2\widetilde{G}(t,x,z), \\
\frac{\partial}{\partial t}\widetilde{G}(t,x,z) &= \Big(i\beta'(t) + i\alpha'(t)(z-x) - ix\lambda(t) - \frac{1}{2t}\Big)\widetilde{G}(t,x,z),
\end{aligned} \tag{7.8}$$

where for the time derivative we used the differential equation (7.6). Hence $\widetilde{G}(t,x,z)$ satisfies

$$i\frac{\partial}{\partial t}\widetilde{G}(t,x,z) = \Big(-\frac{\partial^2}{\partial x^2} + \frac{x-z}{it}\frac{\partial}{\partial x} + \frac{1}{2it} + \lambda(t)x\Big)\widetilde{G}(t,x,z),$$

which is exactly (4.11).

(ii) For any $x \in \mathbb{R}$, we obtain the limit (4.14) as

$$\lim_{t \to 0^+} \frac{G(t,x,x)}{\sqrt{a(t)}} = \frac{1}{\sqrt{i\pi}} \lim_{t \to 0^+} e^{i\beta(t) + it\alpha'(t)x + i\alpha(t)z} = \frac{1}{\sqrt{i\pi}},$$

where we used the initial conditions (7.7) of the parameters α and β.

(iii) The function $\widetilde{G}$ is exponentially bounded by

$$|\widetilde{G}(t,x,y)| \leq \frac{1}{2\sqrt{\pi t}} e^{-\alpha(t)\operatorname{Im}(z)}, \quad t > 0,\, x \in \mathbb{R},\, z \in \mathbb{C}, \tag{7.9}$$

and hence satisfies (4.9a) with the coefficients

$$A_0(t,x) = \frac{1}{2\sqrt{\pi t}} \quad \text{and} \quad B_0(t,x) = |\alpha(t)|.$$

Moreover, the coefficients

$$\frac{A_0(t,x)}{\sqrt{a(t)}} = \frac{1}{\sqrt{\pi}} e^{-\operatorname{Im}(\beta(t)) - t\operatorname{Im}(\alpha'(t))x} \quad \text{and} \quad B_0(t,x) = |\alpha(t)|,$$

are bounded in the limits $t \to 0^+$, again by the initial conditions (7.7). By the explicit form (7.8) of its derivatives, we immediately see, that also the exponential bounds (4.16b) are satisfied.

Hence the Green's function indeed satisfies Assumption 4.4 and Theorem 7.2 is proven. □

7.3. Time dependent harmonic oscillator

One particularly important potential is the harmonic oscillator $V(t,x) = \omega(t)x^2$, with some continuous frequency $\omega : [0,\infty) \to \mathbb{R}$. The reason for this is, that in many approximations the potential gets Taylor expanded up to the quadratic term, which is mainly the first interesting one. This approximation is often called semiclassical approximation or WKB approximation. The particular case of constant frequency $\omega(t) = \omega > 0$ is already treated in [33, 35, 55, 68] and a numerical illustration of the superoscillating solution is displayed in [56]. Also the time dependent case $\omega(t) = \pm t$ was considered in [22, Theorem 2.1 & Theorem 2.3].

The Green's function of the harmonic oscillator is given by

$$G(t,x,y) = \frac{1}{2\sqrt{i\pi\alpha(t)}} e^{-\frac{\alpha'(t)x^2 - 2xy + \beta(t)y^2}{4i\alpha(t)}}, \quad t \in (0,T),\, x,y \in \mathbb{R}. \tag{7.10}$$

where the parameters α and β are solutions of the ordinary differential equations

$$\alpha''(t) = -4\omega(t)\alpha(t) \quad \text{and} \quad \beta''(t) = -4\omega(t)\beta(t), \quad t > 0, \tag{7.11}$$

with initial conditions

$$\alpha(0^+) = \beta'(0^+) = 0 \quad \text{and} \quad \alpha'(0^+) = \beta(0^+) = 1. \tag{7.12}$$

Moreover, $T > 0$ is chosen as the smallest positive zero of either $\alpha(t)$ or $\beta(t)$, or $T = \infty$ if $\alpha(t)$ and $\beta(t)$ have no zeros on $(0, \infty)$. With this choice of T, it follows from the initial conditions that $\alpha(t) > 0$ and $\beta(t) > 0$ for $t \in (0, T)$ and (7.10) is well defined.

In the special case of a time independent harmonic oscillator, we obtain the following explicit forms of the Green's function.

- If $V(t,x) = \omega^2 x^2$ for some $\omega > 0$, the differential equation (7.11) has the two linear independent solutions $\sin(2\omega t)$ and $\cos(2\omega t)$. With the initial values (7.12), this gives the solutions

$$\alpha(t) = \frac{\sin(2\omega t)}{2\omega} \quad \text{and} \quad \beta(t) = \cos(2\omega t).$$

  The smallest positive zero of those functions is $T = \frac{\pi}{4\omega}$, which is then also the maximum time for which our unified approach works. Plugging these solutions into (7.10), gives the Green's function

$$G(t,x,y) = \frac{\sqrt{\omega}}{\sqrt{2i\pi \sin(2\omega t)}} e^{-i\omega xy \tan(\omega t) - \frac{\omega(x-y)^2}{2i\tan(2\omega t)}}, \quad t \in \left(0, \frac{\pi}{4\omega}\right),\ x, y \in \mathbb{R}.$$

  Note, that we obtain the same results as in [7, Remark 5.2], with the only difference, that the upper bound is $\frac{\pi}{2\omega}$ instead of $\frac{\pi}{4\omega}$ there, which is an error in the paper.

- If $V(t,x) = -\omega^2 x^2$ for some $\omega > 0$, the differential equation (7.11) has the two linear independent solutions $\sinh(2\omega t)$ and $\cosh(2\omega t)$. With the initial values (7.12), this gives the solutions

$$\alpha(t) = \frac{\sinh(2\omega t)}{2\omega} \quad \text{and} \quad \beta(t) = \cosh(2\omega t).$$

  Since both functions do not have positive zeros, we get $T = \infty$, which means, that the Green's function is valid for all times $t > 0$. Plugging these solutions into (7.10), gives the Green's function

$$G(t,x,y) = \frac{\sqrt{\omega}}{\sqrt{2i\pi \sinh(2\omega t)}} e^{i\omega xy \tanh(\omega t) - \frac{\omega(x-y)^2}{2i\tanh(2\omega t)}}, \quad t > 0,\ x, y \in \mathbb{R}.$$

- For vanishing potential $\omega(t) = 0$, the initial value problem (7.11) and (7.12) has the solution

$$\alpha(t) = t \quad \text{and} \quad \beta(t) = 1.$$

  These solutions also have no positive zeros, i.e. $T = \infty$, and the Green's function becomes

$$G(t,x,y) = \frac{1}{2\sqrt{i\pi t}} e^{-\frac{(x-y)^2}{4it}}, \quad t > 0,\ x, y \in \mathbb{R},$$

  which means, that we regained the one from the free particle (7.1).

Theorem 7.3. The Green's function (7.10) satisfies Assumption 4.4.

Proof. It is obvious by simply replacing $y \to z$ in (7.10), that $G(t,x,\cdot\,)$ extends to an entire function. Moreover, the decomposition (4.15) is satisfied using $a(t) = \frac{\beta(t)}{4\alpha(t)}$ and

$$\widetilde{G}(t,x,z) = \frac{1}{2\sqrt{i\pi\alpha(t)}} e^{\frac{(\beta(t)-\alpha'(t))x^2+2(1-\beta(t))xz}{4i\alpha(t)}}, \quad t \in (0,T),\ x \in \mathbb{R},\ z \in \mathbb{C}.$$

Next we verify the properties (i)–(iii) of Assumption 4.4.

(i) In order to see, that (7.10) is a solution of the time dependent Schrödinger equation (4.13), or equivalently that $\widetilde{G}$ is a solution of (4.11), we explicitly calculate its derivatives, which are

$$\begin{aligned}
\frac{\partial}{\partial x}\widetilde{G}(t,x,z) &= \frac{(\beta(t)-\alpha'(t))x+(1-\beta(t))z}{4i\alpha(t)}\widetilde{G}(t,x,z),\\
\frac{\partial^2}{\partial x^2}\widetilde{G}(t,x,z) &= \Big(\frac{i(\alpha'(t)-\beta(t))}{4\alpha(t)} - \frac{\big((\beta(t)-\alpha'(t))x+(1-\beta(t))z\big)^2}{8\alpha(t)^2}\Big)\widetilde{G}(t,x,z), \qquad (7.13)\\
\frac{\partial}{\partial t}\widetilde{G}(t,x,z) &= \Big(\frac{(\alpha'(t)^2-1)x^2}{8i\alpha(t)^2} + \frac{(1-\alpha'(t))xz}{4i\alpha(t)^2} - \frac{\alpha'(t)}{4\alpha(t)} - i\omega(t)x^2\Big)\widetilde{G}(t,x,z),
\end{aligned}$$

where for the time derivative we used the differential equations (7.11) of the parameters and that the Wronskian has the constant value

$$\alpha'(t)\beta(t) - \alpha(t)\beta'(t) = 1, \quad t > 0. \tag{7.14}$$

Plugging in $a(t) = \frac{\beta(t)}{4\alpha(t)}$ and using once more the Wronskian (7.14), turns the differential equation (4.11) into

$$i\frac{\partial}{\partial t}\widetilde{G}(t,x,z) = \Big(-\frac{\partial^2}{\partial x^2} + \frac{\beta(t)(x-z)}{i\alpha(t)}\frac{\partial}{\partial x} + \frac{(\beta(t)^2-1)(z-x)^2}{4\alpha(t)^2} + \frac{\beta(t)}{2i\alpha(t)} + \omega(t)x^2\Big)\widetilde{G}(t,x,z).$$

which is clearly satisfied if one plugs in (7.13).

(ii) In order to conclude the initial value (4.14), we first note, that it follows from (7.12) and L'Hospitals rule, that

$$\lim_{t\to 0^+}\frac{1-\beta(t)}{\alpha(t)} = \lim_{t\to 0^+}\frac{-\beta'(t)}{\alpha'(t)} = \frac{0}{1} = 0, \tag{7.15a}$$

$$\lim_{t\to 0^+}\frac{\beta(t)-\alpha'(t)}{\alpha(t)} = \lim_{t\to 0^+}\frac{\beta'(t)-\alpha''(t)}{\alpha'(t)} = \lim_{t\to 0^+}\frac{\beta'(t)+4\omega(t)\alpha(t)}{\alpha'(t)} = 0. \tag{7.15b}$$

Hence, for every $x \in \mathbb{R}$ we obtain the limit

$$\lim_{t\to 0^+}\frac{G(t,x,x)}{\sqrt{a(t)}} = \lim_{t\to 0^+}\frac{1}{\sqrt{i\pi\beta(t)}} e^{\frac{(\beta(t)-\alpha'(t))x^2+2(1-\beta(t))x^2}{4i\alpha(t)}} = \frac{1}{\sqrt{i\pi}}.$$

(iii) Since the parameters α and β are real valued, the absolute value of the function $\widetilde{G}$ equals

$$\big|\widetilde{G}(t,x,z)\big| = \frac{1}{2\sqrt{\pi\alpha(t)}} e^{\frac{(1-\beta(t))x}{2\alpha(t)}\operatorname{Im}(z)}, \quad t \in (0,T),\ x \in \mathbb{R},\ z \in \mathbb{C}, \tag{7.16}$$

and hence satisfies (4.16a) with the coefficients

$$A_0(t,x) = \frac{1}{2\sqrt{\pi\alpha(t)}} \quad \text{and} \quad B_0(t,x) = \frac{|1-\beta(t)||x|}{2\alpha(t)}.$$

It also follows from the above limits (7.15), that

$$\lim_{t\to 0^+} \frac{A_0(t,x)}{\sqrt{a(t)}} = \lim_{t\to 0^+} \frac{1}{\sqrt{\pi\beta(t)}} = \frac{1}{\sqrt{\pi}},$$
$$\lim_{t\to 0^+} B_0(t,x) = \lim_{t\to 0^+} \frac{(1-\beta(t))|x|}{2\alpha(t)} = 0,$$

and hence are bounded as claimed in (4.17). By the explicit form (7.13) of its derivatives, we immediately see, that also the exponential bounds (4.16b) are satisfied.

Hence the Green's function indeed satisfies Assumption 4.4 and Theorem 7.3 is proven. □

7.4. Pöschl-Teller potential

In this section we consider the Pöschl-Teller potential $V(t,x) = -\frac{l(l+1)}{\cosh^2(x)}$, $l \in \mathbb{N}_0$. This potential was already investigated with respect to superoscillations for example in [7]. Using the function R in (A.26) and $\mathcal{P}_l^m$ from (A.30), the Green's function of the Pöschl-Teller potential reads for every $t > 0$, $x, y \in \mathbb{R}$ as

$$G(t,x,y) = \bigg(\frac{1}{2\sqrt{i\pi t}} + \sum_{m=1}^{l} \frac{m(l-m)!}{2(l+m)!} \mathcal{P}_l^m(x)\mathcal{P}_l^m(y) R\big(m^2 t, m(y-x)\big)\bigg) e^{-\frac{(y-x)^2}{4it}}. \tag{7.17}$$

This Green's function can for example be found in [87, Section 6.6.3].

Theorem 7.4. The Green's function (7.17) satisfies Assumption 4.4.

Proof. Since $R(mt^2, \cdot\,)$ is an entire function and $\mathcal{P}_l^m$ is holomorphic on $\mathbb{C} \setminus i\pi(\mathbb{Z} + \frac{1}{2})$, the Green's funcion $G(t,x,\cdot\,)$ trivially admits a holomorphic extension to $S_{\alpha,h}$ from (4.12) for every $\alpha \in (0, \frac{\pi}{2})$, $h \in (0, \frac{\pi}{2})$, by simply replacing $y \to z$ in (7.17). Moreover, the decomposition (4.15) is satisfied using $a(t) = \frac{1}{4t}$ and

$$\widetilde{G}(t,x,z) = \underbrace{\frac{1}{2\sqrt{i\pi t}}}_{=:\widetilde{G}_{\text{free}}(t,x,z)} + \sum_{m=1}^{l} \frac{m(l-m)!}{2(l+m)!} \underbrace{\mathcal{P}_l^m(x)\mathcal{P}_l^m(z) R\big(m^2 t, m(z-x)\big)}_{=:\widetilde{G}_m(t,x,z)}, \quad z \in S_{\alpha,h}. \tag{7.18}$$

Next we verify the properties (i)–(iii) of Assumption 4.4. Note, that we already treated $\widetilde{G}_{\text{free}}$ in Section 7.1, and we will refer to these results in the following.

(i) In the first step, we check whether $\widetilde{G}_m$ is a solution of (4.11). From the differential equation (A.31) of $\mathcal{P}_l^m$ and the derivatives (A.27) of the function R, we immediately derive the derivatives of $\widetilde{G}_m$ by

$$\begin{aligned}
\frac{\partial}{\partial x}\widetilde{G}_m(t,x,z) &= \frac{x-z}{2it}\widetilde{G}_m(t,x,z) + \frac{2}{\sqrt{i\pi t}}\mathcal{P}_l^m(x)\mathcal{P}_l^m(z)\sinh(m(z-x)) \\
&\quad + \mathcal{P}_l^{m\prime}(x)\mathcal{P}_l^m(z)R\big(m^2t, m(z-x)\big) \\
\frac{\partial^2}{\partial x^2}\widetilde{G}_m(t,x,z) &= \Big(\frac{1}{2it} - \frac{(x-z)^2}{4t^2} + m^2 - \frac{l(l+1)}{\cosh^2(x)}\Big)\widetilde{G}_m(t,x,z) \\
&\quad + \frac{1}{\sqrt{i\pi t}}\mathcal{P}_l^m(x)\mathcal{P}_l^m(z)\Big(\frac{x-z}{it}\sinh(m(z-x)) - 2m\cosh(m(z-x))\Big) \\
&\quad + \mathcal{P}_l^{m\prime}(x)\mathcal{P}_l^m(z)\Big(\frac{4}{\sqrt{i\pi t}}\sinh(m(z-x)) + \frac{x-z}{it}R\big(m^2t, m(z-x)\big)\Big) \\
\frac{\partial}{\partial t}\widetilde{G}_m(t,x,z) &= i\Big(m^2 + \frac{(z-x)^2}{4t^2}\Big)\widetilde{G}_m(t,x,z) \\
&\quad + \frac{1}{i\sqrt{i\pi t}}\mathcal{P}_l^m(x)\mathcal{P}_l^m(z)\Big(\frac{x-z}{it}\sinh(m(z-x)) - 2m\cosh(m(z-x))\Big).
\end{aligned} \tag{7.19}$$

These derivatives now show, that $\widetilde{G}_m$ satisfies

$$\begin{aligned}
&\Big(i\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} + \frac{z-x}{it}\frac{\partial}{\partial x} - \frac{1}{2it}\Big)\widetilde{G}_m(t,x,z) \\
&\qquad = -\frac{l(l+1)}{\cosh^2(x)}\widetilde{G}_m(t,x,z) + \frac{4}{\sqrt{i\pi t}}\frac{d}{dx}\Big(\mathcal{P}_l^m(z)\mathcal{P}_l^m(x)\sinh(m(z-x))\Big).
\end{aligned}$$

The derivatives of $\widetilde{G}_{\text{free}}(t,x,z)$ were already calculated in (7.2) and give

$$\Big(i\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} + \frac{z-x}{it}\frac{\partial}{\partial x} - \frac{1}{2it}\Big)\widetilde{G}_{\text{free}}(t,x,z) = 0.$$

Using now the identity (A.33), this immediately shows that indeed

$$\begin{aligned}
&\Big(i\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} + \frac{z-x}{it}\frac{\partial}{\partial x} - \frac{1}{2it}\Big)\widetilde{G}(t,x,z) \\
&\quad = -\frac{l(l+1)}{\cosh^2(x)}\sum_{m=1}^{l}\frac{m(l-m)!}{2(l+m)!}\widetilde{G}_m(t,x,z) \\
&\qquad + \frac{2}{\sqrt{i\pi t}}\frac{d}{dx}\sum_{m=1}^{l}\frac{m(l-m)!}{(l+m)!}\mathcal{P}_l^m(z)\mathcal{P}_l^m(x)\sinh(m(z-x)) \\
&\quad = -\frac{l(l+1)}{\cosh^2(x)}\Big(\widetilde{G}(t,x,z) - \widetilde{G}_{\text{free}}(t,x,z)\Big) + \frac{l(l+1)}{2\sqrt{i\pi t}}\frac{d}{dx}\big(\tanh(z) - \tanh(x)\big) \\
&\quad = -\frac{l(l+1)}{\cosh^2(x)}\widetilde{G}(t,x,z),
\end{aligned}$$

is a solution of (4.11).

(ii) For every $x \in \mathbb{R}$, we obtain the limit

$$\lim_{t \to 0^+} \widetilde{G}_m(t,x,x) = \mathcal{P}_l^m(x)^2(\Lambda(0) - \Lambda(0)) = 0,$$

by the explicit form (A.26) of the function R. In the decomposition (7.18) this then gives the limit

$$\lim_{t \to 0^+} \frac{G(t,x,x)}{\sqrt{a(t)}} = \lim_{t \to 0^+} \frac{G_{\text{free}}(t,x,x)}{\sqrt{a(t)}} = \frac{1}{\sqrt{i\pi}},$$

where the respective limit of G_{free} was taken from (7.3).

(iii) For the estimate (4.16a) of $\widetilde{G}_m$, we will use, that

$$\frac{e^{|\operatorname{Re}(z)|}}{|\cosh(z)|} \leq D_{\alpha,h}, \quad z \in S_{\alpha,h},$$

for some $C_{\alpha,h} \geq 0$. Together with the estimate (A.29) of the function R and the estimate (A.32) of $\mathcal{P}_l^m$, this gives the bounds

$$\begin{aligned} |\widetilde{G}_m(t,x,z)| &\leq 2C_{l,m,\alpha,h}^2 \Lambda\Big(-\frac{m\sqrt{t}}{\sqrt{2}}\Big)\Big(\frac{e^{|\operatorname{Re}(z)|}}{|\cosh(z)|}\Big)^m \Big(\frac{e^{|x|}}{\cosh(x)}\Big)^m \\ &\leq 2C_{l,m,\alpha,h}^2 C_{\alpha,h}^{2m} \Lambda\Big(-\frac{m\sqrt{t}}{\sqrt{2}}\Big). \end{aligned} \tag{7.20}$$

Plugging now (7.20) into the decomposition (7.18) shows, that the estimate (4.16a) is satisfied with

$$A_0(t,x) = \frac{1}{2\sqrt{\pi t}} + \sum_{m=1}^{l} \frac{m(l-m)!}{(l+m)!} C_{l,m,\alpha,h}^2 C_{\alpha,h}^{2m} \Lambda\Big(-\frac{m\sqrt{t}}{\sqrt{2}}\Big) \quad \text{and} \quad B_0(t,x) = 0.$$

With these coefficients the values $\frac{A_0}{\sqrt{a}}$ and B_0 are obviously bounded in the limit $t \to 0^+$ as claimed in (4.10). By the explicit form (7.19) of the derivatives, we immediately see, that also the exponential bounds (4.9b) are satisfied for the derivatives of $\widetilde{G}_m$ and hence also for the derivatives of $\widetilde{G}$.

Hence the Green's function indeed satisfies Assumption 5.2 and Theorem 7.4 is proven. □

The Pöschl-Teller potential was the last example, for which the potential was defined on the whole real line $\mathbb{R}$. In the upcoming Sections 7.5 & 7.6 we consider potentials on $\dot{\mathbb{R}}$, and consequently the Schrödinger equation (5.22) on $(0,T) \times \dot{\mathbb{R}}$ including transmission conditions.

7.5. Centrifugal potential

In this section we consider the strongly singular centrifugal potential $V(t,x) = \frac{\lambda}{x^2}$ of strength $\lambda \in \mathbb{R} \setminus \{0\}$. We want to mention, that the repulsive potential $V(t,x) = \frac{\lambda}{x^2}$, $\lambda > 0$, was already investigated with respect to superoscillations in [67, 22]. In [33] even the combined centrifugal and harmonic oscillator potential $V(t,x) = \frac{\lambda}{x^2} + \omega x^2$ is considered, which would also be possible in our case by combining the ideas of this section with the ones in Section 7.3. However, for simplicity we will omit this discussion and concentrate on the pure centrifugal potential here. The case $\lambda < 0$ on the other hand is not yet treated with respect to stability of superoscillations and is a novelty of this thesis.

For $\lambda > 0$, the Green's function of the centrifugal potential $V(t,x) = \frac{\lambda}{x^2}$ is given by

$$G(t,x,y) = \frac{\Theta(xy)\sqrt{xy}}{2i^{\nu+1}t} e^{-\frac{x^2+y^2}{4it}} J_\nu\Big(\frac{xy}{2t}\Big), \quad t > 0,\ x,y \in \dot{\mathbb{R}}, \tag{7.21}$$

and for $\lambda < 0$ by

$$G(t,x,y) = \frac{\Theta(xy)\sqrt{xy}}{4i^{\nu+1}t} e^{-\frac{x^2+y^2}{4it}} H_\nu^{(2)}\Big(\frac{xy}{2t}\Big), \quad t > 0,\ x,y \in \dot{\mathbb{R}}. \tag{7.22}$$

Here $\Theta(\xi) = \begin{cases} 1, & \text{if } \xi > 0, \\ 0, & \text{if } \xi < 0, \end{cases}$ is the step function, $\nu := \sqrt{\frac{1}{4} + \lambda}$ is either nonnegative or purely imaginary with positive imaginary part, J_ν is the Bessel function of the first kind and $H_\nu^{(2)}$ the Hankel function of the second kind. Note, that the Green's function vanishes for $xy < 0$, which is due to the fact that the $\frac{1}{x^2}$-potential is too singular at $x = 0$ to allow any information exchange between the two halflines. Additionally, this nonintegrable singularity automatically implies a Dirichlet boundary condition of the form $\Psi(t,0^+) = \Psi(t,0^-) = 0$, see for example [76] for justification.

Theorem 7.5.

a) The Green's function (7.21) satisfies the Assumption 5.1 for the repulsive centrifugal potential $V(t,x) = \frac{\lambda}{x^2}$, $\lambda > 0$, and the transition matrices $M = I$, the identity matrix, and $N = 0$.

b) The Green's function (7.22) satisfies Assumption 5.2 for the attractive centrifugal potential $V(t,x) = \frac{\lambda}{x^2}$, $\lambda < 0$, and the transition matrices $M = I$, the identity matrix, and $N = 0$.

Proof of Theorem 7.5 b). First of all note, that the Green's function (7.22) can be written as

$$G(t,x,y) = \frac{\Theta(xy)}{2i^{\nu+1}\sqrt{2t}} e^{-\frac{(x-y)^2}{4it}} \mathcal{H}_\nu^{(2)}\Big(\frac{xy}{2t}\Big), \quad t > 0,\ x,y \in \dot{\mathbb{R}}, \tag{7.23}$$

using the modification (A.36) of the Hankel function. It will now be shown, that G satisfies the Assumption 5.2. First of all, the function $G(t,x,\,\cdot\,)$ holomorphically extends to $\mathbb{C} \setminus i\mathbb{R}$ by

$$G(t,x,z) = \frac{\Theta(\pm x)}{2i^{\nu+1}\sqrt{2t}} e^{-\frac{(x-z)^2}{4it}} \mathcal{H}_\nu^{(2)}\Big(\frac{xz}{2t}\Big), \quad \pm\mathrm{Re}(z) > 0.$$

In particular, $G(t,x,\,\cdot\,)$ is holomorphic in the interior of $D_{\alpha,h}$ in (5.10) for any $\alpha\in(0,\frac{\pi}{2})$, $h>0$. Moreover, the decomposition (5.7) is satisfied using $a(t)=\frac{1}{4t}$ and

$$\widetilde{G}(t,x,z)=\frac{\Theta(\pm x)}{2i^{\nu+1}\sqrt{2t}}\mathcal{H}_\nu^{(2)}\Big(\frac{xz}{2t}\Big),\qquad \pm\operatorname{Re}(z)>0.$$

Next we verify the properties (i)–(iii) of Assumption 5.2.

(i) It is obvious, that for fixed $z\in\mathbb{C}\setminus i\mathbb{R}$ we have $G(\,\cdot\,,\,\cdot\,,z)\in\mathrm{AC}_{1,2}((0,T)\times\dot{\mathbb{R}})$ and in order to see, that it is a solution of the time dependent Schrödinger equation (5.11), it is equivalent to show that $\widetilde{G}$ is a solution of (4.11). Hence, for every $t\in(0,T)$, $x\in\dot{\mathbb{R}}$ and $z\in\mathbb{C}\setminus i\mathbb{R}$ we explicitly calculate its derivatives

$$\begin{aligned}
\frac{\partial}{\partial x}\widetilde{G}(t,x,z)&=\frac{\Theta(\pm x)z}{4i^{\nu+1}\sqrt{2}\,t^{\frac{3}{2}}}\mathcal{H}_\nu^{(2)\prime}\Big(\frac{xz}{2t}\Big),\\
\frac{\partial^2}{\partial x^2}\widetilde{G}(t,x,z)&=\frac{\nu^2-\frac14}{x^2}\widetilde{G}(t,x,z)+\frac{\Theta(\pm x)z^2}{4i^{\nu}\sqrt{2}\,t^{\frac{5}{2}}}\mathcal{H}_\nu^{(2)\prime}\Big(\frac{xz}{2t}\Big),\\
\frac{\partial}{\partial t}\widetilde{G}(t,x,z)&=-\frac{1}{2t}\widetilde{G}(t,x,z)-\frac{\Theta(\pm x)xz}{4i^{\nu+1}\sqrt{2}\,t^{\frac{5}{2}}}\mathcal{H}_\nu^{(2)\prime}\Big(\frac{xz}{2t}\Big),
\end{aligned}\tag{7.24}$$

where for the second spatial derivative we used the differential equation (A.37). This shows, that $\widetilde{G}(t,x,z)$ is a solution of

$$i\frac{\partial}{\partial t}\widetilde{G}(t,x,z)=\Big(-\frac{\partial^2}{\partial x^2}+\frac{x-z}{it}\frac{\partial}{\partial x}+\frac{1}{2it}+\frac{\nu^2-\frac14}{x^2}\Big)\widetilde{G}(t,x,z),$$

which is exactly (4.11) after using $\nu^2-\frac14=\lambda$. Moreover, the boundary values of G in (5.12) follow from (A.45) and are given by

$$G(t,0^\pm,y)=\frac{\Theta(\pm y)}{2i^{\nu+1}\sqrt{2t}}e^{-\frac{y^2}{4it}}\lim_{x\to0^\pm}\mathcal{H}_\nu^{(2)}\Big(\frac{xy}{2t}\Big)=0.$$

(ii) From the first limit (A.45) of the Hankel function, we obtain for every $x\in\dot{\mathbb{R}}$ the initial value (5.13) by

$$\lim_{t\to0^+}\frac{G(t,x,x)}{\sqrt{a(t)}}=\frac{1}{i^{\nu+1}\sqrt{2}}\lim_{t\to0^+}\mathcal{H}_\nu^{(2)}\Big(\frac{x^2}{2t}\Big)=\frac{1}{\sqrt{i\pi}}.$$

(iii) Using (A.44a), the absolute value of $\widetilde{G}$ can be estimated by

$$|\widetilde{G}(t,x,z)|=\frac{\Theta(\pm x)}{2\sqrt{2t}}\Big|\mathcal{H}_\nu^{(2)}\Big(\frac{xz}{2t}\Big)\Big|\le\frac{C_{\nu,\alpha}}{2\sqrt{2t}},\qquad|\operatorname{Arg}(\pm z)|\le\alpha,\tag{7.25}$$

and hence satisfies (5.15a) with the coefficients

$$A_0(t,x)=\frac{C_{\nu,\alpha}}{2\sqrt{2t}}\quad\text{and}\quad B_0(t,x)=0.\tag{7.26}$$

These coefficients are bounded as $x \to 0^{\pm}$ and also $\frac{A_0}{\sqrt{a}}$ and B_0 are also bounded as $t \to 0^+$, as requested in (5.16). Due to the explicit form (7.24) of the derivatives of $\widetilde{G}$ and the additional estimate (A.44b) of the derivative of the Hankel function, one immediately sees, that also the exponential bounds (5.15b) and (5.15c) are satisfied.

Hence the Green's function (7.22) indeed satisfies Assumption 5.2. □

Next we prove the repulsive case a) of Theorem 7.5. Although some parts will be similar, especially the verification of the initial condition is different in the sense that in this case the simple limit (5.13) is no longer satisfied and we really have to check the initial condition (5.6).

Proof of Theorem 7.5 a). First of all, $G(t,x,\cdot\,)$ holomorphically extends to $\mathbb{C}\setminus i\mathbb{R}$ by

$$G(t,x,z) = \frac{\Theta(\pm x)\sqrt{xz}}{2i^{\nu+1}t} e^{-\frac{x^2+z^2}{4it}} J_\nu\Big(\frac{xz}{2t}\Big), \quad \pm\operatorname{Re}(z) > 0.$$

In particular, $G(t,x,\cdot\,)$ is holomorphic in the interior of the double sector S_α, for any $\alpha \in (0,\frac{\pi}{2})$. Moreover, the decomposition (5.7) is satisfied using $a(t) = \frac{1}{4t}$ and

$$\widetilde{G}(t,x,z) = \frac{\Theta(\pm x)\sqrt{xz}}{2i^{\nu+1}t} e^{i\frac{xz}{2t}} J_\nu\Big(\frac{xz}{2t}\Big), \quad \pm\operatorname{Re}(z) > 0.$$

Next we verify the properties (i)–(iii) of Assumption 5.1.

(i) It is obvious, that for fixed $z \in \mathbb{C}\setminus i\mathbb{R}$ we have $G(\,\cdot\,,\cdot\,,z) \in \mathrm{AC}_{1,2}((0,T)\times\dot{\mathbb{R}})$ and in order to see, that it is a solution of the time dependent Schrödinger equation (5.4), it is equivalent to show that $\widetilde{G}$ is a solution of (4.11). Hence, for every $t \in (0,T)$, $x \in \dot{\mathbb{R}}$ and $z \in \mathbb{C}\setminus i\mathbb{R}$ we explicitly calculate its derivatives

$$\frac{\partial}{\partial x}\widetilde{G}(t,x,z) = \Big(\frac{1}{2x} + \frac{iz}{2t}\Big)\widetilde{G}(t,x,z) + \frac{\Theta(\pm x)\sqrt{x}\,z^{\frac{3}{2}}}{4i^{\nu+1}t^2} e^{i\frac{xz}{2t}} J_\nu'\Big(\frac{xz}{2t}\Big), \tag{7.27a}$$

$$\frac{\partial^2}{\partial x^2}\widetilde{G}(t,x,z) = \Big(\frac{iz}{2tx} - \frac{z^2}{2t^2} + \frac{\nu^2-\frac{1}{4}}{x^2}\Big)\widetilde{G}(t,x,z) + \frac{\Theta(\pm x)\sqrt{x}\,z^{\frac{5}{2}}}{4i^{\nu}t^3} e^{i\frac{xz}{2t}} J_\nu'\Big(\frac{xz}{2t}\Big), \tag{7.27b}$$

$$\frac{\partial}{\partial t}\widetilde{G}(t,x,z) = \Big(-\frac{1}{t} - \frac{ixz}{2t^2}\Big)\widetilde{G}(t,x,z) - \frac{\Theta(\pm x)(xz)^{\frac{3}{2}}}{4i^{\nu+1}t^3} e^{i\frac{xz}{2t}} J_\nu'\Big(\frac{xz}{2t}\Big), \tag{7.27c}$$

where for the second spatial derivative we used the differential equation (A.35a). Hence $\widetilde{G}(t,x,z)$ is a solution of

$$i\frac{\partial}{\partial t}\widetilde{G}(t,x,z) = \Big(-\frac{\partial^2}{\partial x^2} + \frac{x-z}{it}\frac{\partial}{\partial x} + \frac{1}{2it} + \frac{\nu^2-\frac{1}{4}}{x^2}\Big)\widetilde{G}(t,x,z),$$

which is exactly (4.11) after using $\nu^2 - \frac{1}{4} = \lambda$. Moreover, the boundary values of G in (5.5) follow from the fact, that $J_\nu(z) \overset{z\to 0}{\longrightarrow} 0$ and are given by

$$G(t,0^{\pm},y) = \frac{\Theta(\pm y)}{2i^{\nu+1}t} e^{-\frac{y^2}{4it}} \lim_{x\to 0^{\pm}} \sqrt{xy}\,J_\nu\Big(\frac{xy}{2t}\Big) = 0, \quad t > 0,\ y \in \dot{\mathbb{R}}.$$

(ii) In order to check the initial condition (5.6), we fix $x \in \dot{\mathbb{R}}$ and without loss of generality we will only consider $x > 0$. The calculations for $x < 0$ are the same. Let now $x_0 > x$ be arbitrary, $F \in \mathcal{H}(\mathbb{C})$ and consider the function

$$\Psi_0(t,x) := \int_{-x_0}^{x_0} G(t,x,y)F(y)dy = \frac{1}{2i^{\nu+1}t}\int_0^{x_0} \sqrt{xy}\, e^{-\frac{x^2+y^2}{4it}} J_\nu\Big(\frac{xy}{2t}\Big)F(y)dy,$$

as well as, motivated by (A.54), the approximated function

$$\widetilde{\Psi}_0(t,x) := \frac{1}{i^{\nu+1}\sqrt{\pi t}}\int_0^{x_0} e^{-\frac{x^2+y^2}{4it}} \cos\Big(\frac{xy}{2t} - \frac{(2\nu+1)\pi}{4}\Big)F(y)dy.$$

In the inequality (A.50), we can interpolate between $|z|^{-\frac{1}{2}}$ and $|z|^{-\frac{3}{2}}$ to get an inequality of the form

$$\Big|J_\nu(z) - \frac{\sqrt{2}}{\sqrt{\pi z}}\cos\Big(z - \frac{(2\nu+1)\pi}{4}\Big)\Big| \le \frac{\widetilde{E}_\nu}{|z|^{\frac{5}{4}}} e^{|\operatorname{Im}(z)|}, \quad \operatorname{Re}(z) > 0.$$

With this one, we can now estimate the error of the approximative function by

$$\begin{aligned}
|\Psi_0(t,x) - \widetilde{\Psi}_0(t,x)| &\le \int_0^{x_0} \frac{\sqrt{xy}}{2t}\Big|J_\nu\Big(\frac{xy}{2t}\Big) - \frac{2\sqrt{t}}{\sqrt{\pi xy}}\cos\Big(\frac{xy}{2t} - \frac{(2\nu+1)\pi}{4}\Big)\Big||F(y)|dy \\
&\le \widetilde{E}_\nu (2t)^{\frac{1}{4}} \int_0^{x_0} \frac{1}{(xy)^{\frac{3}{4}}}|F(y)|dy \\
&\le \frac{4\widetilde{E}_\nu (2tx_0)^{\frac{1}{4}}}{x^{\frac{3}{4}}}\|F|_{[0,x_0]}\|_\infty.
\end{aligned}$$

Since the right hand side converges to zero as $t \to 0^+$, we get

$$\lim_{t\to 0^+} \Psi_0(t,x) = \lim_{t\to 0^+} \widetilde{\Psi}_0(t,x), \tag{7.28}$$

and we reduced the problem (5.6) to the one of the initial value of $\widetilde{\Psi}_0(t,x)$. Writing the cosine as an exponential function we can split up the integral as

$$\begin{aligned}
\widetilde{\Psi}_0(t,x) &= \frac{1}{2i^{\nu+1}\sqrt{\pi t}}\int_0^{x_0} e^{-\frac{x^2+y^2}{4it}}\Big(e^{i\frac{xy}{2t} - i\frac{(2\nu+1)\pi}{4}} + e^{-i\frac{xy}{2t} + i\frac{(2\nu+1)\pi}{4}}\Big)F(y)dy \\
&= \frac{1}{2\sqrt{i\pi t}}\int_0^{x_0}\Big((-1)^{\nu+\frac{1}{2}} e^{-\frac{(x+y)^2}{4it}} + e^{-\frac{(x-y)^2}{4it}}\Big)F(y)dy.
\end{aligned}$$

Using the derivative $\frac{d}{dz}\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}e^{-z^2}$ of the error function and applying integration by parts, one can rewrite this integral as

$$\begin{aligned}
\widetilde{\Psi}_0(t,x) &= \frac{1}{2}\int_0^{x_0} \frac{d}{dy}\Big((-1)^{\nu+\frac{1}{2}}\operatorname{erf}\Big(\frac{x+y}{2\sqrt{it}}\Big) - \operatorname{erf}\Big(\frac{x-y}{2\sqrt{it}}\Big)\Big)F(y)dy \\
&= \frac{(-1)^{\nu+1}}{2}\Big(\operatorname{erf}\Big(\frac{x+x_0}{2\sqrt{it}}\Big)F(x_0) - \operatorname{erf}\Big(\frac{x}{2\sqrt{it}}\Big)F(0) - \int_0^{x_0}\operatorname{erf}\Big(\frac{x+y}{2\sqrt{it}}\Big)F'(y)dy\Big) \\
&\quad - \frac{1}{2}\Big(\operatorname{erf}\Big(\frac{x-x_0}{2\sqrt{it}}\Big)F(x_0) - \operatorname{erf}\Big(\frac{x}{2\sqrt{it}}\Big)F(0) - \int_0^{x_0}\operatorname{erf}\Big(\frac{x-y}{2\sqrt{it}}\Big)F'(y)dy\Big).
\end{aligned}$$

Applying now the limit $t \to 0^+$ and carrying it inside the integral is allowed since the integrand is uniformly bounded. Using also $0 < x < x_0$ as well as the limit $\lim_{\xi\to\pm\infty} \operatorname{erf}(\frac{\xi}{\sqrt{i}}) = \pm 1$ of the error function gives the initial value

$$\begin{aligned}\lim_{t\to 0^+} \widetilde{\Psi}_0(t,x) = \frac{(-1)^{\nu+1}}{2}\Big(\varphi(x_0) - \varphi(0) - \int_0^{x_0} \varphi'(y)dy\Big) \\ - \frac{1}{2}\Big(- \varphi(x_0) - \varphi(0) - \int_0^{x_0} \operatorname{sgn}(x-y)\varphi'(y)dy\Big) = \varphi(x).\end{aligned}$$

Together with (7.28) this proves the initial value (5.6).

(iii) Using (A.49), the absolute value of $\widetilde{G}$ can be estimated by

$$\begin{aligned}|\widetilde{G}(t,x,z)| &\le \frac{\Theta(\pm x)\sqrt{|xz|}}{2t} e^{-\frac{x\operatorname{Im}(z)}{2t}} \Big| J_\nu\Big(\frac{xz}{2t}\Big)\Big| \\ &\le \frac{C_\nu \Theta(\pm x)}{\sqrt{2t}} e^{-\frac{x\operatorname{Im}(z)}{2t}} e^{|\frac{x\operatorname{Im}(z)}{2t}|} \le \frac{C_\nu}{\sqrt{2t}}, \quad \pm z \in S_\alpha^+,\end{aligned}$$

where we used that $|x\operatorname{Im}(z)| = x\operatorname{Im}(z)$ since $\pm z \in S_\alpha^+$ and $\pm x > 0$. Hence $\widetilde{G}$ satisfies the bound (5.8a) with the coefficients

$$A_0(t,x) = \frac{C_\nu}{2\sqrt{2t}} \quad \text{and} \quad B_0(t,x) = 0. \tag{7.29}$$

These coefficients are bound as $x \to 0^\pm$ and also $\frac{A_0}{\sqrt{a}}$ and B_0 are bounded as $t \to 0^+$, as requested in (5.9). Due to the explicit form (7.27) of the derivatives of $\widetilde{G}$ and the additional estimate (A.49) of the derivative of the Bessel function, one immediately sees, that also the exponential bounds (5.8b) and (5.8c) are satisfied.

Hence we also proved (i), namely that the Green's function (7.21) of the repulsive centrifugal potential satisfies Assumption 5.1. □

7.6. Arbitrary point interactions

In this section we consider the classical potential $V(t,x) = 0$ on $(0,\infty)\times\dot{\mathbb{R}}$, and allow all possible self-adjoint singular interactions at the origin. In particular the Dirac δ- and δ'-potential or boundary conditions of Dirichlet-, Neumann- or Robin-type are included. In a mathematical rigorous way, those distributional potentials manifest themselves as interface conditions at the point of interaction $x = 0$.

There are various ways to describe the complete family of self-adjoint interface conditions, see for example [54, 59, 62, 77, 105, 106], but for our purposes it is convenient to use the one from [37, Chapter 2.2], namely

$$(I - J)\begin{pmatrix} \Psi(t,0^+) \\ \Psi(t,0^-)\end{pmatrix} = i(I+J)\begin{pmatrix} \frac{\partial}{\partial x}\Psi(t,0^+) \\ -\frac{\partial}{\partial x}\Psi(t,0^-)\end{pmatrix} \tag{7.30}$$

where I is the 2×2 identity matrix and J is some arbitrary 2×2 unitary matrix, see (7.32). The class of interface conditions (7.30) coincides with the class of self-adjoint interface conditions at $x = 0$. In other words, each unitary matrix $J \in \mathbb{C}^{2\times 2}$ leads to a self-adjoint realization of the Laplacian in $L^2(\mathbb{R})$ with a generalized point interaction supported at $x = 0$, and conversely, for each self-adjoint Laplacian with a generalized point interaction there exists a unitary matrix $J \in \mathbb{C}^{2\times 2}$ such that the interface condition (7.30) is satisfied.

For an arbitrary point interaction (7.30), we can now use this function Λ from (A.16) to write the Green's function as the linear combination

$$\begin{aligned} G(t,x,y) :=& \mu_+^{(x,y)} \Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}} + \omega_+ \sqrt{it}\Big) e^{-\frac{(|x|+|y|)^2}{4it}} \\ &+ \mu_-^{(x,y)} \Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}} + \omega_- \sqrt{it}\Big) e^{-\frac{(|x|+|y|)^2}{4it}} \\ &+ \frac{\mu_0^{(x,y)}}{2\sqrt{i\pi t}} e^{-\frac{(|x|+|y|)^2}{4it}} + \frac{1}{2\sqrt{i\pi t}} e^{-\frac{(x-y)^2}{4it}}, \quad t > 0,\ x, y \in \dot{\mathbb{R}}. \end{aligned} \tag{7.31}$$

The value of the coefficients $\mu_\pm^{(x,y)}$, $\mu_0^{(x,y)}$ and $\omega_\pm$ will be specified in terms of the unitary matrix J in the following. First, we note that any 2×2-unitary matrix can be represented as

$$J = e^{i\phi} \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \tag{7.32}$$

with parameters $\phi \in [0, \pi)$ and $\alpha, \beta \in \mathbb{C}$ satisfying $|\alpha|^2 + |\beta|^2 = 1$. Moreover, it is convenient to use

$$\eta^{(x,y)} := \frac{1}{\sqrt{1-\operatorname{Re}(\alpha)^2}} \begin{cases} -\operatorname{Im}(\alpha), & \text{if } x, y > 0, \\ -i\overline{\beta}, & \text{if } x > 0,\ y < 0, \\ i\beta, & \text{if } x < 0,\ y > 0, \\ \operatorname{Im}(\alpha), & \text{if } x, y < 0, \end{cases} \qquad \text{if } |\operatorname{Re}(\alpha)| \neq 1, \tag{7.33a}$$

$$\eta^{(x,y)} := 0, \qquad \text{if } |\operatorname{Re}(\alpha)| = 1. \tag{7.33b}$$

In order to define now the coefficients of the Green's function (7.31) we distinguish three different cases.

Case I: If $\operatorname{Re}(\alpha) \neq -\cos(\phi)$, then

$$\omega_\pm = \frac{-\sin(\phi) \pm \sqrt{1-\operatorname{Re}(\alpha)^2}}{\cos(\phi) + \operatorname{Re}(\alpha)}, \quad \mu_\pm^{(x,y)} = \frac{\omega_\pm}{2}\big(\Theta(xy) + \eta^{(x,y)}\big), \quad \mu_0^{(x,y)} = \operatorname{sgn}(xy).$$

Case II: If $\operatorname{Re}(\alpha) = -\cos(\phi) \neq -1$, then $\omega_- = \mu_-^{(x,y)} = 0$ and

$$\omega_+ = \cot(\phi), \quad \mu_+^{(x,y)} = -\frac{\omega_+}{2}\big(\Theta(xy) + \eta^{(x,y)}\big), \quad \mu_0^{(x,y)} = \eta^{(x,y)} - \Theta(-xy).$$

Case III: If $\operatorname{Re}(\alpha) = -\cos(\phi) = -1$, then $\omega_\pm = \mu_\pm^{(x,y)} = 0$ and $\mu_0^{(x,y)} = -1$.

These three cases correspondent to the rank of the matrix $I + J$ on the right hand side of the interface condition (7.30). More precisely, in Case I we have $\operatorname{rank}(I + J) = 2$, in Case II we have $\operatorname{rank}(I + J) = 1$ and in Case III we have $\operatorname{rank}(I + J) = 0$.

Theorem 7.6. The Green's function (7.31) satisfies Assumption 5.1 with the vanishing potential $V(t,x)=0$ and the transition matrices $M=I+J$ and $N=i(I+J)$.

Before we do the actual proof of Theorem 7.6, we first verify the transmission condition (5.12) separately.

Lemma 7.7. With the coefficients $\mu_0^{(x,y)}$, $\mu_\pm^{(x,y)}$ and $\omega_\pm$ specified as above, the Green's function G from (7.31) satisfies the transmission condition

$$(I-J)\begin{pmatrix} G(t,0^+,y) \\ G(t,0^-,y) \end{pmatrix} = i(I+J)\begin{pmatrix} \frac{\partial}{\partial x}G(t,0^+,y) \\ -\frac{\partial}{\partial x}G(t,0^-,y) \end{pmatrix}, \quad t>0,\ y\in\dot{\mathbb{R}}. \tag{7.34}$$

Proof. Using the derivative (A.17) of the function Λ, the spatial derivative of the function G is given by

$$\begin{aligned}\frac{\partial}{\partial x}G(t,x,y) &= \mu_+^{(x,y)}\operatorname{sgn}(x)\Big(\omega_+\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}}+\omega_+\sqrt{it}\Big)-\frac{1}{\sqrt{i\pi t}}\Big)e^{-\frac{(|x|+|y|)^2}{4it}}\\ &\quad+\mu_-^{(x,y)}\operatorname{sgn}(x)\Big(\omega_-\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}}+\omega_-\sqrt{it}\Big)-\frac{1}{\sqrt{i\pi t}}\Big)e^{-\frac{(|x|+|y|)^2}{4it}}\\ &\quad-\frac{1}{4it\sqrt{i\pi t}}\Big(\mu_0^{(x,y)}\operatorname{sgn}(x)\big(|x|+|y|\big)e^{-\frac{(|x|+|y|)^2}{4it}}+(x-y)e^{-\frac{(x-y)^2}{4it}}\Big).\end{aligned}$$

For the interface condition (7.34) we have to evaluate G and $\frac{\partial}{\partial x}G$ at $x=0^\pm$. This will be done in a vector form, where the first entry is the limit $x=0^+$ and the second entry the limit $x=0^-$. We have

$$\begin{aligned}\begin{pmatrix} G(t,0^+,y) \\ G(t,0^-,y) \end{pmatrix} &= \left(\begin{pmatrix} \mu_+^{(+,y)} \\ \mu_+^{(-,y)} \end{pmatrix}\Lambda\left(\frac{|y|}{2\sqrt{it}}+\omega_+\sqrt{it}\right)+\begin{pmatrix} \mu_-^{(+,y)} \\ \mu_-^{(-,y)} \end{pmatrix}\Lambda\Big(\frac{|y|}{2\sqrt{it}}+\omega_-\sqrt{it}\Big)\right.\\ &\quad\left.+\frac{1}{2\sqrt{i\pi t}}\begin{pmatrix} \mu_0^{(+,y)}+1 \\ \mu_0^{(-,y)}+1 \end{pmatrix}\right)e^{-\frac{y^2}{4it}},\\ \begin{pmatrix} \frac{\partial}{\partial x}G(t,0^+,y) \\ -\frac{\partial}{\partial x}G(t,0^-,y) \end{pmatrix} &\\ =\left(\begin{pmatrix} \mu_+^{(+,y)} \\ \mu_+^{(-,y)} \end{pmatrix}\omega_+\Lambda\Big(\frac{|y|}{2\sqrt{it}}+\omega_+\sqrt{it}\Big)\right.&+\begin{pmatrix} \mu_-^{(+,y)} \\ \mu_-^{(-,y)} \end{pmatrix}\omega_-\Lambda\Big(\frac{|y|}{2\sqrt{it}}+\omega_-\sqrt{it}\Big)\\ &\quad\left.-\frac{1}{\sqrt{i\pi t}}\begin{pmatrix} \mu_+^{(+,y)}+\mu_-^{(+,y)} \\ \mu_+^{(-,y)}+\mu_-^{(-,y)} \end{pmatrix}-\frac{|y|}{4it\sqrt{i\pi t}}\begin{pmatrix} \mu_0^{(+,y)}-\operatorname{sgn}(y) \\ \mu_0^{(-,y)}+\operatorname{sgn}(y) \end{pmatrix}\right)e^{-\frac{y^2}{4it}},\end{aligned}$$

and since (7.34) has to be satisfied for all $y\in\dot{\mathbb{R}}$, it suffices to compare and match the coefficients corresponding to the terms

$$\Lambda\left(\frac{|y|}{2\sqrt{it}}+\omega_\pm\sqrt{it}\right),\quad \frac{1}{2\sqrt{i\pi t}},\quad\text{and}\quad\frac{|y|}{4it\sqrt{i\pi t}}.$$

This leads to the following four equations

$$
\begin{aligned}
(\mathrm{A}_\pm):\ (I-J)\begin{pmatrix}\mu_\pm^{(+,y)}\\ \mu_\pm^{(-,y)}\end{pmatrix} &= i\omega_\pm (I+J)\begin{pmatrix}\mu_\pm^{(+,y)}\\ \mu_\pm^{(-,y)}\end{pmatrix},\\
(\mathrm{B})\ :\ (I-J)\begin{pmatrix}\mu_0^{(+,y)}+1\\ \mu_0^{(-,y)}+1\end{pmatrix} &= -2i(I+J)\begin{pmatrix}\mu_+^{(+,y)}+\mu_-^{(+,y)}\\ \mu_+^{(-,y)}+\mu_-^{(-,y)}\end{pmatrix},\\
(\mathrm{C})\ :\ \begin{pmatrix}0\\0\end{pmatrix} &= (I+J)\begin{pmatrix}\mu_0^{(+,y)}-\operatorname{sgn}(y)\\ \mu_0^{(-,y)}+\operatorname{sgn}(y)\end{pmatrix}.
\end{aligned}
$$

Since the variable y only appears as $\operatorname{sgn}(y)$ each equation splits up in one for $y>0$ and one for $y<0$. We will consider this by writing ($\mathrm{A}_\pm$), (B), and (C) as matrix equations, where the first column is for $y>0$ and the second column for $y<0$. For a shorter notation we will use the matrices

$$
\mathbb{1} := \begin{pmatrix}1&1\\1&1\end{pmatrix},\quad N := \begin{pmatrix}\eta^{(+,+)}&\eta^{(+,-)}\\ \eta^{(-,+)}&\eta^{(-,-)}\end{pmatrix},\quad M_j := \begin{pmatrix}\mu_j^{(+,+)}&\mu_j^{(+,-)}\\ \mu_j^{(-,+)}&\mu_j^{(-,-)}\end{pmatrix}, \tag{7.35}
$$

where $j\in\{0,\pm\}$. Note that the matrix N satisfies the identity

$$
\sqrt{1-\operatorname{Re}(\alpha)^2}\,N = \begin{pmatrix}-\operatorname{Im}(\alpha)&-i\bar\beta\\ i\beta&\operatorname{Im}(\alpha)\end{pmatrix} \tag{7.36}
$$

by (7.33a) for $|\operatorname{Re}(\alpha)|\neq 1$ and also for $|\operatorname{Re}(\alpha)|=1$, since then $\operatorname{Im}(\alpha)=\beta=0$ due to $|\alpha|^2+|\beta|^2=1$. From (7.36) and $|\alpha|^2+|\beta|^2=1$ it immediately follows that

$$
N^2 = \frac{1}{1-\operatorname{Re}(\alpha)^2}\begin{pmatrix}-\operatorname{Im}(\alpha)&-i\bar\beta\\ i\beta&\operatorname{Im}(\alpha)\end{pmatrix}^2 = \frac{\operatorname{Im}(\alpha)^2+|\beta|^2}{1-\operatorname{Re}(\alpha)^2}\,I = I,\quad \text{if } |\operatorname{Re}(\alpha)|\neq 1,
$$

and, consequently,

$$
(N+I)(N-I) = N^2-N+N-I = N^2-I = 0,\quad \text{if } |\operatorname{Re}(\alpha)|\neq 1, \tag{7.37}
$$

to which we will refer throughout the proof. With the help of the matrices (7.35) we now rewrite the equations ($\mathrm{A}_\pm$), (B), and (C) above in the matrix form

$$
\begin{aligned}
(\mathrm{A}_\pm):&\ (I-J)M_\pm = i\omega_\pm(I+J)M_\pm,\\
(\mathrm{B})\ :&\ (I-J)(M_0+\mathbb{1}) = -2i(I+J)(M_++M_-),\\
(\mathrm{C})\ :&\ 0 = (I+J)(M_0+\mathbb{1}-2I).
\end{aligned}
$$

Plugging in the matrix J from (7.32) and multiplying by $e^{-i\phi}$ these equations turn into

$$
(\mathrm{A}_\pm):\ \begin{pmatrix}e^{-i\phi}-\alpha&\bar\beta\\ -\beta&e^{-i\phi}-\bar\alpha\end{pmatrix}M_\pm = i\omega_\pm\begin{pmatrix}e^{-i\phi}+\alpha&-\bar\beta\\ \beta&e^{-i\phi}+\bar\alpha\end{pmatrix}M_\pm, \tag{7.38a}
$$

$$
(\mathrm{B})\ :\ \begin{pmatrix}e^{-i\phi}-\alpha&\bar\beta\\ -\beta&e^{-i\phi}-\bar\alpha\end{pmatrix}(M_0+\mathbb{1}) = -2i\begin{pmatrix}e^{-i\phi}+\alpha&-\bar\beta\\ \beta&e^{-i\phi}+\bar\alpha\end{pmatrix}(M_++M_-), \tag{7.38b}
$$

$$
(\mathrm{C})\ :\ 0 = \begin{pmatrix}e^{-i\phi}+\alpha&-\bar\beta\\ \beta&e^{-i\phi}+\bar\alpha\end{pmatrix}(M_0+\mathbb{1}-2I). \tag{7.38c}
$$

In the following we will discuss the three cases above Lemma 7.7 separately and verify that in each case with the proper choice of the coefficients $\omega_\pm$ and $\mu_\pm, \mu_0$ the equations ($\mathrm{A}_\pm$), (B) and (C) are satisfied; that is, the jump condition (7.34) holds.

Case I. Observe first that the equation (7.38c) is satisfied since $\mu_0^{(x,y)} = \mathrm{sgn}(xy)$ in this case, and hence we conclude $M_0 = 2I - \mathbb{1}$. Next we use $|\alpha|^2 + |\beta|^2 = 1$ to compute

$$\det \begin{pmatrix} e^{-i\phi} + \alpha & -\bar{\beta} \\ \beta & e^{-i\phi} + \bar{\alpha} \end{pmatrix} = 2e^{-i\phi}\big(\cos(\phi) + \mathrm{Re}(\alpha)\big) \neq 0,$$

where we also used the assumption $\mathrm{Re}(\alpha) \neq -\cos(\phi)$ in Case I. It follows that the matrix on the right hand side of ($\mathrm{A}_\pm$) and (B) is invertible with the inverse

$$\begin{pmatrix} e^{-i\phi} + \alpha & -\bar{\beta} \\ \beta & e^{-i\phi} + \bar{\alpha} \end{pmatrix}^{-1} = \frac{e^{i\phi}}{2(\cos(\phi) + \mathrm{Re}(\alpha))} \begin{pmatrix} e^{-i\phi} + \bar{\alpha} & \bar{\beta} \\ -\beta & e^{-i\phi} + \alpha \end{pmatrix},$$

and this leads to

$$\begin{aligned} &\begin{pmatrix} e^{-i\phi} + \alpha & -\bar{\beta} \\ \beta & e^{-i\phi} + \bar{\alpha} \end{pmatrix}^{-1} \begin{pmatrix} e^{-i\phi} - \alpha & \bar{\beta} \\ -\beta & e^{-i\phi} - \bar{\alpha} \end{pmatrix} \\ &\qquad = \frac{-i}{\cos(\phi) + \mathrm{Re}(\alpha)} \begin{pmatrix} \sin(\phi) + \mathrm{Im}(\alpha) & i\bar{\beta} \\ -i\beta & \sin(\phi) - \mathrm{Im}(\alpha) \end{pmatrix} \\ &\qquad = \frac{-i}{\cos(\phi) + \mathrm{Re}(\alpha)} \big(\sin(\phi)\, I - \sqrt{1 - \mathrm{Re}(\alpha)^2}\, N\big), \end{aligned}$$

where in the last line we used the identity (7.36). Hence the equations (7.38a) and (7.38b) turn into

$$\begin{aligned} (\mathrm{A}_\pm): \quad & \frac{\sin(\phi)\, I - \sqrt{1 - \mathrm{Re}(\alpha)^2}\, N}{\cos(\phi) + \mathrm{Re}(\alpha)} M_\pm = -\omega_\pm M_\pm, \\ (\mathrm{B}) \quad : \quad & \frac{\sin(\phi)\, I - \sqrt{1 - \mathrm{Re}(\alpha)^2}\, N}{\cos(\phi) + \mathrm{Re}(\alpha)} (M_0 + \mathbb{1}) = 2(M_+ + M_-). \end{aligned}$$

Using the explicit form $\omega_\pm = \frac{-\sin(\phi) \pm \sqrt{1 - \mathrm{Re}(\alpha)^2}}{\cos(\phi) + \mathrm{Re}(\alpha)}$ in ($\mathrm{A}_\pm$) and $M_0 = 2I - \mathbb{1}$ in (B) these equations reduce to

$$\begin{aligned} (\mathrm{A}_\pm): \quad & \sqrt{1 - \mathrm{Re}(\alpha)^2}(N \mp I) M_\pm = 0, \\ (\mathrm{B}) \quad : \quad & \frac{\sin(\phi)\, I - \sqrt{1 - \mathrm{Re}(\alpha)^2}\, N}{\cos(\phi) + \mathrm{Re}(\alpha)} = M_+ + M_-. \end{aligned}$$

Since we treat Case I we have $\mu_\pm^{(x,y)} = -\frac{\omega_\pm}{2}\big(\Theta(xy) \pm \eta^{(x,y)}\big)$ and from that we conclude

$$M_\pm = -\frac{\omega_\pm}{2}(I \pm N). \tag{7.39}$$

In particular, this yields

$$M_+ + M_- = -\frac{(\omega_+ + \omega_-)I + (\omega_+ - \omega_-)N}{2} = \frac{\sin(\phi) I - \sqrt{1 - \mathrm{Re}(\alpha)^2}\, N}{\cos(\phi) + \mathrm{Re}(\alpha)},$$

which shows that equation (B) is valid. It remains to check ($\mathrm{A}_\pm$). These equations are obviously valid if $|\operatorname{Re}(\alpha)| = 1$ and if $|\operatorname{Re}(\alpha)| \neq 1$ they follow from the identities (7.37) and (7.39).

Case II. Here we assume $\operatorname{Re}(\alpha) = -\cos(\phi) \neq -1$, which implies, in particular, $\phi \neq 0$ and consequently $\sin(\phi) \neq 0$. The matrices in the equations ($\mathrm{A}_\pm$), (B), and (C) in (7.38) now have the form

$$\begin{aligned}
\begin{pmatrix} e^{-i\phi} - \alpha & \bar{\beta} \\ -\beta & e^{-i\phi} - \bar{\alpha} \end{pmatrix} &= \big(2\cos(\phi) - i\sin(\phi)\big)I + i\begin{pmatrix} -\operatorname{Im}(\alpha) & -i\bar{\beta} \\ i\beta & \operatorname{Im}(\alpha) \end{pmatrix} \\
&= -i\sin(\phi)\big((2i\cot(\phi) + 1)I - N\big), \\
\begin{pmatrix} e^{-i\phi} + \alpha & -\bar{\beta} \\ \beta & e^{-i\phi} + \bar{\alpha} \end{pmatrix} &= -i\sin(\phi)I - i\begin{pmatrix} -\operatorname{Im}(\alpha) & -i\bar{\beta} \\ i\beta & \operatorname{Im}(\alpha) \end{pmatrix} \\
&= -i\sin(\phi)(I + N),
\end{aligned}$$

where in both cases we used (7.36) and $\sqrt{1 - \operatorname{Re}(\alpha)^2} = \sin(\phi)$, which is due to the assumption $\operatorname{Re}(\alpha) = -\cos(\phi)$ in Case II. Using this in (7.38) leads to

$$\begin{aligned}
(\mathrm{A}_\pm) &: \ \big((2i\cot(\phi) + 1)I - N\big)M_\pm = i\omega_\pm(I + N)M_\pm, \\
(\mathrm{B}) &: \ \big((2i\cot(\phi) + 1)I - N\big)(M_0 + \mathbb{1}) = -2i(I + N)(M_+ + M_-), \\
(\mathrm{C}) &: \ 0 = (I + N)(M_0 + \mathbb{1} - 2I).
\end{aligned}$$

Since in Case II we have $\mu_-^{(x,y)} = 0$, that is, $M_- = 0$, the equation (A_-) is trivially satisfied. Furthermore, with our choice $\omega_+ = \cot(\phi)$ the equation (A_+) reduces to

$$(\mathrm{A}_+): \quad (i\cot(\phi) + 1)(I - N)M_+ = 0.$$

By our choice of $\mu_+^{(x,y)}$ we have $M_+ = -\frac{\omega_+}{2}(I + N)$ as in the previous case (cf. (7.39)) and hence we conclude together with (7.37) that equation (A_+) is valid; note that we can apply (7.37) since $\operatorname{Re}(\alpha) \neq -1$ by the assumption in Case II and also because of $\operatorname{Re}(\alpha) = -\cos(\phi) \neq 1$ as $\phi \in [0, \pi)$. Next, we observe that also equation (C) holds by (7.37) and $\mu_0^{(x,y)} = \eta^{(x,y)} - \Theta(-xy)$, which gives $M_0 = N - \mathbb{1} + I$. In order to check (B), we plug in the above values for M_0 and $M_\pm$ and obtain

$$(\mathrm{B}): \quad \big(1 + i\cot(\phi)\big)(I - N)(N + I) = 0,$$

which holds by (7.37).

Case III. Here we assume $\operatorname{Re}(\alpha) = -\cos(\phi) = -1$ and hence $\operatorname{Im}(\alpha) = \beta = \phi = 0$ follows from the condition $|\alpha|^2 + |\beta|^2 = 1$. Therefore, the equations ($\mathrm{A}_\pm$), (B), and (C) in (7.38) have the particularly simple form

$$\begin{aligned}
(\mathrm{A}_\pm) &: \ 2M_\pm = 0, \\
(\mathrm{B}) &: \ M_0 + \mathbb{1} = 0, \\
(\mathrm{C}) &: \ 0 = 0,
\end{aligned}$$

and are all obviously satisfied by the definition of the coefficients in Case III. □

Once we verified the transmission condition of the Green's function, we are ready to prove Theorem 7.6.

Proof. First of all, we split up the Green's function (7.31) into the four parts

$$G(t,x,y) = \mu_+^{(x,y)} G_1(t,x,y;\omega_+) + \mu_-^{(x,y)} G_1(t,x,y;\omega_-) + \mu_0^{(x,y)} G_0(t,x,y) + G_{\text{free}}(t,x,y), \tag{7.40}$$

where

$$\begin{aligned} G_0(t,x,y) &\coloneqq \frac{1}{2\sqrt{i\pi t}} e^{-\frac{(|x|+|y|)^2}{4it}}, \\ G_1(t,x,y;\omega) &\coloneqq \Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}} + \omega\sqrt{it}\Big) e^{-\frac{(|x|+|y|)^2}{4it}}, \\ G_{\text{free}}(t,x,y) &\coloneqq \frac{1}{2\sqrt{i\pi t}} e^{-\frac{(x-y)^2}{4it}}. \end{aligned}$$

These functions then extend holomorphically to $\mathbb{C} \setminus i\mathbb{R}$ by

$$\begin{aligned} G_0(t,x,z) &\coloneqq \frac{1}{2\sqrt{i\pi t}} e^{-\frac{(|x|\pm z)^2}{4it}}, && \pm\operatorname{Re}(z) > 0, \\ G_1(t,x,z;\omega) &\coloneqq \Lambda\Big(\frac{|x|\pm z}{2\sqrt{it}} + \omega\sqrt{it}\Big) e^{-\frac{(|x|\pm z)^2}{4it}}, && \pm\operatorname{Re}(z) > 0, \\ G_{\text{free}}(t,x,z) &\coloneqq \frac{1}{2\sqrt{i\pi t}} e^{-\frac{(x-z)^2}{4it}}, && \operatorname{Re}(z) \neq 0. \end{aligned}$$

Moreover, the reduced Green's functions, according to $a(t) = \frac{1}{4t}$ in the decomposition (5.7), are given by

$$\widetilde{G}_0(t,x,z) = \frac{1}{2\sqrt{i\pi t}} e^{\frac{i\Theta(\pm x)xz}{t}}, \qquad \pm\operatorname{Re}(z) > 0, \tag{7.41a}$$

$$\widetilde{G}_1(t,x,z;\omega) \coloneqq \Lambda\Big(\frac{|x|\pm z}{2\sqrt{it}} + \omega\sqrt{it}\Big) e^{\frac{i\Theta(\pm x)xz}{t}}, \qquad \pm\operatorname{Re}(z) > 0, \tag{7.41b}$$

$$\widetilde{G}_{\text{free}}(t,x,z) \coloneqq \frac{1}{2\sqrt{i\pi t}}, \qquad \operatorname{Re}(z) \neq 0. \tag{7.41c}$$

We will now verify the properties (i)–(iii) of Assumption 5.1.

(i) In order to see, that G is a solution of the Schrödinger equation (5.4), it is equivalent to verify that $\widetilde{G}$ is a solution of (4.11). We will verify this for the functions $\widetilde{G}_0$, $\widetilde{G}_1$ and $\widetilde{G}_{\text{free}}$ separately by explicitly calculating its derivatives. For $\widetilde{G}_0$ they are given by

$$\frac{\partial}{\partial x}\widetilde{G}_0(t,x,z) = \frac{i\Theta(\pm x)z}{t}\widetilde{G}_0(t,x,z), \tag{7.42a}$$

$$\frac{\partial^2}{\partial x^2}\widetilde{G}_0(t,x,z) = -\frac{\Theta(\pm x)z^2}{t^2}\widetilde{G}_0(t,x,z), \tag{7.42b}$$

$$\frac{\partial}{\partial t}\widetilde{G}_0(t,x,z) = \Big(\frac{\Theta(\pm x)xz}{it^2} - \frac{1}{2t}\Big)\widetilde{G}_0(t,x,z). \tag{7.42c}$$

For $\widetilde{G}_1$, we use the derivative (A.17) of the function Λ and get

$$\frac{\partial}{\partial x}\widetilde{G}_1(t,x,z;\omega) = \operatorname{sgn}(x)\Big(\frac{|x|\pm z}{2it}+\omega\Big)\widetilde{G}_1(t,x,z;\omega) - \frac{\operatorname{sgn}(x)}{\sqrt{i\pi t}}, \tag{7.43a}$$

$$\frac{\partial^2}{\partial x^2}\widetilde{G}_1(t,x,z;\omega) = \Big(\Big(\frac{|x|\pm z}{2it}+\omega\Big)^2 + \frac{1}{2it}\Big)\widetilde{G}_1(t,x,z;\omega) - \frac{1}{\sqrt{i\pi t}}\Big(\frac{|x|\pm z}{2it}+\omega\Big), \tag{7.43b}$$

$$\frac{\partial}{\partial t}\widetilde{G}_1(t,x,z;\omega) = i\Big(\frac{(|x|\pm z)^2}{4t^2}+\omega^2\Big)\widetilde{G}_1(t,x,z;\omega) + \frac{i}{\sqrt{i\pi t}}\Big(\frac{|x|\pm z}{2it}-\omega\Big). \tag{7.43c}$$

Since the derivatives of $\widetilde{G}_{\text{free}}$ were already calculated in (7.2), we conclude that all three functions $\widetilde{G}_0$, $\widetilde{G}_1$ and $\widetilde{G}_{\text{free}}$ satisfy

$$i\frac{\partial}{\partial t}\widetilde{G}_j(t,x,z) = \Big(-\frac{\partial^2}{\partial x^2} + \frac{x-z}{it}\frac{\partial}{\partial x} + \frac{1}{2it}\Big)\widetilde{G}_j(t,x,z), \quad j \in \{0,1,\text{free}\},$$

which is exactly (4.11) for $a(t) = \frac{1}{4t}$ and $V(t,x) = 0$. Hence G_0, G_1 and G_{free} satisfy (5.4) and since the coefficients $\mu_0^{(x,y)}$ and $\mu_1^{(x,y)}$ are constant in x on each half line $x > 0$ and $x < 0$, the whole Green's function G is a solution of (5.4) as well.

The fact, that G satisfies the transmission condition (5.5) was proven in Lemma 7.7.

(ii) The initial condition (5.6) will again be proven for the three functions G_0, G_1 and G_{free} separately. Fix $x \in \dot{\mathbb{R}}$ and choose $x_0 > |x|$ arbitrary and $F \in \mathcal{H}(\mathbb{C})$. For G_{free}, we can use the derivative $\frac{d}{d\xi}\operatorname{erf}(\xi) = \frac{2}{\sqrt{\pi}}e^{-\xi^2}$ and apply integration by parts to write the integral as

$$\begin{aligned}\int_{-x_0}^{x_0} G_{\text{free}}(t,x,y)F(y)dy &= \frac{-1}{2}\int_{-x_0}^{x_0}\frac{d}{dy}\operatorname{erf}\Big(\frac{x-y}{2\sqrt{it}}\Big)F(y)dy\\ &= \frac{1}{2}\operatorname{erf}\Big(\frac{x+x_0}{2\sqrt{it}}\Big)F(-x_0) - \frac{1}{2}\operatorname{erf}\Big(\frac{x-x_0}{2\sqrt{it}}\Big)F(x_0) + \frac{1}{2}\int_{-x_0}^{x_0}\operatorname{erf}\Big(\frac{x-y}{2\sqrt{it}}\Big)F'(y)dy.\end{aligned}$$

Applying the limit $t \to 0^+$ and using the limit $\lim_{s\to\pm\infty}\operatorname{erf}(\frac{s}{\sqrt{i}}) = \pm 1$, we get the initial value

$$\lim_{t\to 0^+}\int_{-x_0}^{x_0} G_{\text{free}}(t,x,y)F(y)dy = \frac{F(-x_0)+F(x_0)}{2} + \int_{-x_0}^{x_0}\frac{\operatorname{sgn}(x-y)}{2}F'(y)dy = F(x). \tag{7.44}$$

For the initial value of G_0 we similarly write

$$\begin{aligned}\int_{-x_0}^{x_0} G_0(t,x,y)F(y)dy &= \frac{1}{2}\int_{-x_0}^{x_0}\frac{d}{dy}\operatorname{erf}\Big(\frac{|x|+|y|}{2\sqrt{it}}\Big)\operatorname{sgn}(y)F(y)dy\\ &= \frac{1}{2}\operatorname{erf}\Big(\frac{|x|+|x_0|}{2\sqrt{it}}\Big)F(x_0) - \operatorname{erf}\Big(\frac{|x|}{2\sqrt{it}}\Big)F(0) + \frac{1}{2}\operatorname{erf}\Big(\frac{|x|+|x_0|}{2\sqrt{it}}\Big)F(-x_0)\\ &\quad - \frac{1}{2}\int_{-x_0}^{x_0}\operatorname{erf}\Big(\frac{|x|+|y|}{2\sqrt{it}}\Big)\operatorname{sgn}(y)F'(y)dy.\end{aligned}$$

In the limit $t \to 0^+$ this equations now becomes

$$\lim_{t\to 0^+} \int_{-x_0}^{x_0} G_0(t,x,y)F(y)dy = \frac{F(x_0)}{2} - F(0) + \frac{F(-x_0)}{2} - \int_{-x_0}^{x_0} \frac{\operatorname{sgn}(y)}{2} F'(y)dy = 0. \tag{7.45}$$

Also for the function G_1 we get

$$\begin{aligned}
\int_{-x_0}^{x_0} & G_1(t,x,y;\omega)F(y)dy \\
&= \frac{1}{2}\int_{-x_0}^{x_0} \frac{d}{dy}\operatorname{erf}\Big(\frac{|x|+|y|}{2\sqrt{it}}\Big)\operatorname{sgn}(y)\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}} + \omega\sqrt{it}\Big)F(y)dy \\
&= \frac{1}{2}\operatorname{erf}\Big(\frac{|x|+|x_0|}{2\sqrt{it}}\Big)\Lambda\Big(\frac{|x|+|x_0|}{2\sqrt{it}} + \omega\sqrt{it}\Big)F(-x_0) \\
&\quad + \frac{1}{2}\operatorname{erf}\Big(\frac{|x|+|x_0|}{2\sqrt{it}}\Big)\Lambda\Big(\frac{|x|+|x_0|}{2\sqrt{it}} + \omega\sqrt{it}\Big)F(x_0) \\
&\quad - \operatorname{erf}\Big(\frac{|x|}{2\sqrt{it}}\Big)\Lambda\Big(\frac{|x|}{2\sqrt{it}} + \omega\sqrt{it}\Big)F(0) \\
&\quad - \frac{1}{4\sqrt{it}}\int_{-x_0}^{x_0} \operatorname{erf}\Big(\frac{|x|+|y|}{2\sqrt{it}}\Big)\Lambda'\Big(\frac{|x|+|y|}{2\sqrt{it}} + \omega\sqrt{it}\Big)F(y)dy \\
&\quad - \frac{1}{2}\int_{-x_0}^{x_0} \operatorname{sgn}(y)\operatorname{erf}\Big(\frac{|x|+|y|}{2\sqrt{it}}\Big)\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}} + \omega\sqrt{it}\Big)F'(y)dy.
\end{aligned}$$

Due to the asymptotics (A.22) and the derivative (A.17) we get the following asymptotics of the derivative of the function Λ

$$\Lambda'(z) = 2z\Lambda(z) - \frac{2}{\sqrt{\pi}} = 2z\Big(\frac{1}{\sqrt{\pi}z} + \mathcal{O}\Big(\frac{1}{|z|^2}\Big)\Big) - \frac{2}{\sqrt{\pi}} = \mathcal{O}\Big(\frac{1}{|z|}\Big),$$

as $z \to \infty$, $\operatorname{Re}(z) \geq 0$. Hence all terms in the above representation vanish in the limit $t \to 0^+$, which leads to the initial value

$$\lim_{t\to 0^+} \int_{-x_0}^{x_0} G_1(t,x,y;\omega)F(y)dy = 0. \tag{7.46}$$

Using now the three limits (7.44), (7.45) and (7.46) in the decomposition (7.40) gives the initial value (5.6).

(iii) In order to derive the estimate (5.8a) of the functions (7.41), we use the monotonicity and the estimate (A.21) of the function Λ to get for every $\pm z \in S_\alpha^+$,

$$\begin{aligned}
|\widetilde{G}_0(t,x,z)| &= \frac{1}{2\sqrt{\pi t}} e^{-\frac{\Theta(\pm x)x\operatorname{Im}(z)}{t}} \leq \frac{1}{2\sqrt{\pi t}}, \\
|\widetilde{G}_1(t,x,z;\omega)| &\leq \Lambda\Big(\frac{|x| \pm \operatorname{Re}(z) \pm \operatorname{Im}(z)}{2\sqrt{2t}} + \frac{\omega\sqrt{t}}{\sqrt{2}}\Big)e^{-\frac{\Theta(\pm x)x\operatorname{Im}(z)}{t}} \leq \Lambda\Big(\frac{\omega\sqrt{t}}{\sqrt{2}}\Big), \\
|\widetilde{G}_{\text{free}}(t,x,z)| &= \frac{1}{2\sqrt{\pi t}}.
\end{aligned}$$

Note, that for $\widetilde{G}_0$ and $\widetilde{G}_1$ we used that $\pm\operatorname{Re}(z) \pm \operatorname{Im}(z) \geq 0$, but also the fact that $\Theta(x)x\operatorname{Im}(z) \geq 0$ for all $z \in S_\alpha^+$ and also $\Theta(-x)x\operatorname{Im}(z) \geq 0$ for all $-z \in S_\alpha^+$. Hence the estimate (5.8a) is satisfied with

$$A_{0,0}(t,x) = \frac{1}{2\sqrt{\pi t}} \quad \text{and} \quad B_{0,0}(t,x) = 0,$$
$$A_{0,1}(t,x) = \Lambda\Big(\frac{\omega\sqrt{t}}{\sqrt{2}}\Big) \quad \text{and} \quad B_{0,1}(t,x) = 0,$$
$$A_{0,\mathrm{free}}(t,x) = \frac{1}{2\sqrt{\pi t}} \quad \text{and} \quad B_{0,\mathrm{free}}(t,x) = 0.$$

These coefficients are bounded as $x \to 0^\pm$ and also the boundedness (5.9) in the limit $t \to 0^+$ is satisfied. Moreover, the estimates of the first spatial derivatives (7.42a), (7.43a) and (7.2) are

$$\Big|\frac{\partial}{\partial x}\widetilde{G}_0(t,x,z)\Big| \leq \frac{|z|}{t} A_{0,0}(t,x),$$
$$\Big|\frac{\partial}{\partial x}\widetilde{G}_1(t,x,z;\omega)\Big| \leq \Big(\frac{|x|+|z|}{2t} + |\omega|\Big) A_{0,1}(t,x) + \frac{1}{\sqrt{\pi t}},$$
$$\Big|\frac{\partial}{\partial x}\widetilde{G}_{\mathrm{free}}(t,x,z)\Big| = 0.$$

If we additionally use that $|z| \leq e^{|z|-1}$, the estimates (5.8b) are satisfied with the coefficients

$$A_{1,0}(t,x) = \frac{A_{0,0}(t,x)}{et} \quad \text{and} \quad B_{1,0}(t,x) = 1,$$
$$A_{1,1}(t,x) = \Big(\frac{|x|+\frac{1}{e}}{2t} + |w|\Big) A_{0,1}(t,x) + \frac{1}{\sqrt{\pi t}} \quad \text{and} \quad B_{1,1}(t,x) = 1,$$
$$A_{1,\mathrm{free}}(t,x) = 0 \quad \text{and} \quad B_{1,\mathrm{free}}(t,x) = 0.$$

These coefficients are obviously bounded as $x \to 0^\pm$. Finally, also the second spatial and the time derivatives of $\widetilde{G}_0$, $\widetilde{G}_1$ and $\widetilde{G}_{\mathrm{free}}$ in (7.42), (7.43) and (7.2) are exponentially bounded as claimed in (5.8c).

Hence the Green's function indeed satisfies Assumption 5.1 and Theorem 7.6 is proven. □

Next we want to have a closer look at certain special point interactions and derive the explicit form of the corresponding Green's function in those cases. As an almost trivial case we start with the free particle in Section 7.6.1, discuss the well-known δ and δ'-interactions afterwards in Section 7.6.2 and Section 7.6.3, and in the Sections 7.6.4–7.6.6 we treat decoupled systems with Dirichlet, Neumann, and Robin boundary conditions at the origin, and also show in Section 7.6.7 that these are the only decoupled systems. In each of the examples we first provide the corresponding matrix J for the interface condition (7.30) with parameters ϕ, α, β of the matrix J in (7.32). Then we determine which of the Cases I–III above Lemma 7.7 appears, and finally we compute the coefficients in the Green's function (7.31). The special Green's functions in this section are well known from the mathematical and physical literature.

7.6.1. Free particle

We start with the easiest case, the free particle, where there is no interaction at $x = 0$ happening. This means that both the wave function as well as its derivative are continuous, i.e.

$$\Psi(t, 0^-) = \Psi(t, 0^+) \quad \text{and} \quad \frac{\partial}{\partial x}\Psi(t, 0^-) = \frac{\partial}{\partial x}\Psi(t, 0^+), \quad t > 0.$$

These continuity conditions are described in (7.30) with the unitary matrix

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This matrix is of the form (7.32) with $\alpha = 0$, $\beta = -i$ and $\phi = \frac{\pi}{2}$. In this situation the coefficient $\eta^{(x,y)}$ in (7.33a) is

$$\eta^{(x,y)} = \begin{cases} 0, & \text{if } x, y > 0, \\ 1, & \text{if } x > 0,\ y < 0, \\ 1, & \text{if } x < 0,\ y > 0, \\ 0, & \text{if } x, y < 0, \end{cases} = \Theta(-xy).$$

Since we are in Case II the coefficients of the corresponding Green function in (7.31) have the explicit form

$$\begin{aligned} \omega_- &= 0, & \mu_-^{(x,y)} &= 0, \\ \omega_+ = \cot\left(\frac{\pi}{2}\right) &= 0, & \mu_+^{(x,y)} &= -\frac{\omega_+}{2}\left(\Theta(xy) + \eta^{(x,y)}\right) = 0, \\ & & \mu_0^{(x,y)} &= \eta^{(x,y)} - \Theta(-xy) = 0. \end{aligned}$$

Therefore, the Green's function of the free particle is given by

$$G(t, x, y) = \frac{1}{2\sqrt{i\pi t}} e^{-\frac{(x-y)^2}{4it}},$$

which clearly coincided with the Green's function (7.1) of the free particle on the whole real line.

7.6.2. δ-potential

In the next example we treat the classical δ-point interaction located at the origin. Such singular potentials were studied intensively in the mathematical and physical literature; we refer the interested reader to the standard monograph [31] for a detailed treatment and further references. The particular Green's function that appears below can also be found (sometimes in a slightly different form) in the papers [58, 86, 98].

We consider the standard δ-interaction of strength $2c \in \mathbb{R} \setminus \{0\}$ located at the point $x = 0$. This situation is described by the formal Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(t, x) = \Big(-\frac{\partial^2}{\partial x^2} + 2c\delta(x)\Big)\Psi(t, x), \quad t > 0,\ x \in \mathbb{R},$$

and is made mathematically rigorous in the form

$$
\begin{aligned}
i\frac{\partial}{\partial t}\Psi(t,x) &= -\frac{\partial^2}{\partial x^2}\Psi(t,x), && t>0,\ x\in\dot{\mathbb{R}}, && \text{(7.47a)}\\
\Psi(t,0^+) &= \Psi(t,0^-), && t>0, && \text{(7.47b)}\\
\frac{\partial}{\partial x}\Psi(t,0^+)-\frac{\partial}{\partial x}\Psi(t,0^-) &= 2c\,\Psi(t,0^\pm), && t>0. && \text{(7.47c)}
\end{aligned}
$$

The jump condition (7.47b)–(7.47c) is realized in (7.30) by using the matrix

$$J = \frac{1}{i-c}\begin{pmatrix} c & i \\ i & c \end{pmatrix}.$$

In fact, with this choice of J and multiplication by $(c-i)$ the condition (7.30) reads as

$$\begin{pmatrix} 2c-i & i \\ i & 2c-i \end{pmatrix}\begin{pmatrix} \Psi(t,0^+) \\ \Psi(t,0^-) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x}\Psi(t,0^+) \\ -\frac{\partial}{\partial x}\Psi(t,0^-) \end{pmatrix},$$

or, more explicitly, we have the two equations

$$
\begin{aligned}
(2c-i)\Psi(t,0^+)+i\Psi(t,0^-) &= \frac{\partial}{\partial x}\Psi(t,0^+)-\frac{\partial}{\partial x}\Psi(t,0^-),\\
i\Psi(t,0^+)+(2c-i)\Psi(t,0^-) &= \frac{\partial}{\partial x}\Psi(t,0^+)-\frac{\partial}{\partial x}\Psi(t,0^-).
\end{aligned}
$$

By subtracting these equations from each other we first conclude (7.47b) and adding the equations leads to (7.47c). In order to write the matrix J in the form (7.32), we choose $\phi\in(0,\pi)$ such that $\cot(\phi)=c$. Next we set $\alpha=-\cos(\phi)$ and $\beta=-i\sin(\phi)$. It follows in particular, that

$$\cos(\phi)=\frac{c}{\sqrt{1+c^2}} \quad \text{and} \quad \sin(\phi)=\frac{1}{\sqrt{1+c^2}},$$

and therefore

$$e^{i\phi}\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} = \frac{1}{i-c}\begin{pmatrix} c & i \\ i & c \end{pmatrix} = J.$$

Plugging these values in (7.33a) gives

$$\eta^{(x,y)} = \begin{cases} 0, & \text{if } x,y>0,\\ 1, & \text{if } x>0,\ y<0,\\ 1, & \text{if } x<0,\ y>0,\\ 0, & \text{if } x,y<0, \end{cases} = \Theta(-xy),$$

and since we are in Case II, the coefficients of the Green's function are

$$
\begin{aligned}
\omega_- &= 0, & \mu_-^{(x,y)} &= 0,\\
\omega_+ &= \cot(\phi)=c, & \mu_+^{(x,y)} &= -\frac{c}{2}\big(\Theta(xy)+\Theta(-xy)\big) = -\frac{c}{2},\\
& & \mu_0^{(x,y)} &= \Theta(-xy)-\Theta(-xy)=0.
\end{aligned}
$$

With these quantities we conclude that the Green's function (7.31) of the δ-potential is given by

$$G(t,x,y) = -\frac{c}{2}\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}}+c\sqrt{it}\Big)e^{-\frac{(|x|+|y|)^2}{4it}}+\frac{1}{2\sqrt{i\pi t}}e^{-\frac{(x-y)^2}{4it}}.$$

7.6.3. δ'-potential

Now consider the δ'-interaction of strength $\frac{2}{c} \in \mathbb{R} \setminus \{0\}$ located at the point $x = 0$, which is another popular singular potential that appears in various situations. The formal Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi(t,x) = \Big(-\frac{\partial^2}{\partial x^2} + \frac{2}{c}\delta'(x)\Big)\Psi(t,x), \quad t > 0,\, x \in \mathbb{R},$$

which in a mathematically rigorous form reads as

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(t,x) &= -\frac{\partial^2}{\partial x^2}\Psi(t,x), && t > 0,\, x \in \dot{\mathbb{R}}, \\ \frac{\partial}{\partial x}\Psi(t,0^+) &= \frac{\partial}{\partial x}\Psi(t,0^-), && t > 0, \\ \Psi(t,0^+) - \Psi(t,0^-) &= \frac{2}{c}\frac{\partial}{\partial x}\Psi(t,0), && t > 0. \end{aligned}$$

One verifies in a similar way as for the δ-potential that the jump conditions are realized in (7.30) by the matrix

$$J = \frac{1}{i-c}\begin{pmatrix} i & -c \\ -c & i \end{pmatrix}.$$

This matrix is of the form (7.32) if we choose $\phi \in (0,\pi) \setminus \{\frac{\pi}{2}\}$ such that $\tan(\phi) = -c$ and set $\alpha = \cos(\phi)$ and $\beta = -i\sin(\phi)$. The coefficient $\eta^{(x,y)}$ in (7.33a) then becomes

$$\eta^{(x,y)} = \begin{cases} 0, & \text{if } x,y > 0, \\ 1, & \text{if } x > 0,\, y < 0, \\ 1, & \text{if } x < 0,\, y > 0, \\ 0, & \text{if } x,y < 0, \end{cases} = \Theta(-xy),$$

and since we are in Case I the coefficients of the Green's function are

$$\begin{aligned} \omega_- &= -\tan(\phi) = c, & \mu_-^{(x,y)} &= -\frac{c}{2}\big(\Theta(xy) - \Theta(-xy)\big) = -\frac{c\,\mathrm{sgn}(xy)}{2}, \\ \omega_+ &= 0, & \mu_+^{(x,y)} &= 0, \\ & & \mu_0^{(x,y)} &= \mathrm{sgn}(xy). \end{aligned}$$

It follows that the Green's function (7.31) of the δ'-potential is given by

$$\begin{aligned} G(t,x,y) = &-\frac{c\,\mathrm{sgn}(xy)}{2}\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}} + c\sqrt{it}\Big)e^{-\frac{(|x|+|y|)^2}{4it}} \\ &+ \frac{1}{2\sqrt{i\pi t}}\Big(\mathrm{sgn}(xy)e^{-\frac{(|x|+|y|)^2}{4it}} + e^{-\frac{(x-y)^2}{4it}}\Big). \end{aligned}$$

7.6.4. Dirichlet boundary conditions

Now we turn to generalized point interactions that lead to decoupled systems. In the following examples we discuss Dirichlet, Neumann, and Robin boundary conditions at the origin. For a characterization of all decoupled systems see also Section 7.6.7.

We consider the free Schrödinger equation on the two half lines $\dot{\mathbb{R}}$ with Dirichlet boundary conditions

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(t,x) &= -\frac{\partial^2}{\partial x^2}\Psi(t,x), && t>0,\ x\in\dot{\mathbb{R}}, \\ \Psi(t,0^+) = \Psi(t,0^-) &= 0, && t>0. \end{aligned}$$

These boundary conditions are realized in (7.30) using the matrix

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7.48}$$

that is, with the coefficients $\phi = 0$, $\alpha = -1$ and $\beta = 0$ in (7.32). Hence, Case III applies and the coefficients of the Green's function are given by

$$\omega_\pm = 0, \quad \mu_\pm^{(x,y)} = 0, \quad \text{and} \quad \mu_0^{(x,y)} = -1.$$

This leads to the Green's function

$$G(t,x,y) = \frac{1}{2\sqrt{i\pi t}}\Big(e^{-\frac{(x-y)^2}{4it}} - e^{-\frac{(|x|+|y|)^2}{4it}}\Big). \tag{7.49}$$

7.6.5. Neumann boundary conditions

We consider the free Schrödinger equation on the two half lines $\dot{\mathbb{R}}$ with Neumann boundary conditions

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi(t,x) &= -\frac{\partial^2}{\partial x^2}\Psi(t,x), && t>0,\ x\in\dot{\mathbb{R}}, \\ \frac{\partial}{\partial x}\Psi(t,0^+) = \frac{\partial}{\partial x}\Psi(t,0^-) &= 0, && t>0. \end{aligned}$$

These boundary conditions are realized in (7.30) using the matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.50}$$

that is, with the coefficients $\phi = 0$, $\alpha = 1$ and $\beta = 0$ in (7.32). Hence, Case I applies and the coefficients of the Green's function are given by

$$\omega_\pm = 0, \quad \mu_\pm^{(x,y)} = 0, \quad \text{and} \quad \mu_0^{(x,y)} = \operatorname{sgn}(xy).$$

This leads to the Green's function

$$G(t,x,y) = \frac{1}{2\sqrt{i\pi t}}\Big(e^{-\frac{(x-y)^2}{4it}} + \operatorname{sgn}(xy)e^{-\frac{(|x|+|y|)^2}{4it}}\Big). \tag{7.51}$$

7.6.6. Robin boundary conditions

In the next example we consider the free Schrödinger equation on the two half lines $\dot{\mathbb{R}}$ and Robin boundary conditions at the origin $x = 0$, i.e.

$$\begin{aligned}
i\frac{\partial}{\partial t}\Psi(t,x) &= -\frac{\partial^2}{\partial x^2}\Psi(t,x), && t > 0,\ x \in \mathbb{R}\setminus\{0\},\\
\frac{\partial}{\partial x}\Psi(t,0^+) &= a\,\Psi(t,0^+), && t > 0,\\
\frac{\partial}{\partial x}\Psi(t,0^-) &= b\,\Psi(t,0^-), && t > 0,
\end{aligned}$$

for some $a, b \in \mathbb{R}$.

If one notes, that the minus sign for the derivative at $x = 0^-$ on the right hand side is omitted here, these boundary conditions are realized by (7.30) with the matrix

$$J = \begin{pmatrix} \frac{i+a}{i-a} & 0 \\ 0 & \frac{i-b}{i+b} \end{pmatrix}. \tag{7.52}$$

This matrix is of the form (7.32) with the coefficients

$$\alpha = \operatorname{sgn}(b-a)\frac{(1-ia)(1-ib)}{\sqrt{1+a^2}\sqrt{1+b^2}}, \quad \text{and} \quad \beta = 0,$$

and $\phi \in [0, \pi)$ chosen such that

$$e^{i\phi} = \operatorname{sgn}(b-a)\frac{(1-ia)(1+ib)}{\sqrt{1+a^2}\sqrt{1+b^2}},$$

where we use the convention $\operatorname{sgn}(0) = 1$. One immediately sees that we are always in Case I and the coefficients reduce to

$$\begin{aligned}
\eta^{(x,y)} &= \begin{cases} \operatorname{sgn}(b-a)\operatorname{sgn}(a+b)\operatorname{sgn}(x)\Theta(xy), & \text{if } (a,b) \neq (0,0),\\ 0, & \text{if } (a,b) = (0,0), \end{cases}\\
\omega_\pm &= \frac{(a-b) \pm \operatorname{sgn}(b-a)\operatorname{sgn}(b+a)(a+b)}{2}\\
&= \begin{cases} a, & \text{if } \pm\operatorname{sgn}(b-a)\operatorname{sgn}(b+a) > 0,\\ -b, & \text{if } \pm\operatorname{sgn}(b-a)\operatorname{sgn}(b+a) < 0, \end{cases}\\
\mu_\pm^{(x,y)} &= \begin{cases} \frac{a}{2}\big(1 \pm \operatorname{sgn}(x)\big)\Theta(xy), & \text{if } \pm\operatorname{sgn}(b-a)\operatorname{sgn}(b+a) > 0,\\ \frac{b}{2}\big(1 \mp \operatorname{sgn}(x)\big)\Theta(xy), & \text{if } \pm\operatorname{sgn}(b-a)\operatorname{sgn}(b+a) < 0, \end{cases}\\
&= \begin{cases} a\,\Theta(x)\Theta(y), & \text{if } \pm\operatorname{sgn}(b-a)\operatorname{sgn}(b+a) > 0,\\ b\,\Theta(-x)\Theta(-y), & \text{if } \pm\operatorname{sgn}(b-a)\operatorname{sgn}(b+a) < 0, \end{cases}\\
\mu_0^{(x,y)} &= \operatorname{sgn}(xy).
\end{aligned}$$

Plugging these coefficients into (7.31) gives the Green's function for the Robin boundary conditions

$$\begin{aligned} G(t,x,y) = &\Big(b\,\Theta(-x)\Theta(-y)\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}} - b\sqrt{it}\Big) \\ &\qquad - a\,\Theta(x)\Theta(y)\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}} + a\sqrt{it}\Big)\Big) e^{-\frac{(|x|+|y|)^2}{4it}} \\ &+ \frac{1}{2\sqrt{i\pi t}}\Big(\operatorname{sgn}(xy) e^{-\frac{(|x|+|y|)^2}{4it}} + e^{-\frac{(x-y)^2}{4it}}\Big). \end{aligned} \tag{7.53}$$

Remark 7.8. It is clear, that for $a = b = 0$ the boundary condition and the Green's function reduces to those of the Neumann boundary condition in Section 7.6.5. Moreover, also the boundary condition and Green's function for the Dirichlet decoupling of Section 7.6.4 can be formally recovered by considering the limits $a \to \infty$ and $b \to -\infty$ of the coefficients. In this case the matrix J in (7.52) tends to the one in (7.48) and using the asymptotics (A.22) of the function Λ we also obtain

$$\Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}} + a\sqrt{it}\Big) \sim \frac{1}{a\sqrt{i\pi t}} \quad \text{and} \quad \Lambda\Big(\frac{|x|+|y|}{2\sqrt{it}} - b\sqrt{it}\Big) \sim \frac{-1}{b\sqrt{i\pi t}}.$$

These asymptotics show, that the Green's function (7.53) turns into (7.49).

7.6.7. Decoupled systems

A decoupled system is when the transmission condition (7.30) reduces to one boundary condition for $\Psi(t,0^+)$ and $\frac{\partial}{\partial x}\Psi(t,0^+)$ on the right half line, and a second one for $\Psi(t,0^-)$ and $\frac{\partial}{\partial x}\Psi(t,0^-)$ of the left half line. One can easily see, that this is the case if and only if the matrix J from (7.32) has the value $\beta = 0$. I.e., the matrix J is a unitary diagonal matrix and hence of the form

$$J = \begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix}, \quad \gamma, \delta \in \mathbb{C},\ |\gamma| = |\delta| = 1.$$

Since every complex number, except 1, with absolute value 1, can be written in the form $\frac{a+i}{a-i}$, for some $a \in \mathbb{R}$, we see from Section 7.6.4, Section 7.6.5 and Section 7.6.6, that the only decoupled systems are the Dirichlet, Neumann and Robin boundary conditions.

As the following lemma shows, the situation of a decoupled system also manifests itself in the fact that the Green's function (7.31) vanishes if the values for x and y lie in different half spaces. This property is plausible since in the decoupled case there is no information exchange between the two half spaces and the wave function evolves on each half space independently. See for example also the Green's functions (7.22) and (7.21) where also a decoupling takes place due to the very singular potential $V \sim \frac{1}{x^2}$.

Proposition 7.9. The transmission condition (7.30) is the one of a decoupled system ($\beta = 0$) if and only if for the Green's function G in (7.31) we have

$$G(t,x,y) = 0, \quad t > 0,\ x, y \in \dot{\mathbb{R}} \text{ with } xy < 0.$$

Proof. If we assume that $\beta = 0$, then it follows by definition (7.33), that

$$\eta^{(x,y)} = 0, \quad x, y \in \dot{\mathbb{R}} \text{ with } xy < 0.$$

In all the three Cases I, II and III below (7.33), where we specify the parameters of the Green's function, we immediately see, that

$$\mu_\pm^{(x,y)} = 0 \quad \text{and} \quad \mu_0^{(x,y)} = -1, \quad x, y \in \dot{\mathbb{R}} \text{ with } xy < 0.$$

With this coefficients, the Green's function then vanishes as

$$G(t, x, y) = \frac{1}{2\sqrt{i\pi t}}\Big(- e^{-\frac{(|x|+|y|)^2}{4it}} + e^{-\frac{(x-y)^2}{4it}}\Big) = 0, \quad t > 0,\, x, y \in \dot{\mathbb{R}} \text{ with } xy < 0.$$

For the inverse implication, we assume, that the Green's function (7.31) vanishes for all $x, y \in \dot{\mathbb{R}}$ with $xy < 0$, i.e.

$$\begin{aligned} G(t, x, y) = \bigg(&\mu_+^{(x,y)} \Lambda\Big(\frac{|x| + |y|}{2\sqrt{it}} + \omega_+ \sqrt{it}\Big) \\ &+ \mu_-^{(x,y)} \Lambda\Big(\frac{|x| + |y|}{2\sqrt{it}} + \omega_- \sqrt{it}\Big) + \frac{\mu_0^{(x,y)} + 1}{2\sqrt{i\pi t}}\bigg) e^{-\frac{(x-y)^2}{4it}} \overset{!}{=} 0. \end{aligned}$$

First of all, if $|\operatorname{Re}(\alpha)| = 1$, it immediately follows from the condition $|\alpha| + |\beta| = 1$, that $\beta = 0$ in this case. Because of this we only consider $|\operatorname{Re}(\alpha)| < 1$ in the following. In Case I we note, that $\omega_+ \neq \omega_-$ follows from $|\operatorname{Re}(\alpha)| < 1$, and hence the bracket term of the above Green's function is a linear combination of three linear independent functions

$$\Lambda\Big(\frac{|x| + |y|}{2\sqrt{it}} + \omega_+ \sqrt{it}\Big), \quad \Lambda\Big(\frac{|x| + |y|}{2\sqrt{it}} + \omega_- \sqrt{it}\Big) \quad \text{and} \quad \frac{1}{2\sqrt{i\pi t}}.$$

In order to make the Green's function vanish, it is necessary to make all the coefficients vanish, i.e.

$$\mu_+^{(x,y)} = 0, \quad \mu_-^{(x,y)} = 0 \quad \text{and} \quad \mu_0^{(x,y)} = -1, \quad x, y \in \dot{\mathbb{R}} \text{ with } xy < 0.$$

Since $\mu_0^{(x,y)} = 0$ by definition, it remains to check whether

$$\mu_+^{(x,y)} = \frac{\omega_+}{2}\eta^{(x,y)} = 0 \quad \text{and} \quad \mu_-^{(x,y)} = \frac{\omega_-}{2}\eta^{(x,y)} = 0.$$

Since $\omega_+ \neq \omega_-$, it is only possible for both terms to vanish if $\eta^{(x,y)} = 0$ vanishes. However, this is by definition (7.33a) only possible if $\beta = 0$.

In Case II we already have $\mu_-^{(x,y)} = 0$ by definition. Hence the Green's function is a linear combination of the linear independent terms

$$\Lambda\Big(\frac{|x| + |y|}{2\sqrt{it}} + \omega_+ \sqrt{it}\Big) \quad \text{and} \quad \frac{1}{2\sqrt{i\pi t}}.$$

In order to make both terms vanish we have to make sure that both prefactors vanish independently, i.e.

$$\mu_+^{(x,y)} = 0 \quad \text{and} \quad \mu_0^{(x,y)} = -1, \quad x, y \in \dot{\mathbb{R}} \text{ with } xy < 0.$$

However, already from $\mu_0^{(x,y)} = \eta^{(x,y)} - 1 \overset{!}{=} -1$ it follows that $\eta^{(x,y)} = 0$ has to vanish and hence $\beta = 0$ by definition (7.33a).

Finally, the Case III is not possible since $|\operatorname{Re}(\alpha)| < 1$ and we finished the proof. □

A. Appendix

In this appendix we consider some special functions and derive corresponding properties, which will be used throughout the paper.

A.1. The complex square root

For this section let $\sqrt{\cdot}$ be the complex square root, chosen such that $0 \leq \operatorname{Arg}(\sqrt{\cdot}) < \pi$. In the upcoming Lemma A.1 and Lemma A.2 we derive two basic inequalities which are used to prove the $\mathcal{A}_1$-convergence in Theorem 2.9.

Lemma A.1. For any $a > 1$ and $b := \sqrt{a^2 - 1}$ there holds the lower bound

$$\big|\sqrt{z^2 - 2iaz - 1} - az + i\big| \geq \min\big\{b^2, b(a-b)\big\}, \quad z \in \mathbb{C}. \tag{A.1}$$

Proof. Since we have to prove (A.1) for every $z \in \mathbb{C}$, we can shift the variable $z \mapsto z + ia$ and verify

$$\big|\sqrt{z^2 + b^2} - az - ib^2\big| \geq \min\big\{b^2, b(a-b)\big\}, \quad z \in \mathbb{C}, \tag{A.2}$$

instead. First of all note that our used square root, satisfying $\operatorname{Arg}(\sqrt{w}) \in [0, \pi)$, acts as

$$\sqrt{w} = \frac{\operatorname{sgn}(\operatorname{Im}(w))}{\sqrt{2}} \sqrt{|w| + \operatorname{Re}(w)} + \frac{i}{\sqrt{2}} \sqrt{|w| - \operatorname{Re}(w)}, \quad w \in \mathbb{C}, \tag{A.3}$$

using the convention $\operatorname{sgn}(x) := \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$ Writing $z = x + iy$ and choosing

$$w := z^2 + b^2 = x^2 - y^2 + b^2 + 2ixy, \tag{A.4}$$

we determine the absolute value of $f(z) := \sqrt{w} - az - ib^2$ by

$$\begin{aligned} |f(z)|^2 =& |w| + a^2(x^2 + y^2) + 2ab^2 y + b^4 \\ &- \sqrt{2}\Big(ax \operatorname{sgn}(xy)\sqrt{|w| + \operatorname{Re}(w)} + (b^2 + ay)\sqrt{|w| - \operatorname{Re}(w)}\Big). \end{aligned} \tag{A.5}$$

For a further estimate we distinguish three cases.

- If $y < 0$, we write (A.5) as

$$\begin{aligned} |f(z)|^2 =& |w| + a^2(x^2 + y^2) + 2ab^2 y + b^4 \\ &+ \sqrt{2}\Big(a|x|\sqrt{|w| + \operatorname{Re}(w)} - (b^2 + ay)\sqrt{|w| - \operatorname{Re}(w)}\Big). \end{aligned} \tag{A.6}$$

The bracket term is obviously nonnegative for $x = 0$, but also for $x \neq 0$ we can estimate

$$\begin{aligned} a|x|\sqrt{|w| + \mathrm{Re}(w)} - (b^2 + ay)\sqrt{|w| - \mathrm{Re}(w)} &= \frac{a|x|(|w| + \mathrm{Re}(w)) + 2(b^2 + ay)|x|y}{\sqrt{|w| + \mathrm{Re}(w)}} \\ &\geq \frac{|x|(ay^2 + 2b^2y + ab^2)}{\sqrt{|w| + \mathrm{Re}(w)}} \geq 0, \end{aligned}$$

where in the first inequality we used $|w| \geq 0$ and $\mathrm{Re}(w) \geq b^2 - y^2$ and in the second inequality that the parabola $y \mapsto ay^2 + 2b^2y + ab^2$ attains its minimal value at $y = -\frac{b^2}{a}$ and consequently is bounded from below by $\frac{b^2}{a}$ and hence also by 0. We can then estimate (A.6) by

$$|f(z)|^2 \geq |w| + a^2y^2 + 2ab^2y + b^4 \geq |b^2 - y^2| + a^2y^2 + 2ab^2y + b^4 \geq b^2(a-b)^2,$$

where in the second inequality we used $|w| \geq |b^2 - y^2|$ and the third inequality can easily be verified by seeing that the expression admits its minimal value at $y = -b$. This leaves us with the inequality $|f(z)| \geq b(a-b)$, as stated in (A.1). Note that this lower bound is obtained for $z = -ib$.

- If $y = 0$, the absolute value (A.5) can be estimated as

$$\begin{aligned} |f(z)|^2 &= (1 + a^2)x^2 + b^2 + b^4 - 2ax\sqrt{x^2 + b^2} \\ &= \sqrt{4a^2x^2(x^2 + b^2) + b^4(x^2 - 1)^2} + b^4 - 2ax\sqrt{x^2 + b^2} \geq b^4, \end{aligned}$$

which shows that $|f(z)| \geq b^2$. Note, that this lower bound is obtained for $z = 1$.

- If $y > 0$, the expression (A.5) turns into

$$\begin{aligned} |f(z)|^2 - b^4 =& |w| + a^2(x^2 + y^2) + 2ab^2y \\ &- \sqrt{2}\Big(a|x|\sqrt{|w| + \mathrm{Re}(w)} + (b^2 + ay)\sqrt{|w| - \mathrm{Re}(w)}\Big). \end{aligned} \tag{A.7}$$

In order to show that the right hand side is nonnegative, we note that it is the difference of two nonnegative terms. Hence it is equivalent to consider the difference of the square of the respective terms, namely

$$\delta := \Big(|w| + a^2(x^2 + y^2) + 2ab^2y\Big)^2 - 2\Big(a|x|\sqrt{|w| + \mathrm{Re}(w)} + (b^2 + ay)\sqrt{|w| - \mathrm{Re}(w)}\Big)^2,$$

and prove its nonnegativity. This term reduces to

$$\begin{aligned} \frac{\delta}{b^4} &= (x^2 + y^2)^2 + 2b^2 + 1 + 4a^2y^2 - 2|w| + 4ay(x^2 + y^2 + 1) \\ &\geq (x^2 + y^2 - 1)^2 + 4a^2y^2 + 4y^3 + 4(a-1)y(x^2 + y^2 + 1) \geq 0, \end{aligned}$$

where in the second line we used

$$|w| \leq |\mathrm{Re}(w)| + |\mathrm{Im}(w)| \leq x^2 + y^2 + b^2 + 2|x|y \leq x^2 + y^2 + b^2 + 2(x^2 + 1)y.$$

This shows, that the right hand side of (A.7) is nonnegative and hence $|f(z)| \geq b^2$ also in this case. □

Lemma A.2. For any $a > 1$ there holds the upper bound

$$\left|\operatorname{Im}\left(\sqrt{z^2 - 2iaz - 1} + az - i\right)\right| \leq (a+1)|z|, \quad z \in \mathbb{C}. \tag{A.8}$$

Proof. Writing $z = x + iy$ and choosing $b := \sqrt{a^2 - 1}$ as well as

$$w := z^2 - 2iaz - 1 = x^2 - y^2 + 2ay - 1 + 2ix(y - a),$$

we use the representation (A.3) of the complex square root to write the imaginary part of $f(z) := \sqrt{w} + az - i$ as

$$\operatorname{Im}(f(z)) = ay - 1 + \frac{1}{\sqrt{2}}\sqrt{|w| - \operatorname{Re}(w)}. \tag{A.9}$$

In the *first step* we calculate an upper bound of the imaginary part (A.9) and use the estimate

$$|w| = \sqrt{(x^2 + y^2 - 2ay + 1)^2 + 4b^2x^2} \leq |x^2 + y^2 - 2ay + 1| + 2b|x|, \tag{A.10}$$

to do so.

- In the case $x^2 + y^2 - 2ay + 1 \geq 0$, we can use (A.10) in (A.9) to estimate

  $$\operatorname{Im}(f(z)) \leq ay - 1 + \sqrt{y^2 - 2ay + 1 + b|x|}.$$

  In order to show that the right hand side is bounded by $(a+1)|z|$, i.e.

  $$\sqrt{y^2 - 2ay + 1 + b|x|} \leq (a+1)|z| - ay + 1, \tag{A.11}$$

  we note, that both sides of this inequality are nonnegative and we are allowed to equivalently consider the difference of the respective squares, namely

  $$\begin{aligned}\big((a+1)|z| - ay + 1\big)^2 &- y^2 + 2ay - 1 - b|x| \\ &= (a+1)^2x^2 + 2(a+1)(ay^2 - ay|z| + |z|) - b|x| \\ &\geq (a+1)^2x^2 + 2(a+1)(ay^2 - a|yz| + |z|) - b|x| \\ &\geq (a+1)x^2 + (2a + 2 - b)|x| \geq 0,\end{aligned}$$

  where in the last line we used that $|yz| \leq \frac{x^2}{2} + y^2$ as well as $|z| \geq |x|$. This proves, that the estimate (A.11) is satisfied and hence $\operatorname{Im}(f(z)) \leq (a+1)|z|$.

- If $x^2 + y^2 - 2ay + 1 \leq 0$, the inequality (A.10) used in (A.9) leads to

  $$\operatorname{Im}(f(z)) \leq ay - 1 + \sqrt{b|x| - x^2}.$$

  We want to show that also in this case the right hand side is bounded by $(a+1)|z|$, i.e.

  $$\sqrt{b|x| - x^2} \leq (a+1)|z| - ay + 1.$$

  Since the left hand side is y-independent, we minimize the right hand side with respect to y. Standard analysis shows, that the minimum value is attained at $y = \frac{a|x|}{\sqrt{2a+1}}$ and we need to verify the inequality

  $$\sqrt{b|x| - x^2} \leq |x|\sqrt{2a+1} + 1.$$

  By squaring both sides one immediately sees that this inequality is satisfied. Hence we conclude, that also in this case there holds $\operatorname{Im}(f(z)) \leq (a+1)|z|$.

In the *second step* we derive the lower bound of $\mathrm{Im}(f(z))$. To do so we distinguish two different cases.

- If $|y-a| \geq b$ we can use $|w| \geq x^2 + y^2 - 2ay + 1$ in (A.9) to get

$$\begin{aligned}\mathrm{Im}(f(z)) &\geq ay - 1 + \sqrt{y^2 - 2ay + 1} \\ &\geq \left\{ \begin{array}{ll} 0, & \text{if } y \geq a+b, \\ -by, & \text{if } 0 \leq y \leq a-b, \\ (a-1)y, & \text{if } y \leq 0, \end{array} \right\} \geq -(a+1)|z|.\end{aligned}$$

- If $|y-a| \leq b$ we can use $|w| - \mathrm{Re}(w) \geq 0$ to estimate (A.9) as

$$\mathrm{Im}(f(z)) \geq ay - 1 \geq -by \geq -(a+1)|z|. \qquad \square$$

If we combine the complex square root with the sinus cardinalis

$$\mathrm{sinc}(z) := \begin{cases} \frac{\sin(z)}{z}, & \text{if } z \in \mathbb{C} \setminus \{0\}, \\ 1, & \text{if } z = 0. \end{cases}$$

we obtain the following integral representation.

Lemma A.3. For any $b > 0$, there holds

$$\mathrm{sinc}\left(\sqrt{z^2 + b^2}\right) = \frac{1}{2} \int_{-1}^{1} e^{ikz} J_0(b\sqrt{1-k^2}) dk, \quad z \in \mathbb{C}.$$

where J_0 is the Bessel function of order zero.

Proof. In *Step 1* we consider the function

$$S(z) := \mathrm{sinc}\left(\sqrt{z^2 + b^2}\right), \quad z \in \mathbb{C},$$

which is an entire function due to the power series expansion

$$S(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\sqrt{z^2 + b^2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z^2 + b^2)^n, \quad z \in \mathbb{C}.$$

Since the restriction $S|_{\mathbb{R}}$ is square integrable, its Fourier transform is given by the improper Riemann integral

$$\mathcal{F}[S|_{\mathbb{R}}](k) = \frac{1}{\sqrt{2\pi}} \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} e^{-ikx} S(x) dx, \quad k \in \mathbb{R}.$$

For simplicity we will write $\int_{\mathbb{R}}$ instead of $\lim_{R_1,R_2\to\infty} \int_{-R_1}^{R_2}$ in the following, but always interpret the integral as this limit. Starting with $|k| \leq 1$, the imaginary part of the Fourier transform vanishes due to the symmetry of S, i.e., we get

$$\mathcal{F}[S|_{\mathbb{R}}](k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \cos(kx) S(x) dx.$$

Substituting $x = b\sinh(t)$, we rewrite this integral as

$$\begin{aligned}\mathcal{F}[S|_{\mathbb{R}}](k) &= \frac{b}{\sqrt{2\pi}}\int_{\mathbb{R}} \cos\big(kb\sinh(t)\big)S\big(b\sinh(t)\big)\cosh(t)dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \cos\big(kb\sinh(t)\big)\sin\big(b\cosh(t)\big)dt \\ &= \frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}} \Big(\sin\big(b\cosh(t)+kb\sinh(t)\big)+\sin\big(b\cosh(t)-kb\sinh(t)\big)\Big)dt,\end{aligned} \tag{A.12}$$

where in the last line we used the trigonometric identity $2\sin(u)\cos(v) = \sin(u+v) + \sin(u-v)$. Distinguishing further the two cases $|k| < 1$ and $|k| = 1$ we get

- For $|k| < 1$, there exists some $t_0 \in \mathbb{R}$ such that $e^{2t_0} = \frac{1+k}{1-k}$, and hence we get the hyperbolic identity

  $$\cosh(t) \pm k\sinh(t) = \sqrt{1-k^2}\,\cosh(t \pm t_0),$$

  with which we can write the above integral as

  $$\begin{aligned}\mathcal{F}[S|_{\mathbb{R}}](k) &= \frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}} \Big(\sin\big(b\sqrt{1-k^2}\cosh(t+t_0)\big)+\sin\big(b\sqrt{1-k^2}\cosh(t-t_0)\big)\Big)dt, \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \sin\big(b\sqrt{1-k^2}\cosh(t)\big)dt \\ &= \frac{\sqrt{\pi}}{\sqrt{2}}J_0\big(b\sqrt{1-k^2}\big),\end{aligned}$$

  where we used the Mehline-Sonine integral representation of the Bessel function [2, Eq.(9.1.23)].

- If $|k| = 1$, the integral (A.12) becomes

  $$\mathcal{F}[S|_{\mathbb{R}}](k) = \frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}} \big(\sin(be^t)+\sin(be^{-t})\big)dt = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \sin(be^t)dt.$$

  Substituting $s = be^t$ this integral has the explicit solution

  $$\mathcal{F}[S|_{\mathbb{R}}](k) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{\sin(s)}{s}ds = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

For $k > 1$ we use the Cauchy theorem to change the integration path to a semicircle in the lower half space

$$\mathcal{F}[S|_{\mathbb{R}}](k) = \lim_{R\to\infty}\frac{iR}{\sqrt{2\pi}}\int_\pi^{2\pi} e^{-ikRe^{i\varphi}}S(Re^{i\varphi})e^{i\varphi}d\varphi. \tag{A.13}$$

Choosing $R > b$, the function S in the integrand can be estimated by

$$|S(Re^{i\varphi})| = \left|\frac{\sin(\sqrt{R^2e^{2i\varphi}+b^2})}{\sqrt{R^2e^{2i\varphi}+b^2}}\right| \le \frac{e^{|\operatorname{Im}(\sqrt{R^2e^{2i\varphi}+b^2})|}}{\sqrt{R^2-b^2}}, \quad \varphi \in [\pi, 2\pi]. \tag{A.14}$$

Using the representation (A.3) of the complex square root, we can estimate the exponent by

$$\begin{aligned}
2\Big|\operatorname{Im}\big(\sqrt{R^2e^{2i\varphi}+b^2}\big)\Big|^2 &= |R^2e^{2i\varphi}+b^2| - \operatorname{Re}(R^2e^{2i\varphi}+b^2) \\
&= \sqrt{(R^2\cos(2\varphi)+b^2)^2+R^4\sin^2(2\varphi)} - R^2\cos(2\varphi) - b^2 \\
&= \sqrt{(R^2+b^2)^2-4b^2R^2\sin^2(\varphi)} - R^2\cos(2\varphi) - b^2 \\
&\leq R^2+b^2-R^2\cos(2\varphi)-b^2 \\
&= 2R^2\sin^2(\varphi).
\end{aligned}$$

Hence we can further estimate (A.14) by

$$|S(Re^{i\varphi})| \leq \frac{e^{-R\sin(\varphi)}}{\sqrt{R^2-b^2}}, \quad \varphi\in[\pi,2\pi].$$

Using this inequality in the Fourier transform (A.13) can be estimated by

$$\begin{aligned}
|\mathcal{F}[S|_{\mathbb{R}}](k)| &\leq \lim_{R\to\infty}\frac{R}{\sqrt{2\pi(R^2-b^2)}}\int_\pi^{2\pi} e^{(k-1)R\sin(\varphi)}d\varphi \\
&= \frac{1}{\sqrt{2\pi}}\int_\pi^{2\pi}\lim_{R\to\infty} e^{(k-1)R\sin(\varphi)}d\varphi = 0.
\end{aligned}$$

This proves, that $\mathcal{F}[S|_{\mathbb{R}}](k) = 0$ for $k > 1$. By the symmetry of S we also conclude $\mathcal{F}[S|_{\mathbb{R}}](k) = 0$ for $k < -1$. Altogether we now proved the Fourier transform

$$\mathcal{F}[S|_{\mathbb{R}}](k) = \frac{\sqrt{\pi}}{\sqrt{2}}J_0\big(b\sqrt{1-k^2}\big)\begin{cases} 1, & \text{if } |k|<1, \\ \frac{1}{2}, & \text{if } |k|=1, \\ 0, & \text{if } |k|>1. \end{cases} \tag{A.15}$$

In *Step 2* we now apply the inverse Fourier transform to (A.15) and obtain

$$S(x) = \frac{1}{2}\int_{-1}^{1} e^{ikx}J_0(b\sqrt{1-k^2})dk, \quad x\in\mathbb{R}.$$

Since J_0 is a bounded function, the right hand side extends to an entire function when $x\in\mathbb{R}$ is replaced by $z\in\mathbb{C}$. Since the holomorphic extension is unique, it has to coincide with $S(z)$ and we conclude the stated integral representation

$$\operatorname{sinc}\big(\sqrt{z^2+b^2}\big) = \frac{1}{2}\int_{-1}^{1} e^{ikz}J_0(b\sqrt{1-k^2})dk, \quad z\in\mathbb{C}. \qquad \square$$

A.2. A modification of the error function

In order to write down the Green's function for arbitrary point interactions (7.31) it is convenient to use

$$\Lambda(z) := e^{z^2}(1-\operatorname{erf}(z)), \quad z\in\mathbb{C}, \tag{A.16}$$

where $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}d\xi$ is the well known error function. Some important properties of this function are collected in the following lemma.

Lemma A.4. The function Λ in (A.16) has the following properties:

(i) The function Λ satisfies the differential equation

$$\frac{d}{dz}\Lambda(z) = 2z\Lambda(z) - \frac{2}{\sqrt{\pi}}, \quad z \in \mathbb{C}. \tag{A.17}$$

(ii) The function Λ admits the integral identity

$$\Lambda(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2-2zs} ds, \quad z \in \mathbb{C}. \tag{A.18}$$

(iii) The function Λ admits the power series representation

$$\Lambda(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\frac{n}{2}+1)} z^n, \quad z \in \mathbb{C}. \tag{A.19}$$

(iv) The value of the function Λ at $-z$ is given by

$$\Lambda(-z) = 2e^{z^2} - \Lambda(z), \quad z \in \mathbb{C}. \tag{A.20}$$

(v) The function Λ is monotonically decreasing on $\mathbb{R}$ and its absolute value can be estimated by

$$|\Lambda(z)| \leq \Lambda(\mathrm{Re}(z)), \quad z \in \mathbb{C}. \tag{A.21}$$

(vi) The function Λ asymptotically behaves as

$$\Lambda(z) = \begin{cases} \frac{1}{\sqrt{\pi}z} + \mathcal{O}\big(\frac{1}{|z|^2}\big), & \text{if } \mathrm{Re}(z) \geq 0, \\ 2e^{z^2} + \frac{1}{\sqrt{\pi}z} + \mathcal{O}\big(\frac{1}{|z|^2}\big), & \text{if } \mathrm{Re}(z) \leq 0, \end{cases} \quad \text{as } |z| \to \infty. \tag{A.22}$$

(vii) For every $a > 0$ and $b, c \in \mathbb{C}$ one has the integral identity

$$\int_0^\infty e^{-ax^2-bx} dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \Lambda\Big(\frac{b}{2\sqrt{a}}\Big). \tag{A.23}$$

as well as another identity

$$\int_0^\infty e^{-ax^2-bx} \Lambda\big(\sqrt{a}\,x + c\big) dx = \frac{-1}{2\sqrt{a}} \begin{cases} \frac{\Lambda(c)-\Lambda(\frac{b}{2\sqrt{a}})}{c-\frac{b}{2\sqrt{a}}}, & \text{if } c \neq \frac{b}{2\sqrt{a}}, \\ \Lambda'(c), & \text{if } c = \frac{b}{2\sqrt{a}}. \end{cases} \tag{A.24}$$

Proof.

(i) Using the derivative $\frac{d}{dz}\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2}$ it follows immediately, that

$$\frac{d}{dz}\Lambda(z) = \frac{d}{dz}\Big(e^{z^2}(1-\operatorname{erf}(z))\Big) = 2ze^{z^2}(1-\operatorname{erf}(z)) - \frac{2}{\sqrt{\pi}} = 2z\Lambda(z) - \frac{2}{\sqrt{\pi}}.$$

(ii) Using $\int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}$ in the definition (A.16) gives

$$\Lambda(z) = \frac{2}{\sqrt{\pi}} e^{z^2} \lim_{R\to\infty} \Big(\int_0^R e^{-\xi^2} d\xi - \int_0^z e^{-\xi^2} d\xi \Big).$$

Now we use that the complex integral over the entire function $e^{-\xi^2}$ is path independent to rewrite it as

$$\Lambda(z) = \frac{2}{\sqrt{\pi}} e^{z^2} \lim_{R\to\infty} \Big(\int_z^{z+R} e^{-\xi^2} d\xi - \int_R^{z+R} e^{-\xi^2} d\xi \Big). \tag{A.25}$$

Since the integral along $R \to z+R$ can be estimated as

$$\Big| \int_R^{z+R} e^{-\xi^2} d\xi \Big| = \Big| z \int_0^1 e^{-(R+sz)^2} ds \Big| \leq |z| e^{-R^2} \int_0^1 e^{-2Rs \operatorname{Re}(z) - s^2 \operatorname{Re}(z^2)} ds$$
$$\leq |z| e^{-R^2} e^{2R|\operatorname{Re}(z)| + |\operatorname{Re}(z^2)|} \overset{R\to\infty}{\longrightarrow} 0,$$

and hence vanishes in the limit $R \to \infty$ This means, that in (A.25) only the first integral survives which, after parametrization, gives

$$\Lambda(z) = \frac{2}{\sqrt{\pi}} e^{z^2} \lim_{R\to\infty} \int_0^R e^{-(z+s)^2} ds = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2 - 2zs} ds.$$

(iii) Using the integral representation (A.18) and writing the exponential as a power series gives

$$\Lambda(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} \sum_{n=0}^\infty \frac{(-2sz)^n}{n!} ds = \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-2z)^n}{n!} \int_0^\infty s^n e^{-s^2} ds.$$

Using the integral identity $\int_0^\infty s^n e^{-s^2} ds = \frac{1}{2}\Gamma(\frac{n+1}{2})$ as well as the Legendre duplication formula $\Gamma(\frac{n+1}{2})\Gamma(\frac{n}{2}+1) = \frac{\sqrt{\pi}}{2^n}\Gamma(n+1)$, further reduces this series to

$$\Lambda(z) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{\Gamma(\frac{n+1}{2})}{n!} (-2z)^n = \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(\frac{n}{2}+1)} z^n.$$

(iv) Using the obvious property $\operatorname{erf}(-z) = -\operatorname{erf}(z)$ it follows immediately, that

$$\Lambda(-z) = e^{(-z)^2}(1 - \operatorname{erf}(-z)) = e^{z^2}(1 + \operatorname{erf}(z)) = 2e^{z^2} - \Lambda(z).$$

(v) Firstly, the monotonicity is a direct consequence of (A.18). This integral representation can also be used to estimate the absolute value

$$|\Lambda(z)| \leq \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2 - 2\operatorname{Re}(z)s} ds = \Lambda(\operatorname{Re}(z)).$$

(vi) Applying integration by parts in (A.18) gives

$$\Lambda(z) = -\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{z+s} \frac{d}{ds} e^{-s^2-2zs} ds = \frac{1}{\sqrt{\pi}\, z} - \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{(z+s)^2} e^{-s^2-2zs} ds$$

In the case $\operatorname{Re}(z) \geq 0$ we can use $e^{-2\operatorname{Re}(z)s} \leq 1$ as well as $|z+s| \geq |z|$ to estimate the integral as

$$\Big|\Lambda(z) - \frac{1}{\sqrt{\pi} z}\Big| \leq \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{|z+s|^2} e^{-s^2 - 2\operatorname{Re}(z)s} ds \leq \frac{1}{\sqrt{\pi}\,|z|^2} \int_0^\infty e^{-s^2} ds = \frac{1}{2|z|^2}.$$

The case $\operatorname{Re}(z) \leq 0$ then follows from (A.20).

(vii) For the identity (A.23), we substitute $x = \frac{s}{\sqrt{a}}$ in (A.18) and evaluate at $z = \frac{b}{2\sqrt{a}}$. This exactly gives

$$\Lambda\Big(\frac{b}{2\sqrt{a}}\Big) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2 - \frac{b}{\sqrt{a}}s} ds = \frac{2\sqrt{a}}{\sqrt{\pi}} \int_0^\infty e^{-ax^2 - bx} dx.$$

In order to verify (A.24), we first use (A.17) to obtain the primitive

$$e^{-ax^2-bx} \Lambda\big(\sqrt{a}\,x + c\big) = \frac{d}{dx} e^{-ax^2-bx} \frac{\Lambda\big(\sqrt{a}\,x + c\big) - \Lambda\big(\sqrt{a}\,x + \frac{b}{2\sqrt{a}}\big)}{2\sqrt{a}\,c - b}.$$

Integrating both sides over the interval $(0,\infty)$ gives

$$\begin{aligned}\int_0^\infty e^{-ax^2-bx} \Lambda\big(\sqrt{a}\,x + c\big) dx &= \frac{1}{2\sqrt{a}} e^{-ax^2-bx} \frac{\Lambda\big(\sqrt{a}\,x + c\big) - \Lambda\big(\sqrt{a}\,x + \frac{b}{2\sqrt{a}}\big)}{c - \frac{b}{2\sqrt{a}}}\Bigg|_{x=0}^\infty \\ &= \frac{\Lambda(c) - \Lambda(\frac{b}{2\sqrt{a}})}{b - 2\sqrt{a}\,c},\end{aligned}$$

where the evaluation at $x = \infty$ vanishes due to the asymptotics (A.22). □

For the Green's function (7.17) of the Pöschl-Teller potential we further need

$$R(t,z) := e^z \Lambda\Big(\frac{z}{2\sqrt{it}} - \sqrt{it}\Big) - e^{-z} \Lambda\Big(\frac{z}{2\sqrt{it}} + \sqrt{it}\Big), \quad t > 0,\ z \in \mathbb{C}, \tag{A.26}$$

This function then has the following properties, which are mainly consequences of Lemma A.4.

Lemma A.5. For every $t > 0$, $z \in \mathbb{C}$, the function R admits the following properties.

(i) The derivatives of R are given by

$$\frac{\partial}{\partial t} R(t,z) = i\Big(\frac{z^2}{4t^2} + 1\Big) R(t,z) + \frac{iz\sinh(z)}{\sqrt{\pi}\,(it)^{\frac{3}{2}}} + \frac{2i\cosh(z)}{\sqrt{i\pi t}}, \tag{A.27a}$$

$$\frac{\partial}{\partial z} R(t,z) = \frac{z}{2it} R(t,z) - \frac{2\sinh(z)}{\sqrt{i\pi t}}. \tag{A.27b}$$

(ii) R admits the symmetry

$$R(t,-z) = R(t,z). \tag{A.28}$$

(iii) The function R can be estimated as

$$|R(t,z)| \leq 2e^{|\operatorname{Re}(z)|} \Lambda\Big(-\frac{\sqrt{t}}{\sqrt{2}}\Big). \tag{A.29}$$

Proof.

(i) Using the derivative of the function Λ in (A.17) immediately gives

$$\begin{aligned}\frac{\partial}{\partial t}R(t,z) &= \frac{e^z}{2i\sqrt{it}}\Big(\frac{z}{2it}+1\Big)\Lambda'\Big(\frac{z}{2\sqrt{it}}-\sqrt{it}\Big) - \frac{e^{-z}}{2i\sqrt{it}}\Big(\frac{z}{2it}-1\Big)\Lambda'\Big(\frac{z}{2\sqrt{it}}+\sqrt{it}\Big)\\ &= i\Big(\frac{z^2}{4t^2}+1\Big)R(t,z) - \frac{1}{i\sqrt{i\pi t}}\Big(e^z\Big(\frac{z}{2it}+1\Big)-e^{-z}\Big(\frac{z}{2it}-1\Big)\Big)\\ &= i\Big(\frac{z^2}{4t^2}+1\Big)R(t,z) + \frac{iz\sinh(z)}{\sqrt{\pi}\,(it)^{\frac{3}{2}}} + \frac{2i\cosh(z)}{\sqrt{i\pi t}}.\end{aligned}$$

Using again (A.17) also gives the z-derivative

$$\begin{aligned}\frac{\partial}{\partial z}R(t,z) &= e^z\Lambda\Big(\frac{z}{2\sqrt{it}}-\sqrt{it}\Big) + e^{-z}\Lambda\Big(\frac{z}{2\sqrt{it}}+\sqrt{it}\Big)\\ &\quad + \frac{1}{2\sqrt{it}}\Big(e^z\Lambda'\Big(\frac{z}{2\sqrt{it}}-\sqrt{it}\Big) - e^{-z}\Lambda'\Big(\frac{z}{2\sqrt{it}}+\sqrt{it}\Big)\Big)\\ &= \frac{z}{2it}R(t,z) - \frac{2\sinh(z)}{\sqrt{i\pi t}}.\end{aligned}$$

(ii) Using the property (A.20) of Λ with negative argument we get

$$\begin{aligned}R(t,-z) &= e^{-z}\Lambda\Big(\frac{-z}{2\sqrt{it}}-\sqrt{it}\Big) - e^z\Lambda\Big(\frac{-z}{2\sqrt{it}}+\sqrt{it}\Big)\\ &= -e^{-z}\Lambda\Big(\frac{z}{2\sqrt{it}}+\sqrt{it}\Big) + e^z\Lambda\Big(\frac{z}{2\sqrt{it}}-\sqrt{it}\Big) = R(t,z).\end{aligned}$$

(iii) Due to the symmetry property (A.28), it is sufficient consider values $z \in \mathbb{C}$ with $\mathrm{Re}(z)+\mathrm{Im}(z) \geq 0$. With the estimate (A.21) and monotonicity in Lemma A.4 (iii) we get

$$\begin{aligned}|R(t,z)| &\leq e^{\mathrm{Re}(z)}\Lambda\Big(\mathrm{Re}\Big(\frac{z}{2\sqrt{it}}-\sqrt{it}\Big)\Big) + e^{-\mathrm{Re}(z)}\Lambda\Big(\mathrm{Re}\Big(\frac{z}{2\sqrt{it}}+\sqrt{it}\Big)\Big)\\ &\leq e^{\mathrm{Re}(z)}\Lambda\Big(\frac{\mathrm{Re}(z)+\mathrm{Im}(z)}{2\sqrt{2t}}-\frac{\sqrt{t}}{\sqrt{2}}\Big) + e^{-\mathrm{Re}(z)}\Lambda\Big(\frac{\mathrm{Re}(z)+\mathrm{Im}(z)}{2\sqrt{2t}}+\frac{\sqrt{t}}{\sqrt{2}}\Big)\\ &\leq e^{\mathrm{Re}(z)}\Lambda\Big(-\frac{\sqrt{t}}{\sqrt{2}}\Big) + e^{-\mathrm{Re}(z)}\Lambda\Big(\frac{\sqrt{t}}{\sqrt{2}}\Big) \leq 2e^{|\mathrm{Re}(z)|}\Lambda\Big(-\frac{\sqrt{t}}{\sqrt{2}}\Big).\end{aligned}$$ □

A.3. Legendre Polynomials

In this section we consider for $l \in \mathbb{N}_0$, $m \in \{0,\dots,l\}$ the functions

$$\mathcal{P}_l^m(z) := P_l^m(\tanh(z)), \quad z \in \mathbb{C}\setminus i\pi\Big(\mathbb{Z}+\frac{1}{2}\Big), \tag{A.30}$$

which are a modification of the associated Legendre polynomials P_l^m. Since the Legendre polynomials are holomorphic on $\mathbb{C}\setminus\{\pm 1\}$, see [2, Eq.(8.1.1)], and since tanh never obtains these values ± 1, we only have to exclude the singularities $i\pi(\mathbb{Z}+\frac{1}{2})$ of tanh from the domain of definition. These functions are in particular used to write down the Green's function of the Pöschl-Teller potential in (7.17). Next we will derive some basic properties of the functions (A.30).

Lemma A.6. For every $l \in \mathbb{N}_0$, $m \in \{0, \dots, l\}$ there holds

(i) The function $\mathcal{P}_l^m$ satisfies the differential equation

$$\mathcal{P}_l^{m''}(z) + \Big(\frac{l(l+1)}{\cosh^2(z)} - m^2\Big)\mathcal{P}_l^m(z) = 0, \quad z \in \mathbb{C} \setminus i\pi\Big(\mathbb{Z} + \frac{1}{2}\Big). \tag{A.31}$$

(ii) For every $\alpha \in (0, \frac{\pi}{2})$ and $h \in (0, \frac{\pi}{2})$ there exists some constant $C_{l,m,\alpha,h} \geq 0$, such that

$$|\mathcal{P}_l^m(z)| \leq \frac{C_{l,m,\alpha,h}}{|\cosh(z)|^m}, \quad z \in S_{\alpha,h}, \tag{A.32}$$

where $S_{\alpha,h}$ is the domain from (4.12).

(iii) The functions $\mathcal{P}_l^m$ satisfy for every $x, z \in \mathbb{C} \setminus i\pi\Big(\mathbb{Z} + \frac{1}{2}\Big)$ the identity

$$\sum_{m=1}^{l} \frac{m(l-m)!}{(l+m)!}\mathcal{P}_l^m(z)\sinh\big(m(z-x)\big)\mathcal{P}_l^m(x) = \frac{l(l+1)}{4}\big(\tanh(z) - \tanh(x)\big). \tag{A.33}$$

Proof.

(i) By [2, Eq.(8.1.1)] the functions P_l^m satisfy the Legendre differential equation

$$(1-w^2)P_l^{m''}(w) - 2wP_l^{m'}(w) + \Big(l(l+1) - \frac{m^2}{(1-w^2)}\Big)P_l^m(w) = 0, \quad w \in \mathbb{C} \setminus \{\pm 1\}.$$

After replacing $w = \tanh(z)$, $z \in i\pi(\mathbb{Z} + \frac{1}{2})$, this equation becomes

$$\frac{P_l^{m''}(\tanh(z))}{\cosh^2(z)} - 2\tanh(z)P_l^{m'}(\tanh(z)) + \Big(l(l+1) - m^2\cosh^2(z)\Big)P_l^m(\tanh(z)) = 0.$$

Since the derivatives of $\mathcal{P}_l^m$ are given by

$$\begin{aligned} P_l^{m'}(\tanh(z)) &= \cosh^2(z)\mathcal{P}_l^{m'}(z) \\ P_l^{m''}(\tanh(z)) &= 2\sinh(z)\cosh^3(z)\mathcal{P}_l^{m'}(z) + \cosh^4(z)\mathcal{P}_l^{m''}(z), \end{aligned}$$

we immediately end up with the stated differential equation

$$\mathcal{P}_l^{m''}(z) + \Big(\frac{l(l+1)}{\cosh^2(z)} - m^2\Big)\mathcal{P}_l^m(z) = 0, \quad z \in \mathbb{C} \setminus i\pi\Big(\mathbb{Z} + \frac{1}{2}\Big).$$

(ii) First of all we note, that since $0 < h < \frac{\pi}{2}$, the domain $S_{\alpha,h}$ has positive distance to the singularities $\pm\frac{i\pi}{2}$ and hence the hyperbolic tangens is bounded on $D_{\alpha,h}$ as

$$|\tanh(z)| \leq c_{\alpha,h}, \quad z \in D_{\alpha,h}, \tag{A.34}$$

for some $c_{\alpha,h} \geq 0$. Moreover, by [2, Eq.(8.6.6),(8.6.18)], the associated Legendre polynomials are of the form

$$P_l^m(x) = (1-x^2)^{\frac{m}{2}}\Big(\text{polynomial in } x\Big), \quad x \in (-1, 1).$$

Hence also the modified function $\mathcal{P}_l^m$ is of the form

$$\mathcal{P}_l^m(x) = \frac{1}{\cosh^m(x)}\Big(\text{polynomial in } \tanh(x)\Big), \quad x \in \mathbb{R}.$$

Since the analytic continuation is unique, this representation carries over to complex arguments

$$\mathcal{P}_l^m(z) = \frac{1}{\cosh^m(z)}\Big(\text{polynomial in } \tanh(z)\Big), \quad z \in \mathbb{C} \setminus i\pi\Big(\mathbb{Z} + \frac{1}{2}\Big).$$

Combining this representation with the estimate (A.34) it is now clear, that the inequality (A.32) is valid. □

A.4. Bessel and Hankel functions

In this section we derive asymptotics and estimates of the Bessel function J_ν as well as the Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$, $\mathrm{Re}(\nu) \geq 0$, which are solutions of the differential equations

$$z^2 J_\nu''(z) + zJ_\nu'(z) + (z^2 - \nu^2)J_\nu(z) = 0, \qquad z \in \mathbb{C} \setminus (-\infty, 0], \tag{A.35a}$$

$$z^2 H_\nu^{(1,2)''}(z) + zH_\nu^{(1,2)'}(z) + (z^2 - \nu^2)H_\nu^{(1,2)} = 0, \qquad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{A.35b}$$

These functions will be needed in the investigation of the Green's functions (7.21) and (7.22) of the centrifugal potential. In particular for the Hankel functions it is more suitable to consider the following adaption

$$\mathcal{H}_\nu^{(1)}(z) := \sqrt{z}\, e^{-iz} H_\nu^{(1,2)}(z), \qquad \mathrm{Re}(z) > 0,$$
$$\mathcal{H}_\nu^{(2)}(z) := \sqrt{z}\, e^{iz} H_\nu^{(1,2)}(z), \qquad \mathrm{Re}(z) > 0.$$

For a shorter notation we will introduced the signs $\pm$ or $\mp$ in the sense, that the upper sign always belongs to $\mathcal{H}_\nu^{(1)}$ and the lower sign to $\mathcal{H}_\nu^{(2)}$. For example, the above definition of the functions $\mathcal{H}_\nu^{(1)}$ and $\mathcal{H}_\nu^{(2)}$ looks like

$$\mathcal{H}_\nu^{(1,2)}(z) := \sqrt{z}\, e^{\mp iz} H_\nu^{(1,2)}(z), \quad \mathrm{Re}(z) > 0, \tag{A.36}$$

using this convention. According to (A.35a), these functions then satisfy the differential equation

$$z^2 \mathcal{H}_\nu^{(1,2)''}(z) - 2iz^2 w \mathcal{H}_\nu^{(1,2)'}(z) + \Big(\frac{1}{4} - \nu^2\Big)\mathcal{H}_\nu^{(1,2)}(z) = 0, \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{A.37}$$

We start deriving an integral representation of the Hankel functions, which will be the basis of the upcoming estimates in Lemma A.8.

Lemma A.7. For every $\mathrm{Re}(\nu) \geq 0$ the function $\mathcal{H}_\nu^{(1,2)}$ admits the integral representation

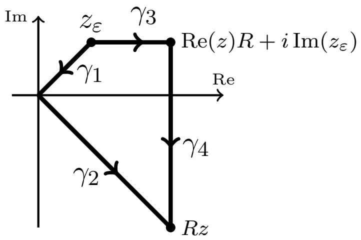
$$\mathcal{H}_\nu^{(1,2)}(z) = \frac{\mp i\sqrt{2}}{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})} \int_0^\infty \Big(\frac{t^2}{2z} \mp it\Big)^{\nu - \frac{1}{2}} e^{-t} dt, \quad \mathrm{Re}(z) > 0. \tag{A.38}$$

Proof. From [53, Equations (10.9.4) & (10.9.5)] and $H_\nu^{(1,2)}(z) = J_\nu(z) \pm iY_\nu(z)$, see [53, Equation (10.4.3)], we get

$$\mathcal{H}_\nu^{(1,2)}(z) = \frac{2^{1-\nu} z^{\nu+\frac{1}{2}} e^{\mp iz}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \left(\int_0^1 (1-t^2)^{\nu-\frac{1}{2}} e^{\pm izt} dt \mp i \int_0^\infty (1+t^2)^{\nu-\frac{1}{2}} e^{-zt} dt \right) \tag{A.39}$$
$$= \frac{2^{1-\nu} z^{\nu+\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} e^{\mp iz} \lim_{\varepsilon\to 0^+} \lim_{R\to\infty} \left(\int_0^{1-\varepsilon} (1-t^2)^{\nu-\frac{1}{2}} e^{\pm izt} dt \mp i \int_0^R (1+t^2)^{\nu-\frac{1}{2}} e^{-zt} dt \right).$$

For sufficiently small $\varepsilon > 0$ and sufficiently large $R \geq 0$ we consider the integration path

$$\gamma_1 := \{ \mp iz(1-t) \mid \varepsilon \leq t \leq 1 \},$$
$$\gamma_2 := \{ tz \mid 0 \leq t \leq R \},$$
$$\gamma_3 := \{ z_\varepsilon + t \mid 0 \leq t \leq \mathrm{Re}(Rz - z_\varepsilon) \},$$
$$\gamma_4 := \{ \mathrm{Re}(z)R + i\,\mathrm{Im}(z_\varepsilon) + it \mid 0 \leq t \leq \mathrm{Im}(Rz - z_\varepsilon) \},$$



where for a shorter notation we defined $z_\varepsilon := \mp iz(1-\varepsilon)$. Using complex path integrals we can write (A.39) as

$$\mathcal{H}_\nu^{(1,2)}(z) = \frac{\pm 2^{1-\nu} z^{\nu-\frac{1}{2}} e^{\mp iz}}{i\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \lim_{\varepsilon\to 0^+} \lim_{R\to\infty} \left(\int_{\gamma_1} \Big(1+\frac{\xi^2}{z^2}\Big)^{\nu-\frac{1}{2}} e^{-\xi} d\xi + \int_{\gamma_2} \Big(1+\frac{\xi^2}{z^2}\Big)^{\nu-\frac{1}{2}} e^{-\xi} d\xi \right).$$

If we interpret the complex power of the integrand as

$$\Big(1+\frac{\xi^2}{z^2}\Big)^{\nu-\frac{1}{2}} = e^{(\nu-\frac{1}{2})\ln(1+\frac{\xi^2}{z^2})},$$

with the complex logarithm holomorphic on $\mathbb{C} \setminus (-\infty, 0]$. In order to make sure that $1 + \frac{\xi^2}{z^2} \notin (-\infty, 0]$ does not cross the negative semiaxis, we need $\xi \neq izr$, for any $r \in \mathbb{R} \setminus (-1, 1)$. However, since we introduced $\varepsilon > 0$, and z_ε respectively, this does not happen in the above integration path. Hence the integrand is holomorphic and we are able to apply Cauchy's theorem to change the integration path to

$$\mathcal{H}_\nu^{(1,2)}(z) = \frac{\pm 2^{1-\nu} z^{\nu-\frac{1}{2}} e^{\mp iz}}{i\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \lim_{\varepsilon\to 0^+} \lim_{R\to\infty} \left(\int_{\gamma_3} \Big(1+\frac{\xi^2}{z^2}\Big)^{\nu-\frac{1}{2}} e^{-\xi} d\xi + \int_{\gamma_4} \Big(1+\frac{\xi^2}{z^2}\Big)^{\nu-\frac{1}{2}} e^{-\xi} d\xi \right). \tag{A.40}$$

The first task is now to show that the integral along γ_4 vanishes in the limit $R \to \infty$. To do so, we start with the estimate

$$\left| \int_{\gamma_4} \Big(1+\frac{\xi^2}{z^2}\Big)^{\nu-\frac{1}{2}} e^{-\xi} d\xi \right|$$
$$= e^{-\mathrm{Re}(z)R} \left| \int_0^{\mathrm{Im}(Rz-z_\varepsilon)} \Big(1 + \frac{(\mathrm{Re}(z)R + i\,\mathrm{Im}(z_\varepsilon) + it)^2}{z^2}\Big)^{\nu-\frac{1}{2}} e^{-it} dt \right| \tag{A.41}$$
$$\leq e^{\pi|\mathrm{Im}(\nu)| - \mathrm{Re}(z)R} \int_0^{|\mathrm{Im}(Rz-z_\varepsilon)|} \left| 1 + \frac{(\mathrm{Re}(z)R + i\,\mathrm{Im}(z_\varepsilon) + it)^2}{z^2} \right|^{\mathrm{Re}(\nu)-\frac{1}{2}} dt,$$

where we used the estimate

$$|a^b| = \left|e^{b\ln(a)}\right| = \left|e^{b\ln|a| + ib\,\mathrm{Arg}(a)}\right| = |a|^{\mathrm{Re}(b)} e^{-\mathrm{Im}(b)\,\mathrm{Arg}(a)} \leq |a|^{\mathrm{Re}(b)} e^{\pi|\mathrm{Im}(b)|}, \tag{A.42}$$

which holds true for every $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$.

- For $0 \leq \mathrm{Re}(\nu) \leq \frac{1}{2}$ we can further estimate the integrand as

$$\Big|1+\frac{(\mathrm{Re}(z)R+i\,\mathrm{Im}(z_\varepsilon)+it)^2}{z^2}\Big| \geq \Big|\frac{\mathrm{Re}(z)R+i\,\mathrm{Im}(z_\varepsilon)+it}{z}\Big|^2-1 \geq \frac{\mathrm{Re}(z)^2R^2}{|z|^2}-1,$$

which leads for $R \geq \frac{|z|}{\mathrm{Re}(z)}$ to the convergence

$$\Big|\int_{\gamma_4}\Big(1+\frac{\xi^2}{z^2}\Big)^{\nu-\frac{1}{2}}e^{-\xi}d\xi\Big| \leq e^{\pi|\mathrm{Im}(\nu)|-\mathrm{Re}(z)R}\Big(\frac{\mathrm{Re}(z)^2R^2}{|z|^2}-1\Big)^{\mathrm{Re}(\nu)-\frac{1}{2}}|\mathrm{Im}(Rz-z_\varepsilon)| \overset{R\to\infty}{\longrightarrow} 0.$$

- For $\mathrm{Re}(\nu) \geq \frac{1}{2}$, we estimate the integrand as

$$\Big|1+\frac{(\mathrm{Re}(z)R+i\,\mathrm{Im}(z_\varepsilon)+it)^2}{z^2}\Big| \leq 1+\frac{(\mathrm{Re}(z)^2R^2+(\mathrm{Im}(z_\varepsilon)+t)^2}{|z|^2},$$

which means, that the integrand in (A.41) grows as $R^{2\,\mathrm{Re}(\nu)-1}$. Since also the upper bound grows of order R, the whole integral grows as $R^{2\,\mathrm{Re}(\nu)}$. Together with the exponential prefactor $e^{-\mathrm{Re}(z)R}$, the whole right hand side of (A.41) converges to zero as $R \to \infty$ and hence also in this case we proved that the γ_4-integral vanishes in the limit $R \to \infty$.

This shows, that in the limit $R \to \infty$ in (A.40) only the γ_3-integral remains. Parametrising the path integral gives

$$\begin{aligned}\mathcal{H}_\nu^{(1,2)}(z) &= \frac{\mp i2^{1-\nu}z^{\nu-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}e^{\mp iz}\lim_{\varepsilon\to0^+}\lim_{R\to\infty}e^{-z_\varepsilon}\int_0^{\mathrm{Re}(z)R-\mathrm{Re}(z_\varepsilon)}\Big(1+\frac{(z_\varepsilon+t)^2}{z^2}\Big)^{\nu-\frac{1}{2}}e^{-t}dt\\ &= \frac{\mp i2^{1-\nu}z^{\nu-\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\lim_{\varepsilon\to0^+}\int_0^\infty\Big(1+\frac{(z_\varepsilon+t)^2}{z^2}\Big)^{\nu-\frac{1}{2}}e^{-t}dt. \end{aligned} \tag{A.43}$$

In the final step we now have to argue why we are allowed to carry the limit $\varepsilon \to 0^+$ inside the integral. In order to apply the dominated convergence theorem, we estimate the integrand as follows.

- If $0 \leq \mathrm{Re}(\nu) \leq \frac{1}{2}$ we use the inequality

$$|1+w| \geq \begin{cases} 1, & \text{if } \mathrm{Re}(w) \geq 0,\\ \frac{|\mathrm{Im}(w)|}{|w|}, & \text{if } \mathrm{Re}(w) < 0,\end{cases}$$

which holds for all $w \in \mathbb{C}$, to estimate the integrand as

$$\Big|1+\frac{(z_\varepsilon+t)^2}{z^2}\Big| = \frac{t^2}{|z|^2}\Big|1\mp i(2-\varepsilon)\frac{z}{t}\Big|\Big|1\pm i\varepsilon\frac{z}{t}\Big| \geq \frac{t^2\,\mathrm{Re}(z)}{|z|^3}.$$

Since the right hand side is independent of ε and the factor e^{-t} ensures convergence on $(0,\infty)$, we found an integrable majorant.

- If $\mathrm{Re}(\nu) \geq \frac{1}{2}$, we simply estimate

$$\Big|1+\frac{(z_\varepsilon+t)^2}{z^2}\Big| \leq 1+\frac{(|z_\varepsilon|+t)^2}{|z|^2} \leq 1+\frac{(|z|+t)^2}{|z|^2}.$$

  Again this upper bound is ε-independent and together with the factor e^{-t} an integrable majorant.

Hence we verified, that the dominated convergence theorem is applicable and we are allowed to carry the limit $\varepsilon \to 0^+$ inside the integral (A.43) to finally end up with the stated integral representation (A.38). □

Next we give some estimates and limits of the modified Hankel function.

Lemma A.8. Let $\mathrm{Re}(\nu) \geq 0$ and $\alpha \in (0, \frac{\pi}{2})$. Then there exist constants $C_{\nu,\alpha}, D_{\nu,\alpha} \geq 0$, such that the functions $\mathcal{H}_\nu^{(1,2)}$ and their derivatives are bounded by

$$\big|\mathcal{H}_\nu^{(1,2)}(z)\big| \leq C_{\nu,\alpha}\begin{cases} 1, & \text{if } \mathrm{Re}(\nu) \leq \frac{1}{2}, \\ 1+\frac{1}{|z|^{\mathrm{Re}(\nu)-\frac{1}{2}}}, & \text{if } \mathrm{Re}(\nu) \geq \frac{1}{2}, \end{cases} \qquad |\mathrm{Arg}(z)| \leq \alpha. \tag{A.44a}$$

$$\big|\mathcal{H}_\nu^{(1,2)\prime}(z)\big| \leq D_{\nu,\alpha}\begin{cases} 1+\frac{1}{|z|}, & \text{if } \mathrm{Re}(\nu) \leq \frac{1}{2} \\ 1+\frac{1}{|z|^{\mathrm{Re}(\nu)+\frac{1}{2}}}, & \text{if } \mathrm{Re}(\nu) \geq \frac{1}{2}. \end{cases} \qquad |\mathrm{Arg}(z)| \leq \alpha. \tag{A.44b}$$

Moreover, for $\mathrm{Re}(\nu) \leq \frac{1}{2}$ they admit the limits

$$\lim_{\substack{z\to\infty \\ |\mathrm{Arg}(z)|\leq\alpha}} \mathcal{H}_\nu^{(1,2)}(z) = \frac{(\mp i)^{\nu+\frac{1}{2}}\sqrt{2}}{\sqrt{\pi}} \quad \text{and} \quad \lim_{\substack{z\to 0 \\ |\mathrm{Arg}(z)|\leq\alpha}} \mathcal{H}_\nu^{(1,2)}(z) = 0. \tag{A.45}$$

Proof. For the proof of the boundedness of $\mathcal{H}_\nu^{(1,2)}$ in (A.44a), we use the integral representation (A.38) and estimate

$$\big|\mathcal{H}_\nu^{(1,2)}(z)\big| \leq \frac{\sqrt{2}\,e^{\pi|\mathrm{Im}(\nu)|}}{\sqrt{\pi}\,|\Gamma(\nu+\frac{1}{2})|}\int_0^\infty \Big|\frac{t^2}{2z} \mp it\Big|^{\mathrm{Re}(\nu)-\frac{1}{2}} e^{-t}dt, \tag{A.46}$$

where we used the identity (A.42) for the complex power. For $\mathrm{Re}(\nu) \leq \frac{1}{2}$, we need the following lower bound of the integrand,

$$\Big|\frac{t}{2z} \mp i\Big|^2 = 1 \pm \frac{t\,\mathrm{Im}(z)}{|z|^2} + \frac{t^2}{4|z|^2} \geq \frac{\mathrm{Re}(z)^2}{|z|^2} = \cos^2(\mathrm{Arg}(z)) \geq \cos^2(\alpha), \quad |\mathrm{Arg}(z)| \leq \alpha, \tag{A.47}$$

where the first inequality comes from minimizing the parabola $t \mapsto 1 \pm \frac{t\,\mathrm{Im}(z)}{|z|^2} + \frac{t^2}{4|z|^2}$. This gives

$$\begin{aligned}\big|\mathcal{H}_\nu^{(1,2)}(z)\big| &\leq \frac{\sqrt{2}\,e^{\pi|\mathrm{Im}(\nu)|}\cos(\alpha)^{\mathrm{Re}(\nu)-\frac{1}{2}}}{\sqrt{\pi}\,|\Gamma(\nu+\frac{1}{2})|}\int_0^\infty t^{\mathrm{Re}(\nu)-\frac{1}{2}}e^{-t}dt \\ &= \frac{\sqrt{2}\,e^{\pi|\mathrm{Im}(\nu)|}\cos(\alpha)^{\mathrm{Re}(\nu)-\frac{1}{2}}\Gamma(\mathrm{Re}(\nu)+\frac{1}{2})}{\sqrt{\pi}\,|\Gamma(\nu+\frac{1}{2})|}.\end{aligned}$$

For $\mathrm{Re}(\nu) \geq \frac{1}{2}$ on the other hand we use the inequality

$$\Big|\frac{t}{2z} \mp i\Big|^{\mathrm{Re}(\nu)-\frac{1}{2}} \leq 2^{\mathrm{Re}(\nu)-\frac{1}{2}}\Big(\Big(\frac{t}{2|z|}\Big)^{\mathrm{Re}(\nu)-\frac{1}{2}} + 1\Big), \tag{A.48}$$

to estimate

$$\begin{aligned}\big|\mathcal{H}_\nu^{(1,2)}(z)\big| &\leq \frac{2^{\mathrm{Re}(\nu)}e^{\pi|\mathrm{Im}(\nu)|}}{\sqrt{\pi}\,|\Gamma(\nu+\frac{1}{2})|}\int_0^\infty \Big(\Big(\frac{t^2}{2|z|}\Big)^{\mathrm{Re}(\nu)-\frac{1}{2}} + t^{\mathrm{Re}(\nu)-\frac{1}{2}}\Big)e^{-t}dt \\ &= \frac{\sqrt{2}\,e^{\pi|\mathrm{Im}(\nu)|}\Gamma(2\,\mathrm{Re}(\nu))}{\sqrt{\pi}\,|\Gamma(\nu+\frac{1}{2})||z|^{\mathrm{Re}(\nu)-\frac{1}{2}}} + \frac{2^{\mathrm{Re}(\nu)}e^{\pi|\mathrm{Im}(\nu)|}\Gamma(\mathrm{Re}(\nu)+\frac{1}{2})}{\sqrt{\pi}\,|\Gamma(\nu+\frac{1}{2})|}.\end{aligned}$$

For the inequality (A.44b) we first use the recurrence relation [2, Eq.(9.1.27)] to write the derivative as

$$\mathcal{H}_\nu^{(1,2)\prime}(z) = \Big(\frac{\nu+\frac{1}{2}}{z} \mp i\Big)\mathcal{H}_\nu(z) - \mathcal{H}_{\nu+1}(z), \quad \mathrm{Re}(z) > 0.$$

Hence we can use the already derived estimate (A.44a) to get

$$\begin{aligned}|\mathcal{H}_\nu^{(1,2)\prime}(z)| &\leq \begin{cases} \Big(\frac{|\nu+\frac{1}{2}|}{|z|}+1\Big)C_{\nu,\alpha} + C_{\nu+1,\alpha}\Big(1+\frac{1}{|z|^{\mathrm{Re}(\nu)+\frac{1}{2}}}\Big), & \mathrm{Re}(z) \leq \frac{1}{2}, \\ \Big(\frac{|\nu+\frac{1}{2}|}{|z|}+1\Big)C_{\nu,\alpha}\Big(1+\frac{1}{|z|^{\mathrm{Re}(\nu)-\frac{1}{2}}}\Big) + C_{\nu+1,\alpha}\Big(1+\frac{1}{|z|^{\mathrm{Re}(\nu)+\frac{1}{2}}}\Big), & \mathrm{Re}(z) \geq \frac{1}{2}. \end{cases} \\ &\leq D_{\nu,\alpha}\begin{cases} 1+\frac{1}{|z|}, & \mathrm{Re}(z) \leq \frac{1}{2} \\ 1+\frac{1}{|z|^{\mathrm{Re}(\nu)+\frac{1}{2}}}, & \mathrm{Re}(z) \geq \frac{1}{2}. \end{cases}\end{aligned}$$

For the proof of the first limit in (A.45), we once more use the integral representation (A.38). With the inequality (A.47), where the right hand side is z-independent, we are allowed to interchange limit and integral and get

$$\begin{aligned}\lim_{\substack{|z|\to\infty \\ |\mathrm{Arg}(z)|\leq\alpha}} \mathcal{H}_\nu^{(1,2)}(z) &= \frac{\mp i\sqrt{2}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\lim_{\substack{|z|\to\infty \\ |\mathrm{Arg}(z)|\leq\alpha}}\int_0^\infty \Big(\frac{t^2}{2z} \mp it\Big)^{\nu-\frac{1}{2}}e^{-t}dt \\ &= \frac{(\mp i)^{\nu+\frac{1}{2}}\sqrt{2}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\infty t^{\nu-\frac{1}{2}}e^{-t}dt \\ &= \frac{(\mp i)^{\nu+\frac{1}{2}}\sqrt{2}}{\sqrt{\pi}}.\end{aligned}$$

For the second limit in, we are allowed to carry the limit inside the integral once more because of the inequality (A.47). This gives

$$\lim_{\substack{|z|\to 0 \\ |\mathrm{Arg}(z)|\leq\alpha}} \mathcal{H}_\nu^{(1,2)}(z) = \frac{\mp i\sqrt{2}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\lim_{\substack{|z|\to 0 \\ |\mathrm{Arg}(z)|\leq\alpha}}\int_0^\infty \Big(\frac{t^2}{2z} \mp it\Big)^{\nu-\frac{1}{2}}e^{-t}dt = 0,$$

where the integrand vanishes in the limit $|z| \to 0$ since $\mathrm{Re}(\nu) < \frac{1}{2}$. □

Also for the Bessel function we obtain the following inequalities.

Lemma A.9. Let $\nu > \frac{1}{2}$ and $\alpha \in (0, \frac{\pi}{2})$. Then there exist constants $C_\nu, D_\nu \geq 0$, such that the Bessel function J_ν is bounded by

$$\big|J_\nu(z)\big| \leq \frac{C_\nu}{\sqrt{|z|}} e^{|\operatorname{Im}(z)|} \quad \text{and} \quad \big|J_\nu'(z)\big| \leq \frac{D_\nu}{|z|^{\frac{3}{2}}} e^{|\operatorname{Im}(z)|}, \quad \operatorname{Re}(z) > 0. \tag{A.49}$$

Moreover, for some $E_\nu \geq 0$, the asymptotics for large arguments of J_ν can be read off the estimate

$$\Big|J_\nu(z) - \frac{\sqrt{2}}{\sqrt{\pi z}} \cos\Big(z - \frac{(2\nu+1)\pi}{4}\Big)\Big| \leq E_\nu \Big(\frac{1}{|z|^{\frac{1}{2}}} + \frac{1}{|z|^{\frac{3}{2}}}\Big) e^{|\operatorname{Im}(z)|}, \quad \operatorname{Re}(z) > 0. \tag{A.50}$$

Proof. According to [2, Eq.(9.2.5)], the Bessel function admits the representation

$$J_\nu(z) = \frac{\sqrt{2}}{\sqrt{\pi z}} \Big(P_\nu(z) \cos\Big(z - \frac{(2\nu+1)\pi}{4}\Big) - Q_\nu(z) \sin\Big(z - \frac{(2\nu+1)\pi}{4}\Big)\Big), \quad \operatorname{Re}(z) > 0, \tag{A.51}$$

using the series

$$P_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(2z)^{2n}} \prod_{k=1}^{2n} \Big(\nu^2 - \frac{(2k-1)^2}{4}\Big), \tag{A.52a}$$

$$Q_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2z)^{2n+1}} \prod_{k=1}^{2n+1} \Big(\nu^2 - \frac{(2k-1)^2}{4}\Big). \tag{A.52b}$$

First of all, since obviously $P_\nu(z)$ and $Q_\nu(z)$ are bounded for $|z| \geq 1$ by some constant $C \geq 0$, we get

$$|J_\nu(z)| \leq \frac{2C\sqrt{2}}{\sqrt{\pi|z|}} e^{|\operatorname{Im}(z)|}, \quad |z| \geq 1,$$

However, since J_ν is continuous this inequality is obviously also fulfilled for $|z| \leq 1$ and hence (A.49) is satisfied. Using the connection formula $J_\nu'(z) = -J_{\nu+1}(z) + \frac{\nu}{z} J_\nu(z)$ from [2, Eq.(9.1.27)] also the second inequality in (A.49) follows immediately.

For the proof of the asymptotic inequality (A.50), we will once more use the representation (A.51) and again start with $|z| \geq 1$. The functions (A.52a) and (A.52b) can be estimated as

$$\begin{aligned}
|P_\nu(z) - 1| &\leq \sum_{n=1}^{\infty} \frac{1}{(2n)!|2z|^{2n}} \prod_{k=1}^{2n} \Big|\nu^2 - \frac{(2k-1)^2}{4}\Big| \\
&= \frac{1}{2|z|} \sum_{n=0}^{\infty} \frac{1}{(2n+2)!|2z|^{2n+1}} \prod_{k=1}^{2n+2} \Big|\nu^2 - \frac{(2k-1)^2}{4}\Big| \\
&\leq \frac{1}{2|z|} \sum_{n=0}^{\infty} \frac{1}{(2n+2)!2^{2n+1}} \prod_{k=1}^{2n+2} \Big|\nu^2 - \frac{(2k-1)^2}{4}\Big| =: \frac{D_\nu^{(1)}}{|z|} \quad |z| \geq 1,
\end{aligned}$$

and as

$$\begin{aligned}|Q_\nu(z)| &\le \sum_{n=0}^{\infty} \frac{1}{(2n+1)!|2z|^{2n+1}} \prod_{k=1}^{2n+1} \left|\nu^2 - \frac{(2k-1)^2}{4}\right| \\ &= \frac{1}{2|z|} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!|2z|^{2n}} \prod_{k=1}^{2n+1} \left|\nu^2 - \frac{(2k-1)^2}{4}\right| \\ &\le \frac{1}{2|z|} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!4^n} \prod_{k=1}^{2n+1} \left|\nu^2 - \frac{(2k-1)^2}{4}\right| =: \frac{D_\nu^{(2)}}{|z|}, \quad |z| \ge 1.\end{aligned}$$

Altogether, this leads to the following estimate of the Bessel function

$$\begin{aligned}&\left|J_\nu(z) - \frac{\sqrt{2}}{\sqrt{\pi z}} \cos\left(z - \frac{(2\nu+1)\pi}{4}\right)\right| \\ &\quad = \frac{\sqrt{2}}{\sqrt{\pi|z|}} \left|(P_\nu(z) - 1)\cos\left(z - \frac{(2\nu+1)\pi}{4}\right) - Q_\nu(z)\sin\left(z - \frac{(2\nu+1)\pi}{4}\right)\right| \\ &\quad \le \frac{\sqrt{2}\,(D_\nu^{(1)} + D_\nu^{(2)})}{\sqrt{\pi}\,|z|^{\frac{3}{2}}} e^{|\operatorname{Im}(z)|}, \quad |z| \ge 1. \qquad \text{(A.53)}\end{aligned}$$

Since J_ν is bounded as already proven in (A.49), we can also estimate

$$\left|J_\nu(z) - \frac{\sqrt{2}}{\sqrt{\pi z}} \cos\left(z - \frac{(2\nu+1)\pi}{4}\right)\right| \le \left(C_\nu + \frac{\sqrt{2}}{\sqrt{\pi|z|}}\right) e^{|\operatorname{Im}(z)|}, \quad |z| \le 1. \qquad \text{(A.54)}$$

Combining now (A.53) and (A.54) gives the stated estimate (A.50). □

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